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On some non-Archimedean normed linear spaces III

by

Pierre Robert

1. Introduction

This paper is the third of a series published under the same title and numbered I, II, III, ... The reader is assumed to be familiar with the definitions, notations and results of Part I and Part II.

Certain spaces of functions, mapping a Hausdorff space into a normed linear space, can be normed in such a way that they become V-spaces. In this Part III we shall describe two methods to generate such V-spaces.

In the first type of V-spaces, the norm considered will characterize the asymptotic behaviour of the functions; the resulting spaces will be called "asymptotic spaces". In the second type, we shall associate with each function a sequence of scalars, called "moments", and the norm of a function will depend on the first non-zero moment; the resulting spaces will be called "moment spaces".

2. The 0 and *o* relations

a) Let Λ be a Hausdorff space and let P and S be arbitrary sets. We consider functions of the three variables $\lambda \in \Lambda$, $p \in P$ and $s \in S$. The variable λ will be called the asymptotic variable; p and s will be called the primary and secondary parameters respectively.

b) Let λ_0 be a fixed non-isolated point of Λ.

The abbreviation "*cd-nbhd of* λ_0" will stand for "closed neighbourhood of λ_0 in Λ, deleted of the point λ_0 itself". A cd-nbhd of λ_0 has non-void interior. A finite intersection of cd-nbhds of λ_0 is a cd-nbhd of λ_0.

c) If f and g are two functions defined on $V \times P \times S$, where V is some cd-nbhd of λ_0, and with range in a (pseudo-) normed space (which may be different for the two functions), then the relation

$$f = 0(g), \quad \lambda \to \lambda_0,$$

will mean that there exists, for each $s \in S$, a positive constant $\alpha(s)$ and a cd-nbhd $V(s)$ of λ_0 such that

$$\text{(III.1)} \qquad ||f(\lambda, p, s)|| \leqq \alpha(s)||g(\lambda, p, s)|| \quad \text{for all } \lambda \in V(s) \text{ and all } p \in P.$$

In this inequality the norms are those of the appropriate range spaces.

Similarly, the relation

$$f = o(g), \quad \lambda \to \lambda_0,$$

will mean that for any $\varepsilon > 0$, there exists, for each $s \in S$, a cd-nbhd $V(s, \varepsilon)$ of λ_0 such that

$$\text{(III.2)} \qquad ||f(\lambda, p, s)|| \leqq \varepsilon||g(\lambda, p, s)|| \quad \text{for all } \lambda \in V(s, \varepsilon) \text{ and all } p \in P.$$

In using the 0 and o symbols, the specification $\lambda \to \lambda_0$ will usually be omitted.

These 0 and o relations have the following properties:

$$\text{(III.3)} \qquad 0(0(f)) = 0(f),$$

$$\text{(III.4)} \qquad 0(o(f)) = o(0(f)) = o(o(f)) = o(f),$$

$$\text{(III.5)} \qquad 0(f)+0(f) = 0(f)+o(f) = 0(f),$$

$$\text{(III.6)} \qquad o(f)+o(f) = o(f),$$

$$\text{(III.7)} \qquad 0(f) \cdot 0(g) = 0(fg),$$

$$\text{(III.8)} \qquad 0(f) \cdot o(g) = o(f) \cdot o(g) = o(fg).$$

Properties (III.7) and (III.8) apply when the range spaces are (pseudo-) normed rings. The proofs are immediate and the formulae can be extended to combinations of any finite number of order symbols. For those and other relations, see [9], Chapter 1.

d) Definition 2.1. A sequence $\{f_n\}$ of functions is called an *asymptotic sequence* (as $\lambda \to \lambda_0$) if

(i) f_n is defined on $V_n \times P \times S$, where V_n is some cd-nbhd of λ_0;
(ii) all f_n have the same range space; and
(iii) $f_{n+1} = o(f_n)$ for each n.

3. Asymptotic spaces: Definition

a) Let Λ, P, S be as in Section 2 and $\Lambda' = \Lambda\backslash\{\lambda_0\}$.

Let B and B_0 be two (pseudo-) normed linear spaces. The (pseudo-) norms on both spaces will be denoted by $\|\cdot\|$.

Let N be a set of integers such that $\sup N = \infty$. The ordering on N is the natural ordering of the integers. For $n \in N$, $\sigma^0(n) = n$, $\sigma^1(n) = \sigma(n)$ denotes the successor of n in N and $\sigma^j(n)$, $j = 2, 3, \ldots$ denotes the j^{th} successor of n in N; the element $m \in N$ such that $\sigma^j(m) = n$ is denoted by $\sigma^{-j}(n)$.

b) Definition 3.1. A family of functions $\Phi = \{\varphi_n : n \in N\}$ is called an *asymptotic scale* (as $\lambda \to \lambda_0$) if for each $n \in N$:

(i) φ_n is defined on $\Lambda' \times P \times S$ and have range in B_0; and

(ii) the sequence $\{\varphi_{\sigma^j(n)} : j = 0, 1, 2, \ldots\}$ is an asymptotic sequence.

In analogy with the terminology of J. G. van der Corput [38], [39] we use the following

Definition 3.2. A function f defined on $V \times P \times S$, where V is some cd-nbhd of λ_0, and with range in a (pseudo-) normed space is said to be *asymptotically finite with respect to an asymptotic scale* $\Phi = \{\varphi_n : n \in N\}$ if there exists $n \in N$ such that $f = 0(\varphi_n)$.

c) Let X be the linear space of all functions defined on $\Lambda' \times P \times S$, with range in B, which are asymptotically finite with respect to a given asymptotic scale Φ.

For each $x \in X$, define

$$\omega(x) = \sup\{n \in N : x = 0(\varphi_n)\}, \tag{III.9}$$

and, for some fixed real ρ, $1 < \rho < \infty$,

$$|x| = \rho^{-\omega(x)}. \tag{III.10}$$

The function defined on X by (III.9) and (III.10) will be called the *Φ-asymptotic norm on X.*

Since the asymptotic behaviour of a function, as $\lambda \to \lambda_0$, is entirely determined by its values on any set of the form $V \times P \times S$, where V is a cd-nbhd of λ_0, the Φ-asymptotic norm of the difference of two functions in X which are identical on $V \times P \times S$ is equal to zero. Conversely, if a function is defined on $V \times P \times S$ and is asymptotically finite with respect to Φ, then it can be arbitrarily extended to all of $\Lambda' \times P \times S$ and the Φ-asymptotic norm of the difference of any two of its extensions is equal to zero.

d) THEOREM 3.3. The space X, under the Φ-asymptotic norm, is a V-space.

PROOF: Using (III.3)—(III.9) one easily verifies that, under the norm (III.10), X has the properties of a pseudo-valued space. The Φ-asymptotic norm satisfies (II.1) of Definition II-1.1 and also (II.3).

To prove the completeness of X, consider an arbitrary Cauchy sequence $\{z_i : i = 0, 1, 2, \ldots\}$. Let

$$y_0 = z_0, \, y_i = z_i - z_{i-1} \quad \text{for} \quad i = 1, 2, \ldots$$

From Theorems I-4.1 and I-4.2, it is sufficient to prove the convergence of a particular rearrangement of the series

$$\sum_{i=0}^{\infty} y_i. \tag{III.11}$$

Without loss of generality, we can assume that none of the y_i's has zero-norm. Let

$$M = \{n \in N : |y_i| = \rho^{-n} \text{ for some } i\},$$
$$q = \inf M.$$

Since $\{z_i\}$ is a Cauchy sequence, it follows that $q > -\infty$. Also, for each $n \in M$, the number of y_i's with norms equal to ρ^{-n} is finite. Let x_n be their sum. It follows that

$$|x_n| \leqq \rho^{-n} \quad \text{for all } n \in N \tag{III.12}$$

and that the series

$$\sum_{\substack{n=q \\ n \in M}}^{\infty} x_n \tag{III.13}$$

can be considered as a rearrangement of (III.11).

We now fix an arbitrary value $s \in S$ for the secondary parameter.

It follows from (III.12) that for each $n \in M$, there exist a constant $\alpha[n, s] > 0$ and a cd-nbhd $V[n, s]$ of λ_0 such that

$$\begin{aligned} ||x_{\sigma(n)}(\lambda, p, s)|| &\leqq \alpha[\sigma(n), s] ||\varphi_{\sigma(n)}(\lambda, p, s)|| \\ &\leqq \tfrac{1}{2}\alpha[n, s] ||\varphi_n(\lambda, p, s)|| \\ &\text{for all } \lambda \in V[\sigma(n), s] \text{ and for all } p \in P. \end{aligned}$$

We can assume without loss of generality that these $V[n, s]$'s are nested and are selected in such a way that their intersection is void. [2]

Then, for all $j \geqq 0$,

$$||x_{\sigma^j(n)}(\lambda, p, s)|| \leqq 2^{-j}\alpha[n, s]||\varphi_n(\lambda, p, s)|| \quad \text{for all } \lambda \in V[\sigma^j(n), s] \text{ and for all } p \in P.$$

We shall now define an element x of X by specifying its values on $\Lambda' \times P \times \{s\}$.

For $\lambda \in \Lambda' \backslash V[q, s]$ we define

$$x(\lambda, p, s) = 0 \text{ for all } p \in P.$$

For $\lambda \in V[q, s]$, there exists an integer $N(\lambda, s)$:

$$N(\lambda, s) = \text{Max}\,\{n \in N : \lambda \in V[n, s]\}$$

If $\lambda \in V[n, s]$ then $N(\lambda, s) \geqq n$.

For $\lambda \in V[q, s]$ we define

$$x(\lambda, p, s) = \sum_{\substack{q \leqq n \leqq N(\lambda, s) \\ n \in M}} x_n(\lambda, p, s) \text{ for all } p \in P.$$

We shall now show that x is a limit of the sequence (III.13) and thus of (III.11).

Let $\varepsilon > 0$ be given; there exists J such that for all $j \geqq J$, $\rho^{-j} < \varepsilon$. We assert that, for all $j \geqq J$, $j \in M$,

$$|x - \sum_{\substack{q \leqq n \leqq j \\ n \in M}} x_n| \leqq \rho^{-j} < \varepsilon,$$

or, equivalently, that

$$(x - \sum_{\substack{q \leqq n \leqq j \\ n \in M}} x_n) = 0(\varphi_j).$$

Indeed, for each $s \in S$, for $\lambda \in V[j, s]$, $(j \in M)$, $\lambda \in V[\sigma(j), s]$, $\lambda \in V[\sigma^2(j), s], \ldots, \lambda \in V[N(\lambda, s), s]$ and

[2] In addition to these requirements, the choice of the cd-nbhds $V[n, s]$ is guided by the condition that for $\lambda \in V[\sigma^j(q), s]$, $j = 1, 2, \cdots$

$$||\varphi_{\sigma^j(q)}(\lambda, p, s)|| \leqq \tfrac{1}{2} \frac{\alpha[\sigma^{j-1}(q), s]}{\alpha[\sigma^j(q), s]} ||\varphi_{\sigma^{j-1}(q)}(\lambda, p, s)|| \quad \text{for all } p \in P;$$

which implies

$$||\varphi_{\sigma^j(q)}(\lambda, p, s)|| \leqq 2^{-j} \frac{\alpha[q, s]}{\alpha[\sigma^j(q), s]} ||\varphi_q(\lambda, p, s)|| \qquad \text{for all } p \in P.$$

$$\begin{aligned}&||x(\lambda, p, s)-\sum_{\substack{q\leqq n\leqq j\\ n\in M}} x_n(\lambda, p, s)||\\ &\quad = ||x_{\sigma(j)}(\lambda, p, s)+x_{\sigma^2(j)}(\lambda, p, s)+\ldots+x_{N(\lambda, s)}(\lambda, p, s)||\\ &\quad \leqq 2\alpha[j, s]||\varphi_j(\lambda, p, s)||, \text{ for all } p\in P.\end{aligned}$$

This completes the proof.

The above proof is modelled after the second part of the proof of [33], Theorem 1 (also [32], Th. 4.2). In [33], the range space B is a Banach space; we have shown that the completeness of X does not require the completeness of B.

If B is a (pseudo-) normed ring, the product xy of two functions $x, y\in X$ is defined by

$$xy(\lambda, p, s) = x(\lambda, p, s)\cdot y(\lambda, p, s) \quad \text{for all } \lambda, p, s.$$

If B_0 is a (pseudo)-normed ring, a similar definition of the product of two elements of Φ can be given.

THEOREM 3.4. Let B and B_0 be (pseudo-) normed rings. If for all $m, n\in N$, $\varphi_m\varphi_n = 0(\varphi_{m+n})$, then X satisfies the properties (II.9), (II.10) and (II.11) of a V-algebra.

PROOF: If $x, y\in X$ and $|x| = \rho^{-m}$, $|y| = \rho^{-n}$, then by (III.7) and (III.3):

$$xy = 0(\varphi_m)\cdot 0(\varphi_n) = 0(\varphi_m\varphi_n) = 0(0(\varphi_{m+n})) = 0(\varphi_{m+n})$$

Hence, $|xy|\leqq\rho^{-(m+n)} = |x|\cdot|y|$ and (II.11) is satisfied.

One verifies easily that (II.9) and (II.10) hold.

e) Let $x\in X$ have, in the Φ-asymptotic norm on X, the following expansion:

(III.15) $\quad x = a_0x_0+a_1x_1+a_2x_2+\ldots, \; x_i\in X, \; \alpha_i\in F.$

The expansion (III.15) is said to be an *expansion of the "Poincaré type"* ([11], pp. 218—219) if the sequence $\{x_n : n = 0, 1, 2, \ldots\}$ is an asymptotic sequence (see Definition 2.1).

When the range spaces B and B_0 are identical, $\Phi\subset X$. A convergent expansion in terms of the elements of Φ is said to be an *expansion "essentially of the Poincaré type"*.

In [10] and [11] the desirability, in a theory of asymptotics, of accepting expansions which are not of the Poincaré type is highly stressed. In an asymptotic space, expansions which are not of the Poincaré type can occur if there exist countable distinguished sets with elements having arbitrarily small norms and which cannot be ordered into an asymptotic sequence. Specific examples will be given in the next sections.

4. Asymptotic spaces: Example I

a) Our first examples of asymptotic spaces are simple and do not involve any primary or secondary parameters: $P = S = \emptyset$.

The Hausdorff space Λ is the real interval $[0, \bar{\lambda}]$, $0 < \bar{\lambda} < \infty$, and $\lambda_0 = 0$. A cd-nbhd of 0 is an interval of the form $(0, \lambda']$, $0 < \lambda' \leqq \bar{\lambda}$. The range spaces B and B_0 are both the space of the real numbers. N is the set of all integers and the asymptotic scale to be used is $\Phi = \{\varphi_n : n \in N\}$, where $\varphi_n(\lambda) = \lambda^n$.

Let $X = \mathscr{R}$ be the space of all real valued functions x on Λ' which are asymptotically finite with respect to Φ and with norm defined by (III.9) and (III.10).

Clearly: $\Phi \subset X$; $\varphi_0(\lambda) = 1$ and $|\varphi_0| = 1$; $\varphi_m \varphi_n = 0(\varphi_{m+n})$ for all $m, n \in N$. From this and Theorems III.3 and III.4, it follows that X is a V-algebra. Direct proofs of this result were given by A. Erdélyi [9], J. Popken [29], J. G. van der Corput [38], [39].

b) To illustrate the results of Parts I and II, we consider the following subsets of $\mathscr{R}$:

$$\text{(III.16)}\quad \begin{aligned} \bar{\Phi} &= \{\varphi_\alpha : -\infty < \alpha < \infty\}, &\quad \varphi_\alpha(\lambda) &= \lambda^\alpha;\\ \Phi_k &= \{\varphi_n : n \in N,\ n \geqq k\}; && \\ E &= \{e_\alpha : -\infty < \alpha < \infty\}, & e_\alpha(\lambda) &= \exp \alpha\lambda;\\ J &= \{J_n : n = 0, 1, 2, \ldots\}, && \end{aligned}$$

where $J_n : \lambda \to J_n(\lambda)$ is the Bessel function of the first kind of order n;

$$G = \{z_n : n = 0, 1, 2, \ldots\},$$

where

$$\text{(III.17)}\qquad z_n(\lambda) = \begin{cases} \lambda^n \sin (n+1)\pi/\lambda & \text{if } n \text{ is even,}\\ \lambda^n \cos (n+1)\pi/\lambda & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to verify that $\bar{\Phi}$, Φ and Φ_k are distinguished subsets of $\mathscr{R}$.

For $n = 0, 1, 2, \ldots$, ([8], Vol. II)

$$\text{(III.18)}\qquad |\varphi_n - 2^n n! J_n| = |\varphi_{n+2}| < |\varphi_n|.$$

Thus, by the Paley-Wiener Theorem (Theorem II-2.4) J is also a distinguished subset of $\mathscr{R}$.

The set G is also a distinguished subset of $\mathscr{R}$ since its elements have distinct norms: $|z_n| = \rho^{-n}$. Since $z_n = o(z_m)$ is not true for any n, $m > 0$, the sequence $\{z_n\}$ is not an asymptotic sequence. An expansion in terms of the elements of G:

$$\text{(III.19)} \quad \alpha_0 \sin \frac{\pi}{\lambda} + \alpha_1 \lambda \cos \frac{2\pi}{\lambda} + \alpha_2 \lambda^2 \sin \frac{3\pi}{\lambda} + \ldots, \; \alpha_n \text{ real},$$

always converges to some element of $\mathscr{R}$. Yet, it is not an expansion of the Poincaré type (see Section 3, e). Expansions such as (III.19) are mentioned in [11].

For all $\beta \neq 0$,

$$\varphi_\alpha + \beta z_n = 0(\varphi_m) \text{ if and only if } m \geqq \min\{\alpha, n\}.$$

Thus $(\bar{\Phi}, G)$ is a distinguished pair of subsets of $\mathscr{R}$. A consequence is that there exists a distinguished basis of $\mathscr{R}$ which contains $\bar{\Phi}$ and G. Another consequence is that a function which is the sum of a series (III.19) cannot admit an expansion in terms of elements of $\bar{\Phi}$, i.e. in terms of powers of λ.

The set E is contained in the closed subspace generated by Φ_0. Indeed, it is well known that for any real number α, the series $\sum_{n=0}^{\infty} \alpha^n \varphi_n / n!$ converges asymptotically, as $\lambda \to 0$, to the function e_α.

E is not a distinguished subset of $\mathscr{R}$, since for $\alpha \neq \beta$:

$$|e_\alpha| = 1, \; |e_\beta| = 1, \; |e_\alpha - e_\beta| = \rho^{-1} < 1.$$

c) Let $\mathscr{G}$ denote the closed subspace generated by G, i.e. the set of functions which admit expansions of the form (III.19). (See Theorem I-5.6.)

Let $\mathscr{P}$ denote the closed subspace generated by Φ, i.e. the set of functions which admit expansions of the form

$$\text{(III.20)} \qquad \alpha_n \varphi_n + \alpha_{n+1} \varphi_{n+1} + \ldots, \; n \in N, \; \alpha_n \text{ real}.$$

$\mathscr{P}$ is a subalgebra of $\mathscr{R}$. A non-trivial element of $\mathscr{P}$ is pseudo-regular. From the above discussion, $\mathscr{G} \cap \mathscr{P}$ is trivial.

Let $\mathscr{P}_k$ be the closed subspace generated by Φ_k, i.e. the set of sums of expansions (III.20) with $n \geqq k$.

E is contained in $\mathscr{P}_k$ for all $k \leqq 0$.

$\mathscr{P}_0$ is a V-algebra. From (III.18) and Theorem II-2.4, J is a distinguished basis of $\mathscr{P}_0$.

E and Φ_0 do not form a distinguished pair of subsets of $\mathscr{P}_0$ but their union is a linearly independent set. Therefore, there exists a Hamel basis of $\mathscr{P}_0$ which contains $E \cup \Phi_0$. In [4], continuous linear functionals on the subspace $(E \cup \Phi_0)$ are studied; see Part VI (to be published in this series of papers).

d) To construct the space $\mathscr{R}$, we selected $\lambda_0 = 0$. Clearly, we would have obtained a similar space by choosing any other finite value for λ_0 and the asymptotic scale $\Phi = \{\varphi_n : n \in N\}$ where $\varphi_n(\lambda) = (\lambda-\lambda_0)^n$.

One may also consider $\Lambda = [\bar{\lambda}, \infty]$, $0 \leqq \bar{\lambda} < \infty$, $\lambda_0 = \infty$ and the asymptotic scale $\Phi = \{\varphi_n : n \in N\}$, where $\varphi_n(\lambda) = \lambda^{-n}$.

e) Spaces such as $\mathscr{R}$ can be constructed in which Λ is some subset of the complex plane and for more sophisticated asymptotic scales. Examples are given in [11], with asymptotic scales such as

$$\Psi = \{\psi_n : n = 0, 1, 2, \ldots\},$$
$$\psi_n(\lambda, z, r, s) = \Gamma(z+n/r)\lambda^{-z-ns};$$

λ is the complex asymptotic variable, $\lambda \to \lambda_0 = \infty$; z is a complex number considered as a primary parameter, i.e. order relations must hold uniformly in z; the positive real numbers r and s are secondary parameters. Proper conditions must be placed on the domains of λ and z. (See [11]; also [9], [10].)

5. Asymptotic spaces: Example II

a) In this section we give two examples involving formal power series in two real or complex variables. The spaces to be constructed will be used in Part IV to obtain asymptotic expansions of some functions defined as two-dimensional Laplace transforms [6].

b) In this example the Hausdorff space Λ is the set of points $\lambda = (u, v)$ of R^2 for which $0 \leqq u, v \leqq \infty$; $\lambda_0 = (0, 0)$. The range spaces B and B_0 are the spaces of complex numbers and of real numbers respectively. N is the set of all non-negative integers and the asymptotic scale to be used is

$$\Phi = \{\varphi_n : n \in N\},\ \varphi_n(u, v) = (u+v)^n.$$

Let $X = Q$ be the space of all complex valued functions on Λ' which are asymptotically finite with respect to Φ.

As for the space $\mathscr{R}$ of Section 4, one verifies that Q is a V-algebra, under the Φ-asymptotic norm defined by (III.9) and (III.10).

Let $x_{lj} \in X$ be the function defined by

$$x_{lj}(u, v) = u^l v^j,\ l, j \in N.$$

For all non-zero complex numbers α, β, and all integers l, j such that $0 \leqq l \leqq n$, $l < j \leqq n$:

(III.21) $x_{n-l,l} = 0(\varphi_m)$ if and only if $m \leqq n$,

(III.22) $\alpha x_{n-l,l}+\beta x_{n-j,j} = 0(\varphi_m)$ if and only if $m \leqq n$.

Indeed, to verify (III.22) suppose firstly that $m \leqq n$. Then:

$$\begin{aligned}|\alpha u^{n-l}v^l+\beta u^{n-j}v^j| &\leqq \operatorname{Max}\{|\alpha|, |\beta|\}\cdot(u^{n-l}v^l+u^{n-j}v^j)\\ &\leqq \operatorname{Max}\{|\alpha|, |\beta|\}(u+v)^n.\end{aligned}$$

Thus:

$$\alpha x_{n-l,l}+\beta x_{n-j,j} = 0(\varphi_n) = 0(\varphi_m)$$

and (III.22) is satisfied if $m \leqq n$. Conversely, suppose that the relation is true for some $m = n+k$, $k > 0$. Since every cd-nbhd of $(0, 0)$ must contain points (u, v) for which $v = u$, there exists a constant $A > 0$ such that for all u small enough

$$|\alpha+\beta|u^n \leqq A \cdot 2^{n+k}u^{n+k}.$$

This is impossible. This completes the verification of (III.22).

Consider the set

$$H_k = \{x_{lj} : l+j \geqq k\},\ k \in N.$$

It follows from (III.21) and (III.22) that H_k is a distinguished subset of Q. Let $\mathscr{H}_k$ denote the closed subspace of Q generated by H_k, i.e. the set of all functions which admit expansions of the form

(III.23) $$\sum_l \sum_j \alpha_{lj}x_{lj},\ l+j \geqq k,\ \alpha_{lj} \text{ complex.}$$

Unlike the space $\mathscr{P}_k$ of Example I (Section 4.c), the elements of a distinguished basis of $\mathscr{H}_k$ do not have distinct norms: from (III.21)

$$|x_{n-j,j}| = \rho^{-n} \text{ for } 0 \leqq j \leqq n.$$

A particular subspace of $\mathscr{H}_k$ is the subspace of functions which admit expansions (III.23) such that for some sequence $\{\alpha_n\}$ of complex numbers ($i^2 = -1$):

$$\alpha_{n-j,j} = \binom{n}{j} i^j\alpha_n \text{ for } 0 \leqq j \leqq n \text{ and all } n.$$

Setting $z = u+iv$, the expansions of such functions are of the form $\sum_{n\geqq k}\alpha_n z^n$.

c) We now let the Hausdorff space Λ be a set of points $\lambda = (z, w)$ where z and w each belong to a subset of the complex Riemann sphere which contains the point at infinity. Let $\lambda_0 = (\infty, \infty)$. N is the set of all non-negative integers and we denote by Ψ the asymptotic scale

$$\Psi = \{\psi_n : n \in N\},\ \psi_n(z, w) = \left(\frac{1}{|z|}+\frac{1}{|w|}\right)^n.$$

Thus, B_0 is the space of the real numbers.

Let B be the space of the complex numbers and consider the space Q' of complex valued functions on Λ' which are asymptotically finite with respect to Ψ.

It can be shown (as in b)) that the set

$$H'_k = \{y_{lj} : l+j \geqq k\},\ k \in N,\ y_{lj}(z, w) = z^{-l}w^{-j},$$

is a distinguished subset of Q'. $\mathscr{H}'_k$ will denote the closed subspace generated by H'_k.

6. Asymptotic spaces: Example III

a) $\mathscr{B}(D)$ will denote the set of all bounded transformations from a closed subset D of a Banach space S into S itself; i.e. the set of transformations from D to S for which $||A||_D < \infty$, where

$$\text{(III.24)} \qquad ||A||_D = \sup_{s_1, s_2 \in D} \frac{||As_1 - As_2||}{||s_1 - s_2||}.$$

The function (III.24) is a pseudo-norm on $\mathscr{B}(D)$. If $||A-B|| = 0$, then $As = Bs+s_0$ for all $s \in D$ and some fixed $s_0 \in S$.

Under the assumption that D is a linear subspace of S, $\mathscr{I}(D)$ will denote the set of all bounded linear transformations from D to S. On $\mathscr{I}(D)$, (III.24) is equivalent to the usual uniform norm:

$$||A||_D = \sup_{\substack{s \in D \\ s \neq 0}} \frac{||As||}{||s||}.$$

b) We now construct an asymptotic space of functions with range in $\mathscr{B}(D)$, i.e. $B = \mathscr{B}(D)$.

The Hausdorff space Λ is the real interval $[0, 1]$ and $\lambda_0 = 0$. The space B_0 is the space of the real numbers. N is the set of all non-negative integers and we use the asymptotic scale

$$\Phi = \{\varphi_n : n \in N\},\ \varphi_n(\lambda) = \lambda^n.$$

Let X be the space of all mappings x defined on $\Lambda' = (0, 1]$, with range in $\mathscr{B}(D)$ and which are asymptotically finite with respect to Φ, i.e. such that for some $n \in N$,

$$\text{(III.25)} \qquad ||x||_D = 0(\varphi_n).$$

Suppose that $y \in X$, y is independent of λ and $||y||_D \neq 0$. Then, one verifies easily that $|y| = 1$. Furthermore, if $x \in X$ satisfies $|y-x| < 1$, then, as a function of λ, x converges uniformly to

y on D, as $\lambda \to 0$, since for λ small enough and some constant $\alpha > 0$

$$||y-x||_D \leqq \alpha\lambda.$$

If $x \in X$ has a Φ-asymptotic norm strictly less than 1 and for λ in some cd-nbhd of 0, x maps D into itself, then $x(\lambda)$ is a contraction mapping on D for all λ in some cd-nbhd of 0, i.e. for some λ', $0 < \lambda' \leqq 1$, $x(\lambda)$ maps D into itself and $||x(\lambda)||_D < 1$ when $\lambda \leqq \lambda'$ ([19], Vol. I, p. 43).

Spaces of this type have been considered by C. A. Swanson and M. Schulzer [32], [33]. In these references, transformations satisfying (III.25) are said to be "of Class Lip (φ_n)". Specific examples are given in [32], pp. 28—38.

c) The space X of b) consists of mappings from (0, 1] to the set $\mathscr{B}(D)$ of bounded transformations from D to S. In the space X' to be constructed now, unbounded transformations will also be considered. The range space B is the Banach space S.

Let Λ, λ_0, N and Φ be as in b). Let X' be the space of all mappings from $(0, 1] \times D$ to S which are asymptotically finite with respect to Φ when $s \in D$ is considered as a secondary parameter.

The Φ-asymptotic norm of $x \in X'$ is less than or equal to ρ^{-n} if for each fixed $s \in D$, there exist a constant $\alpha[s] > 0$ and a cd-nbhd $V[s]$ of 0 such that

$$||x(\lambda, s)|| \leqq \alpha[s]\lambda^n \text{ for all } \lambda \in V[s];$$

equivalently, $|x| < \rho^{-n}$ if for all $s \in D$,

$$||x(\lambda, s)|| \leqq \beta[s] \cdot \lambda^n \cdot ||s||, \text{ for all } \lambda \in V[s],$$

where $\beta[s] = \alpha[s] \cdot ||s||^{-1}$ if $s \neq 0$.

Suppose that $y \in X'$, y is independent of λ and $||y||_D \neq 0$. Then for each $s \in D$, $||y(\lambda, s)|| = ||y(s)||$ for all $\lambda \in \Lambda'$ and, hence, $|y| = 1$. Furthermore, if $x \in X'$ satisfies $|x-y| < 1$, then, as a function of λ, x converges strongly ([7], p. 475) to y on D, when $\lambda \to 0$; indeed, for each $s \in D$, there exists $\alpha[s] > 0$ such that for all λ small enough

$$||x(\lambda, s)-y(\lambda, s)|| \leqq \alpha[s] \cdot \lambda.$$

7. Moment spaces

a) For simplicity we restrict our definition of moment spaces to spaces of real valued functions defined on a finite interval $[a, b]$, $-\infty < a < b < \infty$. All integrals considered are Riemann-Stieltjes integrals ([41], p. 1).

Let α be a real valued function of bounded variation on $[a, b]$ ([41], p. 6). Let $\Phi = \{\varphi_n(t) : n = 0, 1, 2, \ldots\}$ be a sequence of non-zero, real functions on $[a, b]$ such that all integrals

$$\mu_n(1) = \int_a^b \varphi_n(t)d\alpha(t), \quad n = 0, 1, 2, \ldots, \tag{III.26}$$

exist and are finite.

Let X' be the linear space of all real functions x, defined on $[a, b]$ and such that all integrals

$$\mu_n(x) = \int_a^b x(t)\varphi_n(t)d\alpha(t), \quad n = 0, 1, 2, \ldots,$$

exist and are finite. $\mu_n(x)$ is called the *n-th moment of x relative to Φ.*

For $x \in X'$, define

$$\omega(x) = \inf\{n : \mu_n(x) \neq 0\}$$

and, for some fixed ρ, $1 < \rho < \infty$,

$$|x| = \rho^{-\omega(x)}. \tag{III.27}$$

It is immediate that X', with the norm (III.27), is a V-space, except possibly for completeness. In X' the distance of two functions x, y is less than or equal to ρ^{-n} if and only if $\mu_i(x) = \mu_i(y)$ for $i = 0, 1, 2, \ldots, n-1$.

X' admits a distinguished basis (Theorem II-2.2). Two elements of a distinguished basis of X' cannot have the same norm. Indeed, if $|x| = |y| = \rho^{-n}$ for some n, then

$$\mu_i(x) = \mu_i(y) = 0 \quad \text{for } i < n; \ \mu_n(x) \neq 0; \ \mu_n(y) \neq 0;$$

therefore

$$\mu_i(\mu_n(y)x - \mu_n(x)y) = 0 \text{ for } i \leqq n.$$

This implies that $|\mu_n(y)x - \mu_n(x)y| < \rho^{-n}$. Thus x and y are not distinguished.

Let N be the set of integers defined by

$$N = \{n : \text{for some } x_n \in X', \ \mu_i(x_n) = \delta_{in} \text{ for } i \leqq n\}.$$

For each $n \in N$, let x_n be a function such that $\mu_i(x_n) = \delta_{in}$ for $i \leqq n$. The set $H = \{x_n : n \in N\}$ forms a distinguished basis of X'.

The completion X of X', i.e. the set of formal expansions in terms of H, is a V-space (Theorem I-5.6).

V-spaces constructed in the manner described above will be called "*moment spaces*".

b) For the remainder of the Section we suppose that the function α is strictly increasing on $[a, b]$ and that Φ is a linearly independent set of continuous functions contained in X.

These assumptions imply that all the integrals (III.26) and the integrals

$$\int_a^b \varphi_m(t) \cdot \varphi_n(t) \cdot d\alpha(t), \quad m, n = 0, 1, 2, \ldots,$$

exist and are finite ([41], p. 7).

A sequence $\{f_n\}$ of functions defined on $[a, b]$ is said to be orthonormal with respect to α if

$$\langle f_m, f_n \rangle = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots,$$

where

$$\langle f, g \rangle = \int_a^b f(t) \cdot g(t) d\alpha(t).$$

Lemma 7.1. If f is a non-negative continuous function on $[a, b]$ and $\int_a^b f(t)d\alpha(t) = 0$, then $f(t) \equiv 0$.

The proof is identical to that of Proposition (5.2) in [37], p. 41.

Theorem 7.2. There exists a unique sequense of functions $\{p_n\}$ of the form

$$p_n(t) = a_{nn}\varphi_n(t) + a_{n,n-1}\varphi_{n-1}(t) + \ldots + a_{n0}\varphi_0(t), \; a_{nn} > 0,$$

which is orthonormal with respect to α.

The proof is an easy modification of [37], pp. 41—42.

Theorem 7.3.

(i) $|p_n| = \rho^{-n}$, $n = 0, 1, 2, \ldots$.
(ii) $\{p_n\}$ is a distinguished basis of X.
(iii) If $f \in X$, then, in the norm of X,

$$f(t) = \sum_{n=0}^{\infty} \langle f, p_n \rangle \, p_n(t). \tag{III.28}$$

Proof: (i) For all m, there exist coefficients b_{mi} such that

$$\varphi_m(t) = \sum_{i=0}^{m} b_{mi} p_i(t), \quad b_{mm} = \frac{1}{a_{mm}} > 0.$$

Thus, $|p_n| = \rho^{-n}$ follows from

$$\int_a^b p_n(t)\varphi_m(r)d\alpha(t) = \begin{cases} 0 & \text{if } m < n, \\ 1/a_{mm} & \text{if } m = n. \end{cases} \tag{III.29}$$

(ii) follows from (i) and a previous remark, page 13.

(iii) For $m < n$,

$$\begin{aligned}\mu_m(f - \sum_{i=0}^{n-1} \langle f, p_i\rangle p_i) &= \int_a^b f(t)\varphi_m(t)d\alpha(t) - \sum_{i=0}^{n-1} \langle f, p_i\rangle \int_a^b p_i(t)\varphi_m(t)d\alpha(t) \\ &= \sum_{i=0}^{m} b_{mi} \langle f, p_i\rangle - \sum_{i=0}^{m} \langle f, p_i\rangle b_{mi} = 0.\end{aligned}$$

Thus $|f - \sum_{i=0}^{n-1} \langle f, p_i\rangle p_i| \leqq \rho^{-n}$ which proves the convergence.

The series (III.28) is usually called the *Fourier series* of f with respect to $\{p_n\}$ ([37], p. 45). In the moment space X, the distance of two functions f and g is less than or equal to ρ^{-n} if and only if the first n Fourier coefficients of f: $\langle f, p_i\rangle$, $i = 0, 1, 2, \ldots, n-1$, are equal to the corresponding Fourier coefficients $\langle g, p_i\rangle$ of g.

c) Whenever α is strictly increasing on $[a, b]$ and for all n, $\varphi_n(t) = [\varphi(t)]^n$, where φ is a non-constant continuous function on $[a, b]$, the results of b) are valid. Furthermore, we have the following Theorems 7.4 and 7.5.

Theorem 7.4. The orthonormal sequence $\{p_n\}$ satisfies a recurrence formula of the form

$$p_{n+1}(t) = [c_n\varphi(t) + d_n]p_n(t) + e_n p_{n-1}(t), \quad n \geqq 0,$$

where c_n, d_n, e_n are real constants. (Set $p_{-1}(t) = 0$).

The proof is a modification of the proof of Proposition (5.4), [37], p. 43.

Theorem 7.5. $|p_m \cdot p_n| \leqq \rho^{-|m-n|}$ and for some coefficients c_{mni},

$$p_m(t) \cdot p_n(t) = \sum_{i=|m-n|}^{m+n} c_{mni} p_i(t).$$

Proof: Suppose $m \geqq n$. Then

$$\begin{aligned}\mu_i(p_m \cdot p_n) &= \int_a^b p_m(t)p_n(t)\varphi_i(t)d\alpha(t) \\ &= \sum_{r=i}^{n+i} a_{m,r-i} \int_a^b p_m(t)\varphi_r(t)d\alpha(t).\end{aligned}$$

From (III.29), $\mu_i(p_m \cdot p_n) = 0$ for $n+i > m$. The conclusion follows from this and the fact that $p_m p_n$ is a polynomial in φ of degree $m+n$.

For additional properties of the orthonormal sequence, Fourier series and coefficients, see [37], Chapter 5.

d) EXAMPLES. Let $\varphi_n(t) = t^n$ and $[a, b] = [-1, 1]$. The following are three examples of moment spaces (See [37], p. 50. Also [8], [21].)

$d\alpha(t)$	$p_n(t)$	*Reference*
$(1-t^2)^{-\frac{1}{2}}dt$	$\sqrt{1/\pi}\, T_n(t)$	$T_n(t)$: Chebyshev polynomials
$(1+t^2)^{\frac{1}{2}}dt$	$\sqrt{2/\pi}\, U_n(t)$	$U_n(t)$: Chebyshev polynomials of the second kind
dt	$(n+\frac{1}{2})^{\frac{1}{2}} P_n(t)$	$P_n(t)$: Legendre polynomials

To illustrate how a problem can be interpreted within the scope of a moment space, we consider the differential equation

$$\text{(III.30)} \qquad Lx = 2(t+3)\frac{dx}{dt} + x = 0, \quad x(-1) = 1.$$

Two methods have been proposed for the approximation of the solution of a differential equation which take advantage of the special properties (given in b) and c) above) of the Chebyshev polynomials. One is due to Lanczos [20], [21], the other to Clenshaw [2]. See L. Fox [13]. In both methods, the equation (III.30) is replaced by the equation

$$\text{(III.31)} \qquad My(t) = Ly(t) - \tau T_n(t) = 0, \quad y(-1) = 1.$$

It can be shown that for a certain value τ_0 of the parameter τ, (III.31) has a solution of the form

$$y(t) = \tau_0 \sum_{i=0}^{n} \beta_i T_i(t), \text{ with } \tau_0 = \Big(\sum_{i=0}^{n} (-1)^i \beta_i\Big)^{-1}.$$

By Theorem 7.5, if $z \in X$, then Lz and Mz belong to X and

$$|Lz - Mz| \leqq \rho^{-n}.$$

A study of this method of substitution of a perturbed equation for the original one, if conducted within the frame of the theory of moment spaces, may lead to interesting results and interpretations.

(Oblatum 18–3–66)