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MANSOOR AHMAD

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A note on entire functions of infinite order

by

Mansoor Ahmad

It is well-known that for an entire function $f(z)$ of finite order

$$\log M(r) \sim \log u(r),$$

where $M(r)$ denotes the maximum modulus of $f(z)$ and $u(r)$ the maximum term of the power series for $f(z)$, when $|z| = r$.

The object of the note is to prove that the above result and a similar result for the derivatives of $f(z)$ hold for a much wider class of entire functions, which, for practical purposes, can be regarded as the whole class of entire functions. We also prove that Theorem 2 of [1] holds, under the only condition that $f(z)$ is of infinite k-th order. These results are more precise than those of Shah [2] and Shah and Khanna [3].

Let $a(r)$ be any function which is positive and non-decreasing for all positive r and tends to infinity with r. Let $L(r)$ be any positive function which tends to infinity with r and let k denote any fixed positive integer. $a(r)$ is said to be of finite k-th order, with respect to $L(r)$, if there exists a fixed λ', $\lambda' > 1$, such that

$$\overline{\lim_{r\to\infty}} \frac{l_k a(e^{\lambda' r})}{L(r)} < \underline{\lim_{r\to\infty}} \frac{l_1 a(e^r)}{L(r)},$$

where

$$l_0 x = x, \quad l_1 x = \log x, \quad l_2 x = \log\log x,$$
$$l_{-1} x = e^x, \quad l_{-2} x = e^{e^x}, \; \ldots .$$

If we replace r by $\log r$, the above condition takes the form that

$$\overline{\lim_{r\to\infty}} \frac{l_k a(r^{\lambda'})}{L(\log r)} < \underline{\lim_{r\to\infty}} \frac{l_1 a(r)}{L(\log r)}$$

for a fixed λ', $\lambda' > 1$.

Lemma. *If $a(r)$ is of finite k-th order, with respect to $L(r)$, then*

$$\overline{\lim_{r\to\infty}} \frac{\log a(\lambda r)}{a(r)} = 0$$

for every fixed λ, $\lambda > 1$.

PROOF. As a first step, we consider the case when $k = 3$.

By hypothesis, there exists a fixed number H, such that

$$\overline{\lim} \frac{l_3 a(e^{\lambda' r})}{L(r)} < H < \underline{\lim}_{r\to\infty} \frac{l_1 a(e^r)}{L(r)}.$$

Putting $b(r)$ for $a(e^r)$, we have

$$\overline{\lim_{r\to\infty}} \frac{l_3 b(\lambda' r)}{L(r)} < H < \underline{\lim}_{r\to\infty} \frac{l_1 b(r)}{L(r)}.$$

The interval $0 < r \leqq \infty$ can be divided into two sets S_1 and S_2, such that

$$\underline{\lim}_{r\to\infty} \frac{l_2 b(\lambda r)}{L(r)} > H,$$

for every fixed λ, $\lambda > 1$, when $r \in S_1$; and that S_2 can be divided into infinite sequences in such a way that, to every sequence σ, $\sigma \in S_2$, there corresponds, at least, one fixed number λ_σ, $\lambda_\sigma > 1$, which satisfies the condition that

$$\overline{\lim_{r\to\infty}} \frac{l_2 b(\lambda_\sigma \cdot r)}{L(r)} \leqq H,$$

when $r \in \sigma$. One of the two sets S_1 and S_2 may be empty. Since

$$\underline{\lim}_{r\to\infty} \frac{l_1 b(r)}{L(r)} > H,$$

it is easy to see that

$$\overline{\lim_{r\to\infty}} \frac{\log b(\lambda_\sigma \cdot r)}{b(r)} = 0,$$

when $r \in \sigma$. Also, since

$$\underline{\lim}_{r\to\infty} \frac{l_2 b(\lambda r)}{L(r)} > H,$$

for every fixed λ, $\lambda > 1$, when $r \in S_1$, we have

$$\frac{l_2 b(\lambda r)}{L(r)} > H,$$

when $r > r_0(\lambda)$ and $r \in S_1$; and so, it follows easily that there exists, at least, one continuous function $\varphi(r)$ such that $\varphi(r) > 1$ for all r, $0 < r < \infty$ and $\varphi(r) \to 1$, as $r \to \infty$, such that

$$\frac{l_2 b(r \cdot \varphi)}{L(r)} > H,$$

where $\varphi = \varphi(r)$ and $r \in S_1$. Since

$$\overline{\lim_{r\to\infty}} \frac{l_3 b(\lambda' r)}{L(r)} < H,$$

it follows easily that

$$\overline{\lim_{r\to\infty}} \frac{\log b(\lambda' r)}{b(r\cdot\varphi)} = 0,$$

when $r \in S_1$. Consequently, replacing λ_σ and λ' by smaller constants u_σ and u' respectively, we have

$$\overline{\lim_{r\to\infty}} \frac{\log b(u_\sigma \cdot r\varphi)}{b(r\varphi)} = 0,$$

when $r \in \sigma$ and

$$\overline{\lim_{r\to\infty}} \frac{\log b(u' \cdot r \cdot \varphi)}{b(r\cdot\varphi)} = 0,$$

when $r \in S_1$. Let S_1', S_2' and σ' denote the sets which correspond to S_1, S_2 and σ respectively, when r is replaced by $\log r$. We have

$$\overline{\lim_{r\to\infty}} \frac{\log a(r^{u'_\sigma \cdot \psi})}{a(r^\psi)} = 0,$$

where $r \in \sigma'$ and

$$\overline{\lim_{r\to\infty}} \frac{\log a(r^{u' \cdot \psi})}{a(r^\psi)} = 0,$$

when $r \in S_1'$, where $\psi = \varphi\,(\log r)$ and u_σ' corresponds to u_σ. Putting $r^\psi = R$, we have

$$\overline{\lim_{r\to\infty}} \frac{\log a(\lambda\cdot r^\psi)}{a(r^\psi)} = 0$$

for every fixed λ, $\lambda > 1$, there being no restriction on r. Hence putting $r^\psi = R$ the lemma follows.

Similarly, let us consider the case when $k = 4$ and let H be a fixed number such that

$$\overline{\lim_{r\to\infty}} \frac{l_4 a(e^{\lambda' r})}{L(r)} < H < \underline{\lim}_{r\to\infty} \frac{l_1 a(e^r)}{L(r)}.$$

Putting $b(r)$ for $a(e^r)$, we have

$$\overline{\lim_{r\to\infty}} \frac{l_4 b(\lambda' r)}{L(r)} < H < \underline{\lim}_{r\to\infty} \frac{l_1 b(r)}{L(r)}.$$

As before, the interval $0 \leqq r \leqq \infty$ can be divided into two sets S_1 and S_2 such that

$$\underline{\lim}_{r\to\infty} \frac{l_3 b(\lambda r)}{L(r)} > H,$$

for every fixed λ, $\lambda > 1$, when $r \in S_1$ and that S_2 can be divided into infinite sequences in the same way as before. Consequently, we have

$$\overline{\lim}_{r\to\infty} \frac{\log b(\lambda' r)}{b(r \cdot \varphi)} = 0,$$

when $r \in S_1$, where φ has the same meaning as before. The set S_2 can be divided into two sets S_1' and S_2', such that

$$\underline{\lim}_{r\to\infty} \frac{l_2 b(\lambda r)}{L(r)} > H,$$

for every fixed λ, $\lambda > 1$, when $r \in S_1'$; and that S_2' can be divided into infinite sequences in the same way as before. So, it follows easily that there exists, at least, one continuous function $\chi(r)$, satisfying the same conditions as $\varphi(r)$, such that

$$\frac{l_2 b(r\chi)}{L(r)} > H,$$

where $\chi = \chi(r)$ and $r \in S_1'$. Since

$$\overline{\lim}_{r\to\infty} \frac{l_3 b(\lambda_\sigma \cdot r)}{L(r)} \leqq H,$$

when $r \in \sigma \subset S_2$, it follows that

$$\overline{\lim}_{r\to\infty} \frac{\log b(\lambda_\sigma \cdot r)}{b(r \cdot \chi)} = 0,$$

when $r \in \sigma \cap S_1'$. Consequently, we have

$$\overline{\lim}_{r\to\infty} \frac{\log b(\lambda' r)}{b(r \cdot \varphi \cdot \chi)} = 0,$$

when $r \in S_1$ and

$$\overline{\lim}_{r\to\infty} \frac{\log b(\lambda_\sigma \cdot r)}{b(r \cdot \varphi \cdot \chi)} = 0,$$

when $r \in \sigma \cap S_1'$. Now, as before, it follows that

$$\overline{\lim}_{r\to\infty} \frac{\log a(\lambda \cdot r^{\varphi_1 \chi_1})}{a(r^{\varphi_1 \chi_1})} = 0$$

for every fixed λ, $\lambda > 1$, where $\varphi_1 = \varphi(\log r)$ and $\chi_1 = \chi(\log r)$.

Proceeding, just in the same way, it follows that the lemma holds for all k, $k > 1$.

REMARK. If

$$\overline{\lim_{r\to\infty}} \frac{l_k a(r^{\lambda'})}{L(r)} < \underline{\lim_{r\to\infty}} \frac{\lim l_{k_1} a(r)}{L(r)},$$

where k and k_1 are any fixed integers or zero, we put $a_1(r) = l_{k_1-1}a(r)$ and so, $a_1(r)$ is a function of finite $(k-k_1+1)$-th order. Therefore, by the lemma, we have

$$\overline{\lim_{r\to\infty}} \frac{\log a_1(\lambda r)}{a_1(r)} = 0$$

for every fixed λ, $\lambda > 1$; and thus it follows that

$$\overline{\lim_{r\to\infty}} \frac{\log a(\lambda r)}{a(r)} \leqq 1.$$

THEOREM 1. *If $f(z)$ is an entire function and if either* $\log M(r)$ *is of finite k-th order, with respect to $L(r)$, or $M(r)$ is of finite k-th order, with respect to $L(r)$, and*

$$\underline{\lim_{r\to\infty}} \frac{l_1 M(r)}{L(\log r)} = \infty,$$

then

(i) $\log M(r) \sim \log u(r)$

(ii) $\log \left(r^q M^q(r)\right) \sim \log u(r)$,

where $M^q(r)$ denotes the maximum modulus of the q-th differential coefficient of $f(z)$, when $|z| = r$.

PROOF OF (i). For an entire function, we have [5, § 4]

$$\begin{aligned}\log u(r) &\leqq \log M(r) \leqq \{1+o(1)\} \log u(r) + 2 \log \nu(hr)\\ &\leqq \{1+o(1)\} \log u(r) + 2 \log\log u(h'r)\\ &\leqq \{1+o(1)\} \log u(h'r) + 2 \log\log u(h'r)\\ &= \{1+o(1)\} \log u(h'r) \qquad (1)\end{aligned}$$

for all large r, h and h' being fixed numbers such that $h' > h > 1$. If $M(r)$ is of finite k-th order, with respect to $L(r)$, we have

$$\overline{\lim_{r\to\infty}} \frac{l_k M(r^{\lambda'})}{L_1(r)} < H < \underline{\lim_{r\to\infty}} \frac{l_1 M(r)}{L_1(r)},$$

where $L_1(r) = L(\log r)$. Therefore, if λ'' is a fixed number such that $\lambda' > \lambda'' > 1$, by (1), we have

$$\overline{\lim_{r\to\infty}} \frac{l_k u(h'^{\lambda''} r^{\lambda''})}{L_1(r)} \leqq \overline{\lim_{r\to\infty}} \frac{l_k M(r^{\lambda'})}{L_1(r)} < \underline{\lim_{r\to\infty}} \frac{l_1 M(r)}{L_1(r)} \leqq \underline{\lim_{r\to\infty}} \frac{l_1 u(h'r)}{L_1(r)}.$$

Since, by hypothesis,

$$\underline{\lim_{r\to\infty}} \frac{l_1 M(r)}{L_1(r)} = \infty,$$

by (1) it follows that

$$\underline{\lim_{r\to\infty}} \frac{l_1 u(h'r)}{L_1(r)} = \infty$$

and so, by the method of proof of the lemma, it follows that

$$\overline{\lim_{r\to\infty}} \frac{l_2 u(\lambda r)}{l_1 u(r)} = \overline{\lim_{r\to\infty}} \frac{l_2 u(\lambda h' r)}{l_1 u(h'r)} = 0$$

for every fixed λ, $\lambda > 1$. The rest of the proof, now, follows easily by (1).

Proof of (ii). Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We have

$$u(r) \leqq M(r) \leqq \sum_{n=1}^{q-1} |a_n| r^n + r^q M^q(r) < 2r^q M^q(r)$$

for all $r > r_0$, A being independent of r; and

$$r^q M^q(r) \leqq \sum_{n=q}^{\infty} n(n-1)\cdots(n-q+1)|a_n| r^n.$$

Also, in the notations of [5, § 4], for $n \geqq p$, we have

$$\begin{aligned} n(n-1)\cdots(n-q+1)|a_n|r^n &\leqq n(n-1)\cdots(n-q+1)e^{-G_n}r^n \\ &\leqq n(n-1)\cdots(n-q+1)u(r)\left(\frac{r}{R_p}\right)^{n-p+1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} r^q M^q(r) \leqq u(r) &\sum_{q}^{p-1} n(n-1)\cdots(n-q+1) \\ &+u(r)\sum_{p}^{\infty} n(n-1)\cdots(n-q+1)\left(\frac{r}{R_p}\right)^{n-1}. \end{aligned}$$

Now, if we take

$$p = \nu\left(r + \frac{1}{r\nu^2(r)}\right) + 1,$$

we can easily prove that

$$r^q M^q(r) \leqq Ap^{q+1}u(r)+Bp^q(q+1)!\,\nu(r)^{2q+2},$$

A and B being independent of r.

Since

$$p = \nu\left(r+\frac{1}{r\nu^2(r)}\right)+1 < \nu(2r)+1 < C \log u(3r)+o(1),$$

C being independent of r, the rest of the proof follows the same lines as before.

THEOREM 2. *For an entire function which satisfies the condition*

$$\overline{\lim_{r\to\infty}} \frac{l_{k+1}M(r)}{\log r} = \infty,$$

$$\underline{\lim}_{r\to\infty} \frac{l_1M(r)\, l_2M(r) \cdots l_kM(r)}{\nu(r)} = 0,$$

where k is fixed.

PROOF. By [5, § 4], we have

$$\log u(r) \leqq \log M(r) \leqq \{1+o(1)\} \log u(r)+2 \log \nu(k'r)$$

for all $r > r_0$, k' being any fixed number greater than 1. Also, we have

$$\nu(br) \log \frac{1}{b} < \log u(r),$$

b being any fixed positive numberless than 1. Consequently, we have

$$\begin{aligned}\log u(r) \leqq \log M(r) &\leqq \{1+o(1)\} \log u(r)+\log\log u(ar)\\ &\leqq \{1+o(1)\} \log u(ar) < 2 \log u(ar)\end{aligned}$$

for all $r > r_1$, a being any fixed number greater than 1. Therefore, we have

$$\overline{\lim_{r\to\infty}} \frac{l_{k+1}M(r)}{\log r} = \overline{\lim_{r\to\infty}} \frac{l_{k+1}u(r)}{\log r}\,.$$

Now, by [1, § 4, (3)], we have

$$\underline{\lim}_{n\to\infty} \frac{l_1u(R_n)l_2u(R_n) \cdots l_ku(R_n)}{\nu(R_n)} = 0.$$

Given ε, let E denote the set of all positive integers n_p ($p=1, 2, \cdots$) such that

$$\frac{l_1u(R_m)l_2u(R_m) \cdots l_ku(R_m)}{\nu(R_m)} < \varepsilon \qquad (m = n_1, n_2 \cdots).$$

By [2, § 2], in Case A, we have

$$
\begin{aligned}
l_1 M(R_m) &< \{1+o(1)\} l_1 u(R_m) + 2 l_1 \nu(R_m) \\
&< 4 l_1 \beta(R_m)
\end{aligned}
$$

for $m > m_0$, where $\beta(R_m) = \max\big(u(R_m), \nu(R_m)\big)$, and so

$$
l_\alpha M(R_m) < l_\alpha \beta(R_m) + o(1),
$$

where α is any fixed integer greater than 1.

Since $\beta(R_m) = u(R_m)$ or $\nu(R_m)$, it follows easily that

$$
\varliminf_{m\to\infty} \frac{l_1 M(R_m) l_2 M(R_m) \cdots l_k M(R_m)}{\nu(R_m)} = 0.
$$

In Case B, if $R_{m+1} > R_m$, we have

$$
\begin{aligned}
&l_1 u(R_{m+1}) < l_1 u(R_m) + \frac{1}{mR_m} < l_1 u(R_m)\left(1 + \frac{1}{mR_m}\right) \\
&l_2 u(R_{m+1}) < l_1\left(l_1 u(R_m) + \frac{1}{mR_m}\right) < l_2 u(R_m)\left(1 + \frac{1}{mR_m}\right) \\
&\cdots \\
&l_k u(R_{m+1}) < l_k u(R_m) + \frac{1}{mR_m} < l_k u(R_m)\left(1 + \frac{1}{mR_m}\right)
\end{aligned}
$$

for $m > m_1$.

Since

$$
\left(1 + \frac{1}{mR_m}\right)^k < 1 + \frac{1}{m}
$$

if

$$
k \log\left(1 + \frac{1}{mR_m}\right) < \log\left(1 + \frac{1}{m}\right)
$$

or if

$$
\frac{k}{mR_m} < \frac{1}{m} - \frac{1}{2m^2}
$$

or if

$$
\frac{k}{R_m} < 1 - \tfrac{1}{2} = \tfrac{1}{2},
$$

which is true, if $m > m_0(k)$, we have

$$
\begin{aligned}
\frac{l_1 u(R_{m+1}) l_2 u(R_{m+1}) \cdots l_k u(R_{m+1})}{m+1} &< \frac{m}{m+1}\left(1 + \frac{1}{mR_m}\right)^k \frac{l_1 u(R_m)}{m} \cdots \\
&< \varepsilon
\end{aligned}
$$

and so $m+1 \in E$. Similarly $m+2, m+3, \cdots \in E$. The rest of the proof is the same as in [2, § 2].

THEOREM 3. *For an entire function of infinite order*

$$\varlimsup_{r\to\infty} \frac{\log M\left(r+\dfrac{\lambda r \log u(r)}{\nu^2(r)H(r)}\right)}{\nu(r)} = 0,$$

where $H(r)$ is any positive function such that

$$\sum_{m=1}^{\infty} \frac{1}{\nu(R_m)H(R_m)}$$

is convergent and $H(r) = o(\nu(r))$, λ being any fixed positive number.

PROOF. By [2, § 2], we have

$$\frac{\log u(R_m)}{\nu(R_m)} < \varepsilon \qquad (m = n_1, n_2, \cdots).$$

Either [Case A] there exists a subsequence of integers K_t $(t = 1, 2, \ldots)$ tending to infinity such that

$$R_{m+1} > R_m \left(1+\frac{\lambda' \log u(R_m)}{\nu^2(R_m)H(R_m)}\right) \qquad (m = K_t, \lambda' > \lambda)$$

in which case

$$\nu\left(R_m+\frac{\lambda' R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}\right) = \nu(R_m), \tag{2}$$

or [Case B] for all large m, say $m > N$, where $m \in n_p$ $(p = 1, 2, \cdots)$,

$$R_{m+1} \leqq R_m \left(1+\frac{\lambda' \log u(R_m)}{\nu^2(R_m)H(R_m)}\right)$$

in which case either $R_{m+1} = R_m$ and then $m+1 \in n_p$ $(p = 1, 2, 3, \cdots)$ or $R_{m+1} > R_m$,

$$\begin{aligned}
\frac{\log u\,(R_{m+1})}{\nu(R_{m+1})} &\leqq \frac{1}{m+1}\left\{\log u(R_m)+\int_{R_m}^{R_{m+1}} \frac{\nu(x)}{x}\,dx\right\} \\
&\leqq \frac{1}{m+1}\left\{\log u(R_m)+m \log\left(1+\frac{\lambda' \log u(R_m)}{\nu^2(R_m)H(R_m)}\right)\right\} \\
&< \frac{1}{m+1}\left\{\log u(R_m)+\lambda' \frac{\log u(R_m)}{mH(R_m)}\right\} \\
&< \frac{1}{m+1}\left\{\log u(R_m)+\frac{\log u(R_m)}{m}\right\} \\
&= \frac{\log u(R_m)}{m} < \varepsilon,
\end{aligned}$$

and so $m+1 \in n_p$ $(p = 1, 2, \ldots)$. Similarly

$$m+2,\ m+3,\ \cdots \in n_p \qquad (p = 1, 2, \cdots).$$

Let $m \in n_p$ $(p = 1, 2, \ldots)$ and $m > N$. Then

$$\begin{aligned} R_{m+p} &\leqq R_m \prod_{n=m}^{m+p-1} \left(1 + \frac{\lambda' \log u(R_n)}{\nu^2(R_n)H(R_n)}\right) \\ &< R_m \prod_{n=m}^{m+p-1} \left(1 + \frac{\lambda' \in \nu(R_n)}{\nu^2(R_n)H(R_n)}\right) \\ &< a \text{ constant} \end{aligned}$$

which leads to a contradiction. Proving thereby that Case B is untenable and (2) holds

Now, putting

$$p = \nu\left(r + \frac{1}{r\nu^3(r)}\right) + 1$$

in the inequality

$$M(r) \leqq u(r)\left(p + \frac{r}{R_p - r}\right),$$

we have

$$\log M(r) \leqq \{1+o(1)\} \log u(r) + 3 \log \nu\left(r + \frac{1}{r\nu^3(r)}\right).$$

Since $H(r) = o(\nu(r))$, by (2), we have

$$\begin{aligned} &\log M\left(R_m + \frac{\lambda R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}\right) \\ &\leqq \{1+o(1)\}\left\{\log u(R_m) + \int_{R_m}^{R_m + \frac{\lambda R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}} \frac{\nu x}{x}\, dx \right. \\ &\qquad \left. + 3 \log \nu\left(R_m + \frac{\lambda' R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}\right)\right\} \\ &\leqq \{1+o(1)\}\left\{\log u(R_m) + \nu\left(R_m + \frac{\lambda R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}\right)\right. \\ &\qquad \cdot \log \nu\left(R_m + \frac{\lambda' R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}\right) \\ &\qquad \left. + 3 \log \nu\left(R_m + \frac{\lambda' R_m \log u(R_m)}{\nu^2(R_m)H(R_m)}\right)\right\} \\ &\leqq \{1+o(1)\}\left\{\log u(R_m) + \frac{\lambda \log u(R_m)}{\nu(R_m)H(R_m)} + 3 \log \nu(R_m)\right\}. \end{aligned}$$

REMARKS.

(i) It is easy to see that, if $f(z)$ is an entire function for which $\log M(r)$ is a function of finite k-th order, with respect to $L(r)$, and if $\varphi(r)$ is any positive function which is continuous for all positive r and differentiable in adjacent intervals; and which tends steadily to infinity with r, such that

$$\varlimsup_{r\to\infty} \frac{\log\log M(r)}{\varphi(r)} = \infty,$$

then, since

$$\log M(r) \sim \log u(r),$$

$$\varlimsup_{r\to\infty} \frac{\nu(r)}{r\varphi'(r)\log u(r)} \geqq \varlimsup_{r\to\infty} \frac{\log\log u(r)}{\varphi(r)} = \varlimsup_{r\to\infty} \frac{\log\log M(r)}{\varphi(r)}$$

and, consequently, we have

$$\varliminf_{r\to\infty} \frac{r\varphi'(r)\log M(r)}{\nu(r)} = 0,$$

where $\varphi'(r)$ denotes the differential coefficient of $\varphi(r)$ at all the points where it exists. For this class of functions, this result is more general than that of Shah [2, Theorem 1]

(ii) Theorem 1 of [4] can be put in a more general form as follows. If $f(z)$ is an entire function for which $T(r, f)$ is of finite k-th order, with respect to $L(r)$; and if $\varphi(r)$ satisfies the same conditions as in (i), such that

$$\varlimsup_{r\to\infty} \frac{\log\left(\int_{r_0}^{r} \frac{T(x, f)}{x}\, dx\right)}{\varphi(r)} = \rho > 0;$$

and if $f_1(z)$ is an entire function such that $T(r, f_1) = o(T(r, f))$, then

$$\varliminf_{r\to\infty} \frac{r\varphi'(r)\cdot\int_{r_0}^{r} \frac{T(x, f)}{x}\, dx}{N(r, f-f_1)} \leqq \frac{2}{\rho}$$

for every entire function $f_1(z)$, with one possible exception.

This can be easily proved by using the lemma, the method of (i) and the form of the second fundamental theorem of Nevanlinna, given in [4, (4)].

(iii) Theorem 3 of [4] can be put in a more general form as follows. If $f(z)$ is an entire function for which $T(r, f)$ is of finite

k-th order, with respect to $L(r)$, if $f_1(z)$ is an entire function such that $T(r, f_1) = o(T(r, f))$ and if r_m $(m = 1, 2, \ldots)$ is any positive sequence which tends steadily to infinity with m, then

$$\varliminf_{m\to\infty} \frac{T(r_m, f)}{N(r_m, f-f_1)} \leqq 2$$

for every entire function $f_1(z)$, with one possible exception.

(iv) Similar modifications can be made in Theorems 2 (i), 5, 6, 7 (i) and 8 (i) of [4] and Theorems 3, 4 and 5 of [1].

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(Oblatum 1-4-64)

Aminabad, Lucknow (India)