COMPOSITIO MATHEMATICA

PIERRE ROBERT

# On some non-archimedean normed linear spaces. II

*Compositio Mathematica*, tome 19, nº 1 (1968), p. 16-27
<http://www.numdam.org/item?id=CM_1968__19_1_16_0>

# On some non-Archimedean normed linear spaces II

by

Pierre Robert

## Introduction

This paper is the second of a series published under the same title and numbered I, II, . . .. The reader is assumed to be familiar with the definitions, notations and results of Part I.

This Part II is devoted to the study of $V$-spaces (see Part I, Introduction). In the last Section, $V$-algebras are introduced and results to be used in the theory of operators on $V$-spaces (Part IV) are stated.

## 1. Definitions

A systematic study of non-Archimedean normed linear spaces has been made by A. F. Monna ([24], [25]). Other references are [3], [12], [17].

Monna obtains interesting results when the norm range of the non-Archimedean normed linear space is assumed to have at most one accumulation point: 0. We shall retain this assumption. In most of his work, Monna requires that the valuation of the field of scalars be non-trivial; this, of course, is impossible in the case of a valued space.

DEFINITION 1.1. A *$V$-space* $X$ is a pseudo-valued or a valued space which is complete in its norm topology and for which there exists a set of integers $\omega(X)$ and a real number $\rho > 1$, such that

$$\text{(II.1)} \qquad \Omega(X) = \{0\} \cup \{\rho^{-n} : n \in \omega(X)\}.$$

DEFINITION 1.2. A *discrete $V$-space* is a $V$-space such that the set $\omega(X)$ of Definition 1.1 satisfies

$$\text{(II.2)} \qquad \sup \omega(x) = M \text{ for some } M < \infty.$$

The topology of a discrete $V$-space is discrete. A $V$-space such that

$$\text{(II.3)} \qquad \sup \omega(X) = \infty,$$

has a proper sequence convergent to $\theta$ and hence its topology is not discrete.

CONVENTIONS. (i) In all of this work, the symbol "$\rho$" will retain the meaning attached to it in Definition 1.1.

(ii) In the sequel, whenever two or more $V$-spaces will be considered simultaneously, it will be assumed that the value of $\rho$ is the same for all of these spaces.

The definitions, theorems and remarks of the following sections of this chapter, except Th. 2.4 (ii), do not depend on the completeness of the $V$-space. They remain valid for any space which satisfies Definition 1.1 except for the completeness requirement. Part (ii) of Theorem 2.4 requires completeness.

## 2. Existence of distinguished bases

In this section we prove that a $V$-space admits a distinguished basis (Theorem 2.2).

The proof of this statement is analogous, in part, to the proof of the existence of an orthonormal basis in a Hilbert space ([7], p. 252). It is made possible by the following improvement over Riesz's Lemma (Theorem I-6.1).

LEMMA 2.1. Let $Y$ be a proper, closed subspace of a $V$-space $X$. There exists $z \in X$ such that

$$|y-z| \geqq |z|, \text{ for all } y \in Y.$$

PROOF: Let $\alpha$ satisfy $\rho^{-1} < \alpha < 1$. Then

$$|x| > \alpha|z| \text{ implies } |x| \geqq |z|$$

for any pair $x$, $z \in X$.

By Theorem I-6.1, there exists $z \in X \backslash Y$, such that

$$|y-z| > \alpha|z|, \text{ for all } y \in Y.$$

Thus $|y-z| \geqq |z|$ for all $y \in Y$.

THEOREM 2.2. A $V$-space admits a distinguished basis.

PROOF: Let $D$ be the family of all distinguished subsets of a $V$-space $X$. $D$ is not empty since a single point with non-zero norm forms a distinguished subset of $X$. Let $D$ be ordered by set inclusion. It is easy to see that a linearly ordered subfamily of $D$ satisfies the conditions of Zorn's Lemma. Therefore $D$ contains at least one maximal element $H$.

We shall show that $[H] = X$ (see Section I.5). Suppose the contrary. Then by Lemma 2.1 there exists $z \in X \backslash [H]$ such that

$$|y-z| \geqq |z| \text{ for all } y \in [H].$$

a) If for each $y \in (H)$, $|y| \neq |z|$, then

$$|y+z| = \text{Max } \{|y|, |z|\} \text{ for all } y \in (H).$$

b) If a) fails, then for each $y \in (H)$ such that $|y| = |z|$, we have

$$-\frac{\alpha}{\beta} y \in [H] \text{ for all } \alpha, \beta \in F, \; \alpha \neq 0, \; \beta \neq 0,$$

and by the above inequality:

$$|\alpha y+\beta z| = \left|-\frac{\alpha}{\beta} y-z\right| = |z| = \text{Max } \{|y|, |z|\}.$$

From a) and b) it follows that $H \cup \{z\}$ is a distinguished subset of $X$, contradicting the maximality of $H$. Hence $[H] = X$ and $H$ is a distinguished basis of $X$.

The same argument applies as usual to yield the following:

Corollary 2.3. A $V$-space admits a distinguished basis which contains any given distinguished set.

In a Banach space $B$ a complete basis is a sequence $\{b_n\}$ such that for every $b \in B$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $b = \sum_n \alpha_n \beta_n$. The classical Paley-Wiener theorem ([1], [27], [30]) asserts that every sequence in $B$, which is "sufficiently close" to a complete basis, is itself a basis.

Arsove [1] has extended the Paley-Wiener theorem to arbitrary complete metric linear spaces over the real or complex field with the usual valuation. Theorem 2.4 (ii) is the $V$-space analogue of Arsove's Theorem 1 ([1], p. 366). We do not require that $H$ be countable.

Theorem 2.4. Let $H$ be a distinguished subset of a $V$-space $X$ and $f$ be a mapping of $H$ into $X$ such that, for each $h \in H$,

$$|h-\alpha_h f(h)| < |h| \tag{II.4}$$

for some scalar $\alpha_h \neq 0$. Then

(i) $f(H)$ is a distinguished set of $X$;

(ii) $f(H)$ is a distinguished basis if $H$ is a distinguished basis.

PROOF: (i) $|f(h)| = |h|$ for all $h \in H$. Let $\{h_i : i = 1, 2, \ldots, n\}$ be a subset of $H$ such that $|f(h_i)| = r$ for $i = 1, 2, \ldots, n$ and some $r > 0$. Let $\{\beta_i : i = 1, 2, \ldots, n\}$ be any set of non-zero scalars. Then, from (2.4), $|h_i| = |f(h_i)| = r$ and

$$\left|\left(\frac{\beta_1}{\alpha_1} h_1 + \frac{\beta_2}{\alpha_2} h_2 + \ldots + \frac{\beta_n}{\alpha_n} h_n\right) - (\beta_1 f(h_1) + \ldots + \beta_n f(h))\right|$$
$$= \left|\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} (h_i - \alpha_i f(h_i))\right| < r.$$

Since $H$ is distinguished, $|\sum_{i=1}^{n} \beta_i/\alpha_i\, h_i| = r$. It follows that

$$|\sum_{i=1}^{n} \beta_i f(h_i)| = r.$$

This proves that $f(H)$ is a distinguished set.

(ii) The proof is a rewording of Arsove [1], page 367, in which "$y_n$" and "$\lambda$" must be replaced by "$\alpha_{h_n} f(h_n)$" and "$\rho^{-1}$" respectively. Note that the proof requires the completeness of the space. (See the last paragraph of Section II-1).

## 3. Distinguished families of subsets

The notion of distinguishability was introduced for subsets of a (pseudo-) valued space. It will now be extended to families of subsets.

In all of this section $X$ is a $V$-space.

A set will be called *trivial* if it is a subset of $[\theta]$, i.e. if all its elements have norms equal to zero.

DEFINITION 3.1. A family $\{A_\alpha\}$ of subsets of $X$ is a *distinguished family of subsets* of $X$ if

(i) $A_{\alpha_1} \cap A_{\alpha_2}$ is trivial for $\alpha_1 \neq \alpha_2$.

(ii) every non-empty subset $B$ of $\bigcup_\alpha A_\alpha$ such that

a) $|x| \neq 0$ for each $x \in B$,

b) no two elements of $B$ belong to the same $A_\alpha$,

is a distinguished subset of $A$.

Clearly, a trivial set and any other subset of $X$ form a distinguished family. Also, if $\{A_\alpha\}$ is a distinguished family, $\{B_\alpha\}$ is a distinguished family of subsets if $B_\alpha \subset A_\alpha$ for all $\alpha$.

The following theorems give characterizations of distinguished families of non-trivial subspaces of $X$.

THEOREM 3.2. A family $\{X_\alpha\}$ of non-trivial (closed or open) subspaces of $X$ is a distinguished family of subsets of $X$ if and only if:

(i) $X_{\alpha_1} \cap X_{\alpha_2}$ is trivial for $\alpha_1 \neq \alpha_2$,

(ii) any union of distinguished subsets of some or all of the $X_\alpha$'s is a distinguished subset of $X$.

PROOF: The sufficiency is obvious. To prove the necessity, let $B = \bigcup_\alpha B_\alpha$, where $B$ is not empty and $B_\alpha$ is either empty or a distinguished subset of $X_\alpha$.

Consider a finite linear combination of elements of $B$:

$$x = \sum_{i=1}^{n} \sum_{j=1}^{p_i} \alpha_{ij} x_{ij},$$

where no $\alpha_{ij}$ is equal to 0 and where for each $i \in \{1, 2, \ldots, n\}$, $x_{ij} \in B_{\alpha_i}$ for $j = 1, 2, \ldots, p_i$.

Define $x_i = \sum_{j=1}^{p_i} \alpha_{ij} x_{ij}$. Then $x_i \in B_{\alpha_i}$.

Since the $B_{\alpha_i}$'s are distinguished sets and $\{x_1, x_2, \ldots, x_n\}$ is by (ii) of Definition 3.1 a distinguished set:

$$|x| = |\sum_{i=1}^{n} x_i| = \operatorname*{Max}_{1 \leqq i \leqq n} |x_i| = \operatorname*{Max}_{1 \leqq i \leqq n} \{ \operatorname*{Max}_{1 \leqq j \leqq p_i} |x_{ij}| \}.$$

This shows that $B$ is a distinguished set.

THEOREM 3.3. A finite family $\mathscr{A} = \{X_1, X_2, \ldots, X_n\}$ of non-trivial, closed subspaces of a $V$-space $X$ forms a distinguished family of subspaces of $X$ if and only if there exists *a* family $\mathscr{B} = \{H_1, H_2, \ldots, H_n\}$ such that:

(i) $H_i$ is a distinguished basis of $X_i$; $i = 1, 2, \ldots, n$,

(ii) $H_i \cap H_j$ is empty for $i \neq j$,

(iii) $H_0 = \bigcup_{i=1}^{n} H_i$ is a distinguished basis for the closed subspace $X_0 = [X_1 \cup X_2 \cup \ldots \cup X_n]$.

PROOF: Necessity. For each $i$, $X_i$ is a $V$-space and admits a distinguished basis, $H_i$. By (i) of Definition 2.5, the assumption that $\mathscr{A}$ is a distinguished family implies that $X_i \cap X_j$ is trivial for $i \neq j$. Since distinguished bases do not contain any trivial element, (ii) is satisfied. By (ii) of Theorem 3.2, $H_0$ is a distinguished set. Clearly $[H_0] = X_0$.

Sufficiency. It is easy to see that $X_i \cap X_j$ is trivial for $i \neq j$. We must show that (ii) of Definition 3.1 is satisfied.

Let $\{x_1, x_2, \ldots, x_m\}$, $m \leqq n$, be such that $|x_i| \neq 0$ and assume that the $X_i$'s are reindexed in such a way that $x_i \in X_i$ for $i = 1, 2, \ldots, m$.

For each fixed $i$, $1 \leqq i \leqq m$, there exists a non-increasing expansion of $x_i$ in terms of $H_i$:

$$x_i = \sum_{j \geqq 1} \alpha_{ij} y_{ij}, \quad \alpha_{ij} \neq 0.$$

According to Lemma I-5.5

$$|x_i| = \sup_j |y_{ij}|.$$

Suppose that

$$|y_{ik}| = \sup_j |y_{ij}| \text{ for } k \leqq p_i,$$

$$|y_{ik}| < \sup_j |y_{ij}| \text{ for } k > p_i.$$

$p_i$ is necessarily finite.

Consider now any set of scalars $\{\beta_1, \beta_2, \ldots, \beta_m\}$. We can assume without loss of generality that $\beta_i \neq 0$ for each $i \leqq m$. Let $x = \sum_{i=1}^{m} \beta_i x_i$. If we can show that

$$\text{(II.5)} \qquad |x| = \operatorname*{Max}_{1 \leqq i \leqq m} |x_i|,$$

we will have proved that $\mathscr{A}$ is a distinguished family of subsets of $X$.

Since $H_0$ is a distinguished set by assumption, we have:

$$\text{(II.6)} \qquad |x| = |\sum_{i=1}^{n} \sum_{j=1}^{p_i} \beta_i \alpha_{ij} y_{ij} + \sum_{i=1}^{n} \sum_{j=p_i+1}^{\infty} \beta_i \alpha_{ij} y_{ij}|,$$

$$\text{(II.7)} \qquad |\sum_{i=1}^{n} \sum_{j=1}^{p_i} \beta_i \alpha_{ij} y_{ij}| = \operatorname*{Max}_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq p_i}} |y_{ij}| = \operatorname*{Max}_{1 \leqq i \leqq n} |x_i|,$$

$$\text{(II.8)} \qquad |\sum_{i=1}^{n} \sum_{j=p_i+1}^{\infty} \beta_i \alpha_{ij} y_{ij}| < \operatorname*{Max}_{\substack{1 \leqq i \leqq n \\ 1 \leqq j \leqq p_i}} |y_{ij}| = \operatorname*{Max}_{1 \leqq i \leqq n} |x_i|.$$

The equality (II.7) is guaranteed by the fact that no cancellation of terms can arise in the finite sum $\sum_{i=1}^{n} \sum_{j=1}^{p_i}$ since the $H_i$'s are assumed to be disjoint.

From (II.6), (II.7) and (II.8), it follows that (II.5) is true, and the proof of sufficiency is completed.

If a finite family $\{X_1, X_2, \ldots, X_n\}$ of non-trivial, closed subspaces of a $V$-space is a distinguished family, the closed subspace $[X_1 \cup X_2 \cup \ldots \cup X_n]$ will be called the *direct sum* of $X_1, X_2, \ldots, X_n$.

The following corollary to Theorem 3.3 is easily proved:

Corollary 3.4. Let $\{X_1, X_2, \ldots, X_n\}$ be a distinguished family of non-trivial closed subspaces of $X$. The decomposition of any point of their direct sum as a sum of elements of these $X_i$'s is unique except for order and addition of trivial elements.

In this section we have exhibited some analogy between Hilbert spaces and $V$-spaces. The analogy is a consequence of the similarity of the rôles played by the concepts of orthogonality and distinguishability in the two types of spaces.

In the next section an important difference between the two structures will become apparent.

## 4. Distinguished complements

In a Hilbert space, the orthogonal complement of a set $A$ is defined to be the set of all the elements of the space which are orthogonal to all the elements of $A$. In a $V$-space, we introduce a corresponding notion: The distinguished adjunct $A^d$ of a subset $A$ of a $V$-space $X$ is defined by

$$A^d = \{x \in X : \{x\} \text{ and } A \text{ form a distinguished pair of subsets of } X\}.$$

In Theorem 4.1, simple properties of distinguished adjuncts are stated. (ii) expresses the fact that the distinguished adjunct of a set $A$ is the largest set which forms with $A$ a distinguished pair of subsets of $X$. Parts (i), (iii), (iv) and (v) should be compared with Theorems 1, 2, 3, 4 of [14], p. 24. The proofs of the statements follow directly from our definitions and are omitted.

THEOREM 4.1. If $A$ and $B$ are subsets of a $V$-space, then each of the following statements is valid:

(i) $A \cap A^d$ is trivial.

(ii) If $(A, B)$ is a distinguished family of subsets of $X$, then $A \subset B^b$ and $B \subset A^d$.

(iii) $A \subset A^{dd}$.

(iv) If $A \subset B$, then $B^d \subset A^d$.

(v) $A^d = A^{ddd}$.

In a Hilbert space the orthogonal complement of any set is a closed subspace. However, the same is not true of the distinguished adjunct of a set in a $V$-space. The following example illustrates this fact. Let $A = S(\theta, r)$ and $|z| > r$; then $z \in A^d$ and for all $y \in A$, $y+z \in A^d$; clearly $y = (y+z)-z$ does not belong to $A^d$. Thus, $A^d$ is not, in general, a subspace of the $V$-space, even when $A$ itself is a subspace.

Two consequences of the discrepancy just mentioned are, first, that the notion of distinguished adjunct will not be useful in the sequel; and, secondly, that the non-uniqueness of the distinguished complement of a closed subspace (as described in the following definition) will have a rôle in the theory.

Definition 4.2. Two closed subspaces $Y$, $Z$, of a $V$-space $X$ are said to be *distinguished complements* of one another if

(i) $Y$ and $Z$ form a distinguished pair of subsets of $X$,

(ii) $X$ is the direct sum of $Y$ and $Z$.

It is clear that the only distinguished complement of $[\theta]$ is $X$, and conversely. If $Y$ is a non-trivial, closed, proper subspace of $X$, $Y$ admits a distinguished complement, but, in general, it is not unique. This is expressed in Theorem 4.4, in which we use the following terminology.

Definition 4.3. Let $Y$ and $Z$ be closed subspaces of a $V$-space, with distinguished bases $H(Y)$ and $H(Z)$ respectively. $H(Z)$ is called an *extension of* $H(Y)$ *to* $Z$ if $H(Y) \subset H(Z)$. (This implies $Y \subset Z$).

Theorem 4.4. Let $Y$ be a non-trivial, closed, proper subspace of a $V$-space $X$. Let $H(Y)$ be any distinguished basis of $Y$ and $H$ be any extension of $H(Y)$ to $X$. The subspace $[H \backslash H(Y)]$ is a non-trivial, closed, proper subspace of $X$ which is a distinguished complement of $Y$.

The theorem is easily proved, using Corollary 2.3 and Theorem 3.3. It does not state that two different pairs $(H_1(Y), H_1)$ and $(H_2(Y), H_2)$ necessarily will generate distinct distinguished complements of $Y$.

As a simple example, let $X$ have a distinguished basis formed by three elements $x_1$, $x_2$, $x_3$ with $|x_1| = |x_2| = |x_3|$. Let $Y = [x_1]$ and $H(Y) = \{x_1\}$. Three possible extensions of $H(Y)$ to $X$ are:

$$H_1 = \{x_1, x_2, x_3\},\ H_2 = \{x_1, x_1+x_2, x_3\},\ H_3 = \{x_1, x_2+x_3, x_3\}.$$

The distinguished complements of $Y$ generated by the pairs $(H(Y), H_1)$ and $(H(Y), H_3)$ are both equal to $[x_2, x_3]$ but that generated by $(H(Y), H_2)$ differs from $[x_2, x_3]$.

## 5. Notes

(i) The concept of distinguishability has been introduced by Monna under a different name and through another formal definition.

In his early papers, [24], Monna uses the term "pseudo-orthogonal"; in his later work [25], he uses the word "orthogonal". In a strongly non-Archimedean normed linear space, a point $x$ is said to be orthogonal to a point $y$ if the distance from $x$ to the linear subspace $(y)$ is equal to the norm of $x$. ([24], V, p. 197; [25], I. p. 480). It is easily verified that $y$ is then orthogonal to $x$.

The equivalence of this definition of orthogonality and of our definition of distinguishability is indicated in the following theorem (cf. Monna, [24]):

THEOREM 5.1. Let $A$ be a subset of a $V$-space $X$. For $x \in A$, let $A_x$ denote the linear subspace $(A \backslash \{x\})$. Then, $A$ is a distinguished subset of $X$ if and only if, for all $x \in A$:

$$x \notin [\theta] \text{ and distance } (x, A_x) = |x|.$$

The proof is omitted.

From the notion of orthogonality, Monna constructs a theory of orthogonal sets and orthogonal complements quite analogous to our theory of distinguished sets and complements.

In [25], it is assumed that the valuation of the field of scalars is not trivial. Use is not made very extensively of "complete orthogonal" (distinguished) bases, which exist only under special assumptions, such as local compactness and separability.

An important tool used by Monna is the concept of a "projection". We have postponed the introduction of projections in our theory until linear operators are studied.

(ii) Ingleton ([17], p. 42; see also [25], I, p. 475) defines a spherically complete totally non-Archimedean metric space (field) as a totally non-Archimedean metric space (field) in which every family of closed balls linearly ordered by inclusion has non-void intersection. Spherical completeness implies completeness ([25], I, p. 476). In general, completeness does not imply spherical completeness, but, if the norm (valuation) satisfies (2.1) and (2.3) of Definition 1.1, then completeness implies spherical completeness ([25], II, p. 486). Therefore, a $V$-space is spherically complete.

Monna ([25], III, p. 466) has shown that the existence of a complete orthogonal (distinguished) basis in a non-Archimedean normed linear space is related to the completeness of the space and the spherical completeness of the field of scalars, when the valuation of the field is non-trivial.

It is possible that a reformulation of the arguments of Monna to the case of a field of scalars provided with a trivial valuation could lead to a proof of existence of a distinguished basis for a $V$-space. Our proof (Theorem 2.2) is more direct and shows that completeness conditions are unnecessary.

## 6. $V$-algebras

In this section we shall consider $V$-spaces $X$ on which a multiplication is defined, i.e. such that to each pair $(x, y) \in X \times X$ there corresponds a unique "product" $xy \in X$.

REMARK: We shall define some elements of a $V$-space by use of sequences and series. Since a $V$-space can be a pseudo-normed space, the limit of a sequence or the sum of a series are not necessarily unique. For this reason we make the following notational convention:

CONVENTION. In the sequel, the relation "$x = y$" means that $|x-y| = 0$; strict identity between $x$ and $y$ is indicated by the symbol "$x \equiv y$".

The definitions and theorems of this Section are simple modifications of the definitions and theorems of the classical theory of normed rings ([22], [26]).

DEFINITION 6.1. A $V$-space $X$ with a multiplication is called a *$V$-algebra* if for all $x, y \in X$ and all scalars $\alpha$:

$$\alpha(xy) \equiv (\alpha x)y \equiv x(\alpha y), \tag{II.9}$$

$$x(yz) \equiv (xy)z, \tag{II.10}$$

$$x(y+z) \equiv xy+xz, \quad (y+z)x \equiv yx+zx, \tag{II.11}$$

$$|xy| \leqq |x| \cdot |y|. \tag{II.12}$$

We also assume the existence of an identity, i.e. of an element $e$ such that

$$xe \equiv ex \equiv x \text{ for all } x \in X, \tag{II.13}$$

$$|e| = 1. \tag{II.14}$$

$X$ is said to be a *commutative* $V$-algebra if

$$xy \equiv yx \text{ for all } x, y \in X. \tag{II.15}$$

As usual, we denote by $x^n$ the product $xx \dots x$ of $n$ elements equal to $x$. $x^0 \equiv e$ for all $x \in X$.

DEFINITION 6.2. An element $e'$ will be called a pseudo-identity if $e' = e$.

Definition 6.3. Let $x$ be an element of a $V$-algebra $X$.

(i) $x$ is said to be *pseudo-regular* if there exists an element $x^{-1}$ such that

$$xx^{-1} = x^{-1}x = e.$$

$x^{-1}$ is called a *pseudo-inverse* of $x$.

(ii) $x$ is said to be *regular* if there exists an element $x^{-1}$ such that

$$xx^{-1} \equiv x^{-1}x \equiv e.$$

$x^{-1}$ is called the *inverse* of $x$. (It can be proved that such an element is unique.)

(iii) If $x$ is not (pseudo-) regular it is said to be *singular*.

No element of $[\theta]$ is (pseudo-) regular for otherwise

$$1 = |xx^{-1}| \leqq |x| \cdot |x^{-1}| = 0.$$

Theorem 6.4. Let $x^{-1}$ be a pseudo-inverse of an element $x$ of a $V$-algebra $X$. Then $y$ is a pseudo-inverse of $x$ if and only if $y = x^{-1}$. Consequently, any two pseudo-inverses of $x$ have equal norms. (The proof is omitted).

Lemma 6.5. Let $X$ be a $V$-algebra, $x \in X$ and $|x| < 1$. Then

(i) the sums of the series $\sum_{n=0}^{\infty} x^n$ are pseudo-inverses of $(e-x)$;

(ii) for every pseudo-inverse $x'$ of $(e-x)$:

$$x+(e-x') = x(e-x'),$$
$$x+(e-x') = (e-x')x.$$

The proof is obtained by direct verification. (See [22], pp. 64—66).

Theorem 6.6. Let $Y$ denote the set of pseudo-regular points of $X$. If $y \in Y$, $x \in X$ and $|x-y| < |y^{-1}|^{-1}$, then

(i) $x \in Y$;

(ii) $|x^{-1}| = |y^{-1}|$;

(iii) $|x^{-1}-y^{-1}| \leqq |y^{-1}|^2 \cdot |x-y| < |y^{-1}|$;

(iv) $Y$ is an open subset of $X$ and the mapping $y \to y^{-1}$, defined on $Y$, is continuous.

The proof is similar to that of Theorem 4.1-D of [36], p. 164.

Definition 6.7. The spectrum $\sigma(x)$ of an element $x$ of a $V$-algebra $X$ is the set of scalars $\lambda$ for which $(x-\lambda e)$ is singular.

THEOREM 6.8. Let $x \in X$, $0 < |x| \leqq 1$. If for some scalar $\mu$, $|x-\mu e| < 1$, then

(i) $\sigma(x)$ is empty or $\sigma(x) = \{\mu\}$;

(ii) for $\lambda \neq \mu$, the pseudo-inverses of $(x-\lambda e)$ are the sums of the series

$$-\sum_{n=0}^{\infty} \frac{(x-\mu e)^n}{(\lambda-\mu)^{n+1}}, \tag{II.16}$$

and satisfy

$$|(x-\lambda e)^{-1}| = |x-\lambda e| = 1. \tag{II.17}$$

PROOF: We note first that this theorem is an extension of Lemma 6.5. Indeed, if $|x| < 1$, we take $\mu = 0$.

(i) is a consequence of (ii).

(ii) Since $|(x-\mu e)^n| \leqq |x-\mu e|^n$, the series (II.16) converges. Using the continuity of the multiplication, we verify directly that if $y$ is a sum of (II.16),

$$y(x-\mu e)-(\lambda-\mu)y = y(x-\lambda e) = e,$$
$$(x-\mu e)y-(\lambda-\mu)y = (x-\lambda e)y = e.$$

(II.17) follows from (II.16).

A direct proof of the following theorem is similar to that of Theorem 6.8.

THEOREM 6.9. Let $x \in X$, $|x| > 1$. If for some scalar $\mu$, $(x-\mu e)$ is pseudo-regular and $|(x-\mu e)^{-1}| < 1$, then

(i) $\sigma(x)$ is empty;

(ii) for $\lambda \neq \mu$, the pseudo-inverses of $(x-\lambda e)$ are the sums of the series

$$\sum_{n=0}^{\infty} (\lambda-\mu)^n[(x-\mu e)^{-1}]^{n+1}$$

and satisfy

$$|(x-\lambda e)^{-1}| = |(x-\mu e)^{-1}|.$$

REMARK. As in the classical theory of normed rings, the singularity or regularity of an element of a $\Omega$-algebra depends on its belonging to some maximal ideal of the algebra.

The theory of Banach algebras ([22], [26]) is partially based on the fact that any Banach field is completely isomorphic to the field of the complex numbers (with its usual topology). From this result one attemps to characterize the maximal ideals of the algebra. There does not seem to be any interesting analogue of this theory in the case of $V$-algebras.

(Oblatum 18–3–66) Université de Montréal, Canada