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**The coarseness of a graph**

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# The coarseness of a graph[1]

by

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## Introduction

A graph is *planar* if it can be drawn in the plane (or on a sphere) so that no two of its edges intersect. The *thickness* $t(G)$ of a graph $G$, introduced by Tutte [3], is the minimum number of edge-disjoint planar subgraphs whose union is $G$. In contrast to the thickness is the *coarseness* [2] $c(G)$ defined as the maximum number of edge-disjoint nonplanar subgraphs contained in $G$. Obviously, $G$ is planar if and only if $c(G) = 0$.

We present some results here concerning the coarseness of a graph. In particular, bounds on the coarseness of complete graphs are given.

## Basic results

A *subdivision* of a graph $G$ is a graph obtained from $G$ by replacing an edge $x = uv$ of $G$ by a new vertex $w$ and the two new edges $uw$ and $wv$. Two graphs $G_1$ and $G_2$ are *homeomorphic* if there exists a graph $G_3$ which can be obtained from each of $G_1$ and $G_2$ by a sequence of subdivisions. Following Kuratowski [2], we call a graph *skew* if it is homeomorphic to either the complete graph $K_5$ or the complete bigraph $K_{3,3}$. These graphs are shown in Figure 1. The classic result of Kuratowski then states that a graph is planar if and only if it contains no skew subgraph.
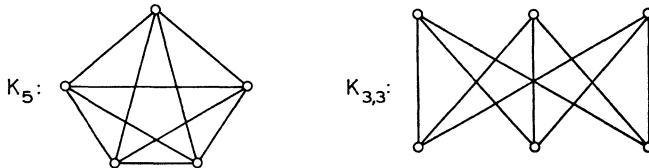


$K_5$:    $K_{3,3}$:

Figure 1

[2] This concept was suggested by Professor Paul Erdös.

If a graph $G$ has coarseness $n$, then the $n$ edge-disjoint nonplanar subgraphs contained in $G$ may all be chosen to be skew, for if one were not skew, then by Kuratowski's theorem, some extraneous edges could be deleted to obtain a skew graph.

A *block of a graph* $G$ is a maximal connected subgraph of $G$ which contains no cut vertices. It is clear that any skew subgraph of $G$ must be a subgraph of some block of $G$; thus we arrive at the following result.

THEOREM 1. *The coarseness of a graph is the sum of the coarsenesses of its blocks.*

This implies that it is sufficient to determine the coarseness of graphs having no cut vertices. Another observation which can be made concerns the number of edges in a graph. The smallest number of edges which a nonplanar graph can possess is 9 (and then only if the graph is $K_{3,3}$). Therefore, it immediately follows that if a graph $G$ has $q$ edges, then

$$(1) \qquad\qquad c(G) \leqq \left[\frac{q}{9}\right].$$

In [1] it was shown that every graph is homeomorphic to a graph having thickness 1 or 2. As we shall now see, however, coarseness is invariant under homeomorphism.

THEOREM 2. *Homeomorphic graphs have the same coarseness.*

PROOF. Let $G_1$ and $G_2$ be homeomorphic graphs. Then there exists a graph $G_3$ which can be obtained from each of $G_1$ and $G_2$ by a sequence of subdivisions. To show that $c(G_1) = c(G_2)$, it is clearly sufficient to prove that the coarseness of one of $G_1$ and $G_2$, say $G_1$, equals that of $G_3$. To prove this, however, it is sufficient to show that $c(G_1) = c(G_1')$, where $G_1'$ is a subdivision of $G_1$. Thus, there is an edge $x = uv$ of $G_1$ which has been replaced by a new vertex $w$ and the two new edges $uw$ and $wv$ to obtain $G_1'$. Assume $c(G_1) = n$. Consider a set of $n$ edge-disjoint skew subgraphs of $G_1$. If none of these subgraphs contains $x$, then this is a set of subgraphs of $G_1'$, so $c(G_1') \geqq n$. Should one of these subgraphs $H$ contain $x$, then by replacing $x$ by the vertex $w$ and the edges $uw$ and $wv$, a skew subgraph $H'$ of $G_1'$ is produced, which together with the other $n-1$ subgraphs constitutes a collection of $n$ edge-disjoint skew subgraphs of $G_1'$. Hence, in any case, $c(G_1') \geqq n$. To show that $c(G_1') > n$ is not possible, suppose that $G_1'$ contains $n+1$ edge-disjoint skew subgraphs. If neither of the edges $uv$ nor $wv$ is contained in any of these subgraphs, then these skew graphs

are also subgraphs of $G_1$, but this cannot happen since $c(G_1) = n$. Thus, one of these edges, say $uw$, must belong to some skew graph $S$. Because no skew graph contains a vertex of degree 1, the edge $wv$ is also necessarily in $S$. But then, a skew graph $S'$ can be obtained by replacing the edges $uw$ and $wv$ and the vertex $w$ by the edge $x$. The graph $S'$ along with the other $n$ skew graphs form a set of $n+1$ edge-disjoint skew subgraphs of $G_1$, but this is also a contradiction.

By taking a sufficiently large number of subdivisions of a graph $G$ with high coarseness, a graph $G'$ can be obtained whose thickness is 2. By the previous theorem, however, $c(G) = c(G')$. We state this as a corollary.

COROLLARY 2a. *For every positive integer $n$, there exists a graph $G$ such that $c(G)-t(G) = n$.*

More specifically, consider the nonplanar complete bigraph $K_{3,3n}$. It is easily observed that $K_{3,3n}$ is the edge-disjoint union of $K_{1,3n}$ and $K_{2,3n}$, each of which is planar. Thus $t(K_{3,3n}) = 2$. One also sees that $K_{3,3n}$ contains $n$ edge-disjoint copies of $K_{3,3}$ so that $c(K_{3,3n}) \geqq n$, but since this graph has $9n$ edges, by inequality (1), $c(K_{3,3n}) = n$. These facts have another, perhaps somewhat unexpected interpretation: For any positive integer $n$, it is possible to find two planar graphs which can be combined into a graph $G$, which then in turn can be decomposed into $n$ edge-disjoint nonplanar graphs.

## On the coarseness of complete graphs

The complete graph $K_p$ with $p$ vertices has $p(p-1)/2$ edges; therefore by (1), we have

$$(2) \qquad\qquad c(K_p) \leqq p(p-1)/18.$$

In this section we also present lower bounds for $c(K_p)$. The determination of a formula for $c(K_p)$ appears to be a very difficult problem. The known values for $c(K_p)$ are given in Table 1. Thus the smallest value of $p$ for which $c(K_p)$ is unknown is 13. Also unknown is the smallest positive integer $n$ for which there exists no $p$ such that $c(K_p) = n$.

For several of the entries in Table 1, a simple construction and a degree argument provides the answer. We illustrate this for $p = 9$. By dividing the 9 vertices into 3 sets of 3 vertices each, one sees that every two of these sets induces a copy of $K_{3,3}$ so that

| $p$ | 1 — 4 | 5 — 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| $c(K_p)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

$c(K_9) \geqq 3$. Since $K_9$ has 36 edges, it follows, by (2), that $c(K_9) \leqq 4$. However, for $c(K_9)$ to have the value 4, it would necessarily have to contain 4 edge-disjoint copies of $K_{3,3}$, but each vertex of $K_9$ has degree 8 so at least two edges meeting at each vertex cannot be in any of these copies of $K_{3,3}$ implying that 3 is the maximum number of edge-disjoint nonplanar skew graphs contained in $K_9$.

For $p = 11$ and $p = 12$ an argument similar to that for $p = 9$ can be given, but for $p = 10$ a different technique is employed. We label the vertices of $K_{10}$ as $v_1, v_2, \cdots, v_{10}$. Then $c(K_{10}) \geqq 4$ since $K_{10}$ contains 4 edge-disjoint skew graphs, e.g., those shown in Figure 2.

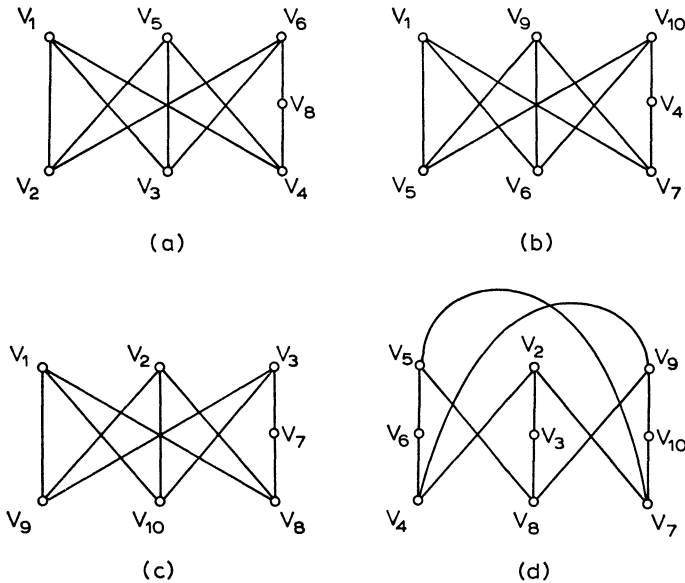

Figure 2

By (2), however, $c(K_{10}) \leqq 5$. If $c(K_{10}) = 5$, then, since $K_{10}$ has 45 edges, $K_{10}$ must contain 5 edge disjoint copies $G_i$ $(1 \leqq i \leqq 5)$ of the graph $K_{3,3}$. Necessarily, each vertex appears in 3 of these copies. We now relabel the vertices of $K_{10}$ by $u_1, u_2, \cdots, u_{10}$ so

that $u_1$ appears in $G_1$, $G_2$, and $G_3$ and so that the vertices to which $u_1$ is adjacent are as indicated in Figure 3.
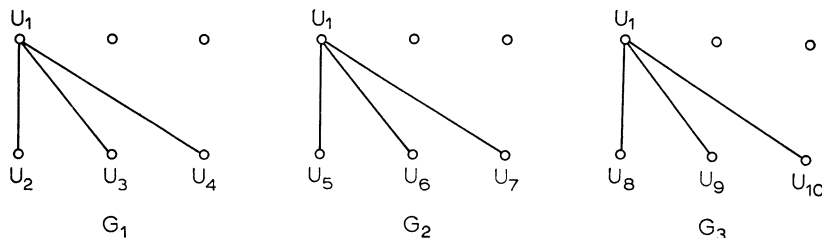


Figure 3

The two unlabeled vertices in $G_1$ clearly belong to the set $\{u_5, u_6, \cdots, u_{10}\}$. If both these vertices are in the set $\{u_5, u_6, u_7\}$ or in $\{u_8, u_9, u_{10}\}$, then these two vertices must appear together in some copy of $K_{3,3}$ since the edge determined by them is in some $G_i$. However, no such graph can exist, for these vertices must be mutually adjacent to two other vertices in this $G_i$. Hence, of the two unlabeled vertices in $G_1$, one must be in $\{u_5, u_6, u_7\}$ and the other in $\{u_8, u_9, u_{10}\}$, say $u_5$ and $u_8$. Similarly, of the two unlabeled vertices in $G_2$, one must belong to the set $\{u_2, u_3, u_4\}$ and the other to $\{u_8, u_9, u_{10}\}$. However, no vertex of $\{u_2, u_3, u_4\}$ can be in $G_2$ since each of these vertices is already adjacent to $u_5$ in $G_1$. This is a contradiction; thus $K_{10}$ does not contain 5 edge-disjoint copies of $K_{3,3}$ so that $c(K_{10}) = 4$. This fact will be useful in obtaining a lower bound for $c(K_p)$.

THEOREM 3. For the complete graph $K_p$,

$$c(K_p) \geqq \begin{cases} \dbinom{r}{2} & \text{if } p = 3r \\ \dbinom{r}{2} + \left[\dfrac{r}{3}\right] & \text{if } p = 3r+1 \\ \dbinom{r}{2} + \left[\dfrac{3r+1}{4}\right] & \text{if } p = 3r+2. \end{cases}$$

PROOF. For $p = 3r$, the vertices of $K_p$ can be divided into $r$ sets of 3 vertices each. Every two of these sets clearly determine a copy of $K_{3,3}$. Since there are $\binom{r}{2}$ such pairs of sets, $K_{3r}$ contains $\binom{r}{2}$ copies of $K_{3,3}$ and thus $c(K_{3r}) \geqq \binom{r}{2}$.

Assume $p = 3r+1$. Let $v$ be one vertex of $K_p$ and divide the remaining vertices into $r$ sets of 3 vertices each. As before, one

gets $\binom{r}{2}$ copies of $K_{3,3}$. However, if one takes 3 triples of vertices together with $v$, 4 skew graphs can be obtained as in the decomposition of $K_{10}$. These can replace the 3 copies of $K_{3,3}$ which had been obtained originally. There are $\left[\frac{r}{3}\right]$ disjoint collections of 3 triples, and it is possible to produce an additional skew graph for each such collection. Hence, $c(K_{3r+1}) \geq \binom{r}{2} + \left[\frac{r}{3}\right]$.

For $p = 3r+2$, we again take $r$ sets of 3 vertices each and denote the remaining vertices by $u$ and $v$. On one of these sets together with $u$ and $v$, a copy of $K_5$ can be formed. Let $C$ denote the collection of $3\left[\frac{r+2}{4}\right]$ vertices in $\left[\frac{r+2}{4}\right]$ of the remaining $r-1$ sets. A skew graph homeomorphic to $K_5$ can then be constructed for each of the $r-1-\left[\frac{r+2}{4}\right]$ still remaining sets by using such a set with $u$ and $v$ and assigning a vertex of $C$ as a vertex of degree 2 between $u$ and $v$. Since

$$r-1-\left[\frac{r+2}{4}\right] = r + \left[-\frac{r+3}{4}\right] \leq \frac{3r-3}{4} \text{ and } 3\left[\frac{r+2}{4}\right] \geq 3\left(\frac{r-1}{4}\right),$$

there are enough vertices to accomplish this. Therefore,

$$c(K_{3r+2}) \geq \binom{r}{2} + 1 + \left(r-1-\left[\frac{r+2}{4}\right]\right) = \binom{r}{2} + \left[\frac{3r+1}{4}\right].$$

In terms of $p$, the bounds provided by (2) and Theorem 3 can be stated as follows:

$$\frac{p^2-3p}{18} \leq c(K_p) \leq \frac{p^2-p}{18} \qquad\qquad p \equiv 0 \mod 3$$

(3)
$$\frac{p^2-5p+4}{18} + \left[\frac{p-1}{9}\right] \leq c(K_p) \leq \frac{p^2-p}{18} \quad p \equiv 1 \mod 3$$

$$\frac{p^2-7p+10}{18} + \left[\frac{p-1}{4}\right] \leq c(K_p) \leq \frac{p^2-p}{18} \quad p \equiv 2 \mod 3.$$

The inequalities (3) immediately establish an estimate of $c(K_p)$ for large $p$.

COROLLARY 3a. The coarseness of the complete graph $K_p$ is asymptotic to $p^2/18$.

The lower bound $a(p)$ and the upper bound $b(p)$ given by (3) are presented in Table 2 for $1 \leq p \leq 40$.

**TABLE 2**

Lower Bounds $a(p)$ and Upper Bounds $b(p)$ on $c(K_p)$, $1 \leq p \leq 40$

| $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a(p)$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 4 |
| $b(p)$ | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 |

| $p$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a(p)$ | 5 | 6 | 7 | 9 | 10 | 11 | 14 | 15 | 17 | 19 |
| $b(p)$ | 6 | 7 | 8 | 10 | 11 | 13 | 15 | 17 | 19 | 21 |

| $p$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a(p)$ | 21 | 23 | 26 | 28 | 30 | 34 | 36 | 39 | 43 | 45 |
| $b(p)$ | 23 | 25 | 28 | 30 | 33 | 36 | 39 | 42 | 45 | 48 |

| $p$ | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a(p)$ | 48 | 52 | 55 | 58 | 63 | 66 | 70 | 75 | 78 | 82 |
| $b(p)$ | 51 | 55 | 58 | 62 | 66 | 70 | 74 | 78 | 82 | 86 |

The amount by which $a(p)$ and $b(p)$ can differ is now considered.

**THEOREM 4.** The lower bound $a(p)$ and the upper bound $b(p)$ of $c(K_p)$, given in (3), satisfy the inequality:

$$b(p)-a(p) \leq \left[\frac{p+2}{9}\right].$$

**PROOF.** We consider the 3 cases $p = 3r$, $3r+1$, and $3r+2$ individually.

First, assume $p = 3r$. Then

$$b(p)-a(p) \leq \frac{3r(3r-1)}{18} - \frac{r(r-1)}{2} = \frac{r}{3} = \frac{p}{9}.$$

Next, suppose $p = 3r+1$. Then

$$b(p)-a(p) \leq \frac{(3r+1)3r}{18} - \frac{r(r-1)}{2} - \frac{r-2}{2} = \frac{r+2}{3} = \frac{p+5}{9}.$$

However, for $p = 3r+1$,

$$\left[\frac{p+5}{9}\right] > \left[\frac{p+2}{9}\right] \text{ implies that } \left[\frac{r+2}{3}\right] > \left[\frac{r+1}{3}\right].$$

This inequality is satisfied only when $r \equiv 1 \pmod 3$, i.e., when $r = 3s+1$. But then $p = 9s+4$, in which case

$$b(p)-a(p) \leqq \frac{(9s+4)(9s+3)}{18} - \frac{(3s+1)3s}{2} - s = \frac{3s+2}{3} = \frac{p+2}{9}.$$

Finally, assume $p = 3r+2$. Then

$$b(p)-a(p) \leqq \frac{(3r+2)(3r+1)}{18} - \frac{r(r-1)}{2} - \frac{3r-2}{4} = \frac{9r+22}{36} = \frac{3p+16}{36},$$

and

$$\frac{3p+16}{36} \leq \frac{p+2}{9} \qquad \text{for all } p \geqq 8.$$

Because

$$b(p)-a(p) \leqq \frac{p+2}{9} \qquad \text{for } p = 2 \text{ and } p = 5$$

also, the inequality is satisfied for all $p = 3r+2$.

Since the difference $b(p)-a(p)$ is always an integer, the theorem follows.

In conclusion, we present bounds for the number $c(K_{m,n})$. Since $K_{m,n}$ has $mn$ edges, we have, by (1), $c(K_{m,n}) \leq \left[\frac{mn}{9}\right]$. We now obtain a lower bound for $c(K_{m,n})$. Assume the vertex set $V$ of $K_{m,n}$ is partitioned as $V_1 \cup V_2$, where $[V_1] = m$, $[V_2] = n$, and every edge joins a vertex of $V_1$ with a vertex of $V_2$. The set $V_1$ has $\left[\frac{m}{3}\right]$ distinct triples of vertices while $V_2$ has $\left[\frac{n}{3}\right]$ such triples. Thus, $K_{m,n}$ has at least $\left[\frac{m}{3}\right]\left[\frac{n}{3}\right]$ copies of $K_{3,3}$. These observations are summarized below.

THEOREM 5. For the complete bigraph $K_{m,n}$,

$$\left[\frac{m}{3}\right]\left[\frac{n}{3}\right] \leqq c(K_{m,n}) \leqq \left[\frac{mn}{9}\right].$$

COROLLARY 5a. If $m \equiv n \equiv 0 \bmod 3$, then $c(K_{m,n}) = mn/9$.

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