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by

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Introduction

This paper is the sixth of a series published under the same title and numbered I, II, ... The reader is assumed to be familiar with most of the definitions, notations and results of the first four papers.

This Part VI is devoted to the study of continuous linear functionals on V-spaces.

1. Dual space

In this Part, X is a V-space over the field of scalars F. F is a V-space over itself and is given the discrete topology induced by its trivial valuation. [1]

The term "functional on X" will be used to denote an operator from X to F.

Definition 1.1. The space $X^* = (X, F)$ of bounded linear functionals on X is called the *dual space of* X.

Theorem 1.2. (i) X^* is a V-space.

(ii) Every continuous linear functional on X is bounded and belongs to X^*.

(iii) For each $x \in X$ and $f \in X^*$ there exists $r > 0$ such that $f(S(x, r)) = f(x)$.

Proof: (i) and (ii) are special cases of Theorem IV-3.1 and Theorem IV-3.5, respectively. (iii) follows from the continuity of f and the discreteness of F.

A direct proof of the validity of the Hahn-Banach Theorem (Th. 1.3(i) below; [36], p. 186) in V-spaces has been given by

[1] Since $[0] = \{0\}$, the symbols "$\equiv$" and "$=$" have the same meaning in the space F.

A. F. Monna ([24], Part III, pp. 1137—1138). A. W. Ingleton [17] constructed a proof based on the notion of spherical completeness (see II-5, (ii)). Another proof is due to I. S. Cohen [13], p. 696. Monna has also proved (same reference) the existence of the linear functionals referred to in (ii) of the following theorem.

THEOREM 1.3. (i) Let Z be a subspace of X. To each linear functional $f_1 \in Z^*$ there corresponds at least one linear functional $f_2 \in X^*$ such that

(VI.1) $\quad |f_2|_X = |f_1|_Z$ and $f_2(x) \equiv f_1(x)$ for all $x \in Z$.

(ii) For $x_0 \in X$, $|x_0| \neq 0$ and every scalar $\alpha \in F$, $\alpha \neq 0$, there exists $f \in X^*$ such that

$$f(x_0) \equiv \alpha \text{ and } |f| = |x_0|^{-1}.$$

PROOF: See the references quoted above.

We give a new proof of (i), using Theorem 4-4.1. Let H' be a distinguished basis of Z and H be an arbitrary extension of H' to all of X (see Definition II-4.3). On H, define

$$f_2(h) \equiv \begin{cases} f_1(h) & \text{for } h \in H' \\ 0 & \text{for } h \in H \backslash H'. \end{cases}$$

It follows from Theorem IV-4.1 that f_2 is determined on X by its values on H and that (VI.1) is satisfied.

To prove (ii), define $f_1(x_0) \equiv \alpha$ and extend f_1 by linearity to the subspace $[x_0]$. Then $|f_1|_{[x_0]} = |x_0|^{-1}$. The conclusion follows from (i).

THEOREM 1.4. One of X and X^* is a bounded space if and only if the other one is a discrete space.

PROOF. It follows from Theorem 1.3(ii) that if X is not discrete, i.e. if there are points in X with arbitrarily small non-zero norms, then X^* is unbounded. The same theorem implies that if X is unbounded there exist linear functionals of arbitrarily small non-zero norms.

Suppose that X^* is unbounded. Then, for any integer $K > 0$ there exists $f \in X^*$ with $|f| > K$. Since there must be a point $x \in X$ for which

$$|f(x)| = 1 = |f||x|,$$

there must be non-trivial points of X with norms less than K^{-1}. Hence, X is not discrete.

Finally, suppose that X is bounded, i.e. for some $M > 0$, $|x| \leqq M < \infty$ for all $x \in X$. For all $f \in X^*$, $|f| \neq 0$, we have

$$|f(x)| = 1 \leqq |f| \cdot |x| \text{ for all } x \text{ such that } f(x) \neq 0.$$

Thus, $|f| \geqq (1/M)$ and X^* is discrete.

2. The * norm on (*H*)

Let $H = \{h_i : i \in J\}$, where J is some index set, be a distinguished basis of X. (H) denotes the set of all finite linear combinations of elements of H.

In this section we shall define a new norm on the elements of (H). In the next section we shall use this new norm to establish the relationship between X^* and (H).

The symbol "$(x, h)_H$" was introduced in I-5 (after Th. 5.6).

DEFINITION 2.1. For $x \in X$,
(i) $J(x) = \{i \in J : (x, h_i)_H \neq 0\}$;
(ii) $\omega(x)$ is defined by the relation: $|x| = \rho^{-\omega(x)}$;
(iii) $l(x) = \sup_{i \in J(x)} \{\omega(h_i)\}$, $l(\theta) = (-\infty)$.
For $x \in X$, $J(x)$ is countable and $x = \sum_{i \in J(x)} (x, h_i)_H h_i$.
For $y \in (H)$, $J(y)$ is finite. It is easily verified that for all y, $z \in (H)$:

$$\text{(VI.2)} \quad \begin{aligned} &\text{(i)} \quad l(\alpha y) = l(y) \text{ for all } \alpha \in F,\ \alpha \neq 0; \\ &\text{(ii)} \quad l(y+z) \begin{cases} \leqq \operatorname{Max}\{l(y), l(z)\} \\ = \operatorname{Max}\{l(y), l(z)\} \text{ whenever } l(y) \neq l(z). \end{cases} \end{aligned}$$

The two sets of integers $\{\omega(h_i) : h_i \in H\}$ and $\{l(h_i) : h_i \in H\}$ are identical since for each $h_i \in H$, $\omega(h_i) = l(h_i)$. The set $\{\omega(h_i) : h_i \in H\}$ is bounded above if and only if X is a discrete space; it is bounded below if and only if X is bounded in its norm.

DEFINITION 2.2. The function which assigns to each point y of (H) the non-negative real number

$$\text{(VI.3)} \qquad |y|^* = \rho^{l(y)}$$

will be called the * *norm on* (H).

THEOREM 2.3. (i) Under the *norm, (H) has all the defining properties of a V-space, except possibly when X is unbounded, in which case (H) may not be complete.

(ii) One of the spaces X and (H), under the *norm, is bounded if and only if the other is discrete.

(iii) The set H is a distinguished Hamel basis of (H) under the *norm.

Proof: Except for the completeness requirement, (i) is easily proved from (VI.2) and (VI.3).

(ii) follows from the remark preceding Definition 2.2, and the fact that the set $\{l(h_i) : h_i \in H\}$ is bounded above if and only if (H) is bounded under the *norm; — is bounded below if and only if (H) is a discrete space under the *norm.

If X is bounded, the completeness of (H) follows from its discreteness.

(iii) follows from the fact that for all $y \in (H)$ such that $|y|^* \neq 0$:

$$\begin{aligned} |y|^* = \rho^{l(y)} &= \rho \cdot \exp\,[\operatorname*{Max}_{i \in J(y)} \{\omega(h_i)\}] &&= \operatorname*{Max}_{i \in J(y)} \{\rho^{\omega(h_i)}\} \\ &= \operatorname*{Max}_{i \in J(y)} \{\rho^{l(h_i)}\} &&= \operatorname*{Max}_{i \in J(y)} \{|h_i|^*\}. \end{aligned}$$

3. *H*-inner product and representation theorems

To the notations, definitions and hypotheses of the previous section, we add the assumption that the field of scalars, F, is the field of the real or complex numbers.

Definition 3.1. (i) $J(x, y) = J(x) \cap J(y)$.

(ii) The scalar valued function, defined on $X \times (H)$ by

$$\langle x, y \rangle_H = \begin{cases} 0 & \text{if } J(x, y) = \emptyset, \\ \sum_{i \in J(x,y)} (x, h_i)_H \cdot (y, h_i)_H & \text{if } J(x, y) \neq \emptyset, \end{cases}$$

$x \in X$, $y \in (H)$, will be called the *H-inner product on X*.

The following properties of the H-inner product are easily verified: For all $u, v \in X$, all $y, z \in (H)$ and all $\alpha, \beta \in F$:

$$\langle y, z \rangle_H = \langle z, y \rangle_H;$$
$$\langle \alpha u, \beta y \rangle_H = \alpha\beta \langle u, y \rangle_H;$$
$$\langle u+v, y+z \rangle_H = \langle u, y \rangle_H + \langle v, y \rangle_H + \langle u, z \rangle_H + \langle v, z \rangle_H.$$

The analogy with the usual inner product is evident. An important difference is that the H-inner product depends on H. Indeed, given two distinct distinguished bases H_1 and H_2 of X, if $y \in (H_1)$ and $y \notin (H_2)$, then, for all $x \in X$, $\langle x, y \rangle_{H_1}$ is defined but $\langle x, y \rangle_{H_2}$ is not; if $y \in (H_1) \cap (H_2)$, then there may exist $x \in X$ such that $\langle x, y \rangle_{H_1} \neq \langle x, y \rangle_{H_2}$. To pursue the analogy, we shall establish a relationship between the H-inner product and the bounded linear functionals on X.

THEOREM 3.2. There exists an isometric isomorphism φ_H between (H) with its *norm and a subspace of X^*; for all $y \in (H)$, $\varphi_H(y) \equiv f_y$ is such that

$$f_y(x) = \langle x, y\rangle_H \text{ for all } x \in X. \tag{VI.4}$$

Furthermore, the set $\varphi_H(H)$ is a distinguished Hamel basis for the subspace $\varphi_H((H))$ of X^*; for $f_y \in \varphi_H((H))$,

$$f_y \equiv \sum_{i \in J(y)} (y, h_i)_H f_{h_i} \tag{VI.5}$$

in the norm of X^*.

PROOF: For each fixed $y \in (H)$ it is easy to verify that the mapping defined by (VI.4) is a linear functional on X. Let φ_H be the operator on (H) defined by $\varphi_H(y) \equiv f_y$. φ_H is linear since the H-inner product is linear in y.

a) If $J(x, y) = \emptyset$, then $|f_y(x)| = 0$.

b) If $J(x, y) \neq \emptyset$, then

$$\omega(x) \leqq \omega(h_i) \text{ for all } i \in J(x),$$
$$\omega(h_i) \leqq l(y) \quad \text{for all } i \in J(y).$$

Therefore, $\omega(x) \leqq l(y)$ and

$$|f_y(x)| = 1 \leqq \rho^{l(y)} \cdot \rho^{-\omega(x)} = |y|^* \cdot |x|.$$

c) Since $l(y)$ is finite, there exists $i \in J$ such that $l(y) = l(h_i) = \omega(h_i)$, and

$$|f_y(h_i)| = 1 = \rho^{l(y)}\rho^{-\omega(h_i)} = |y|^* \cdot |h_i|.$$

d) If $y, z \in (H)$ and $y \neq z$, there exists $j \in J$ such that

$$(y, h_j)_H \neq (z, h_j)_H \text{ and, hence } \langle h_j, y\rangle_H \neq \langle h_j, z\rangle.$$

Thus, $\varphi_H(y) \neq \varphi_H(z)$.

a), b), c) show that φ_H is an isometric, and therefore continuous, operator from (H) with its *norm to X^*. d) shows that φ_H is one-to-one.

The latter part of the theorem follows from the linearity of φ_H and Theorem 2.3 (iii).

DEFINITION 3.3. A subset A of a V-space is called locally finite if for every integer n, there is at most a finite number of elements of A with norms equal to ρ^n.

LEMMA 3.4. Let $f \in X^*$. If the subset H' of H on which f is non-zero is bounded and locally finite, then

(i) H' is a finite set;
(ii) there exists $y_f \in (H)$ such that $\varphi_H(y_f) \equiv f$.

Proof: (i) Since f is a continuous linear mapping into the discrete space F, there exists an integer m such that $f(x) \neq 0$ implies $|x| \geqq \rho^m$. Thus, H' is bounded below, bounded above and locally finite; hence it is finite.

(ii) Let

$$y_f \equiv \sum_{h' \in H'} f(h')h'. \tag{VI.6}$$

For all $h \in H$,

$$f(h) = \langle h, y_f \rangle_H = \varphi_H(y_f)(h).$$

From Theorem VI-4.1, it follows that $\varphi_H(y_f) \equiv f$.

Theorem 3.5. The operator φ_H is an isometric isomorphism between (H) with its *norm and X^* if and only if X is bounded and H is locally finite.

Proof: If X is bounded and H is locally finite, every subset H' of H satisfies the hypotheses of Lemma 3.4. Therefore, φ_H maps (H) onto X^*.

For the converse, suppose that X is unbounded or that H is not locally finite. Then, for some integer n there exists an infinite subset H' of H such that for all $h' \in H'$:

$|h'| \geqq \rho^n$, when X is unbounded,
or $|h'| = \rho^n$, when H is not locally finite.

From Theorem IV-4.1, there exists $f \in X^*$ such that

$$f(h') = 1 \text{ for all } h' \in H', \; f(h) = 0 \text{ for all } h \in H \backslash H'.$$

Should there exist $y_f \in (H)$ such that $\varphi_H(y_f) \equiv f$, y_f would have to have the infinite expansion (VI.6). This is impossible.

Corollary 3.6. If X is unbounded and H is locally finite, then φ_H is an isometric isomorphism between (H) with its *norm and the subspace of X^* formed by the continuous linear functionals which vanish outside of a bounded subset of X.

Corollary 3.7. If X is bounded and admits a locally finite distinguished basis, then X and X^* have the same dimension.

The proof follows from Theorems 3.2 and 3.5.

Examples: The spaces $\mathscr{P}_k$ of III-4 and Q_k of III-5 are bounded and admit locally finite distinguished bases. Thus, the spaces

$\mathscr{P}_k^*$ and Q_k^* are equivalent to the spaces of polynomials (VI.7) and (VI.8) respectively:

$$\text{(VI.7)}\quad \begin{cases} \xi(\lambda) \equiv 0, & |\xi|^* = 0, \\ \xi(\lambda) = \sum_{i=p}^{n} \alpha_i \lambda^i, & |\xi|^* = \rho^n,\ \alpha_n \neq 0,\ k \leqq p \leqq n. \end{cases}$$

$$\text{(VI.8)}\quad \begin{cases} \xi(u, v) = 0 \quad |\xi|^* = 0, \\ \xi(u, v) = \sum_{i=p}^{n} \sum_{j=0}^{i} \alpha_{ij} u^{i-j} v^j, \\ \qquad |\xi| = \rho^n,\ \sum_{j=0}^{n} |\alpha_{nj}| \neq 0,\ k \leqq p \leqq n. \end{cases}$$

According to (VI.5), a continuous linear functional on $\mathscr{P}_k$ is a finite linear combination of the functionals f_{φ_n}:

$$f_{\varphi_n}(x) = \text{coefficient of } \lambda^n \text{ in the expansion of } x(\lambda) \text{ in powers of } \lambda.$$

This result was proved directly by H. F. Davis [4], p. 91, for the space $\mathscr{P}_0$.

It was shown in III-4 that $\mathscr{P}_0$ admits as distinguished bases the sets Φ_0 and J of (III.16), Part III. Consider the continuous linear functional f defined on Φ_0 by

$$f(\varphi_n) = \begin{cases} \alpha_n \neq 0 & \text{for } 0 \leqq n \leqq N, \\ 0 & \text{for } n > N. \end{cases}$$

The isometric isomorphism Ψ_{Φ_0} between (Φ_0) and $\mathscr{P}_0^*$ is such that

$$\Psi_{\Phi_0}^{-1}(f) = \sum_{n=0}^{N} \alpha_n \varphi_n.$$

The isometric isomorphism Ψ_J between (J) and $\mathscr{P}_0^*$ is such that

$$\Psi_J^{-1}(f) = \sum_{n=0}^{N} \beta_n J_n,$$

where the coefficients β_n, determined from the power series expansions of the J_n's ([8]), are the solutions of the system:

$$\sum_{i=0}^{p} (-1)^{p-i} \frac{\beta_{2i}}{2^{2p}(p-i)!\,(p+i)!} = \alpha_{2p}, \quad p = 0, 1, \ldots, \left[\frac{n}{2}\right],$$

$$\sum_{i=0}^{p} (-1)^{p-i} \frac{\beta_{2i+1}}{2^{2p+1}(p-i)!\,(p+i+1)!} = \alpha_{2p+1}, \quad p = 0, 1, \ldots, \left[\frac{n-1}{2}\right].$$

Clearly, in $\mathscr{P}_0$, $\Psi_{\Phi_0}^{-1}(f) \neq \Psi_J^{-1}(f)$. This inequality reflects the dependence of the H-inner product on the distinguished basis H.

(Oblatum 18–3–66)