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by

A. J. Stam

1. Notations and Results

By P, Q, R and the same letters with indices attached, we denote probability measures on the Borel sets of the real line. The convolution of any two finite signed measures M and N on the Borel sets of the real line will be written MN, iterated convolutions being written as powers. The inequalities

$$(1.1) \qquad ||M+N|| \leqq ||M||+||N||,$$

$$(1.2) \qquad ||MN|| \leqq ||M|| \, ||N||,$$

where $||M||$ denotes total absolute variation, will be used repeatedly.

The probability measure degenerate at a is denoted by U_a. Occasionally we will write I for U_0. If P may be written in the form

$$(1.3) \qquad P = \sum_{k=-\infty}^{+\infty} p_k U_{b+kc},$$

it will be called a lattice distribution. If moreover P is nondegenerate, the largest c for which (1.3) holds, is called the span. The p_k will be called the probabilities of the lattice distribution.

For absolutely continuous P unimodality is defined here as the existence of ξ such that a probability density of P is nondecreasing on $(-\infty, \xi)$ and nonincreasing on (ξ, ∞). For a lattice distribution unimodality will mean that the sequence $\{p_k\}$ in (1.3) is nondecreasing for $k \leqq k_0$ and nonincreasing for $k \geqq k_0$ for some k_0.

In a preceding paper, Stam [7], henceforward cited as I, the author studied the set L_0 consisting of those a for which

$$(1.4) \qquad \lim_{n\to\infty} ||P^n - U_a P^n|| = 0.$$

From the notations introduced above it follows that $U_a P^n$ is the n-fold convolution of P shifted over a distance a. It was shown (I, theorem 2) that $L_0 = (-\infty, +\infty)$ if and only if some P^m has an absolutely continuous component.

In I also a weaker form of (1.4) was considered, viz.

$$\lim_{n\to\infty} ||P^n Q - U_a P^n Q|| = 0 \tag{1.5}$$

for every absolutely continuous Q, which holds for every a if P is not a lattice distribution (I, theorem 5).

The subject of this paper is the order of convergence in (1.4) and (1.5), if present, the dependence on a and questions on uniformity with respect to Q in (1.5). The following results will be obtained.

A necessary condition for convergence of prescribed order in (1.4) and (1.5) is derived (theorem 7). If $a \in L_0$, then $||P^n - U_a P^n|| \leqq cn^{-\frac{1}{2}}$ (theorem 2). If P^m for some m has an absolutely continuous component and if P has finite absolute moment of order $2+\delta$ for some $\delta \in (0, 1]$, then (theorem 3)

$$||P^n - U_a P^n|| \sim 2|a|(2\pi n\sigma^2)^{-\frac{1}{2}}, \tag{1.6}$$

where σ^2 is the variance of P. A similar theorem holds for lattice distributions (theorem 4).

If P has infinite second moment, the situation is more complex. It will be shown (theorem 8) that under the condition

$$\left|\int e^{iux}\, dP(x)\right| \leqq 1-\gamma|u|^\delta, \qquad |u| < \varepsilon, \tag{1.7}$$

with $\delta > 0$, $\gamma > 0$, $\varepsilon > 0$, if P^m for some m has an absolutely continuous component and P has a finite moment of some positive order,

$$||P^n - U_a P^n|| \leqq c(\alpha, a, P)n^{-1/\alpha}, \qquad n = 1, 2, \ldots \tag{1.8}$$

for any $\alpha > \delta$. A similar result holds for lattice distributions (theorem 9). The condition (1.7) is necessary for (1.8) with $\alpha = \delta$.

In special cases, e.g. if all P^n are unimodal, the above results may be sharpened (theorems 5 and 6).

In section 5 the relation (1.5) will be studied. It will turn out (theorems 10 and 11) that the convergence in (1.5) cannot be uniform with respect to Q and the order of convergence cannot be independent of Q, except when $a \in L_0$. However, the results for absolutely continuous P extend to (1.5) for non-lattice P, for suitable Q arbitrarily close to U_0 in the sense of weak convergence (theorem 12).

In section 6 (theorem 13) it will be shown that if some P^m has an absolutely continuous component and

$$||P^n - U_a P^n|| \leqq c b_n^{-1}, \qquad n = 1, 2, \ldots$$

for a single a, then under certain conditions on the sequence $\{b_n\}$

$$\limsup_{u \to \infty} b_n ||P^n - U_a P^n|| = c_2 a,$$

for every $a \in (-\infty, +\infty)$.

We say that P contains P_1, if

$$P = \alpha P_1 + \beta P_2$$

with $\alpha > 0$, $\beta \geqq 0$, $\alpha + \beta = 1$, P, P_1 and P_2 being probability measures. If P^m for some m contains P_1, the convergence to zero of $||P^n R - U_a P^n R||$ is at least of the same order as $||P_1^n R - U_a P_1^n R||$, if certain mild conditions are satisfied (theorem 1). This fact, an extension of I, lemma 5, will play an important rôle in our proofs.

2. Comparison of orders of convergence

In theorem 1 below we have to restrict ourselves to convergence of order h_n^{-1}, with $b_n = f(n)$, the function f on $[1, \infty)$ satisfying

$$f(1) = 1. \tag{2.1}$$

$$f(x) \leqq f(y), \quad 1 \leqq x \leqq y, \tag{2.2}$$

$$\lim_{n \to \infty} f(x) = +\infty, \tag{2.3}$$

$$\lambda(c) \overset{df}{=} \sup_{x \geqq 1} \frac{f(cx)}{f(x)} < \infty, \qquad c \geqq 1. \tag{2.4}$$

The condition (2.1) and the choice of 1 as left endpoint of the domain of f are inessential.

Lemma 1. *Under* (2.1)—(2.4) *we have*

$$\lambda(\xi\eta) \leqq \lambda(\xi)\lambda(\eta), \quad \xi \geqq 1, \ \eta \geqq 1, \tag{2.5}$$

$$\lambda(\xi) \leqq c\xi^r, \qquad \xi \geqq 1, \tag{2.6}$$

$$f(x) \leqq cx^r, \qquad x \geqq 1, \tag{2.7}$$

for some positive constants r *and* c.

Proof. The relation (2.5) follows from

$$\frac{f(\xi\eta x)}{f(x)} = \frac{f(\xi\eta x)}{f(\eta x)} \frac{f(\eta x)}{f(x)}.$$

Let $\varphi(u) \overset{df}{=} \log \lambda(e^u)$, $u \geqq 0$. The relation (2.5) then becomes

$$\varphi(u+v) \leqq \varphi(u)+\varphi(v), \quad u \geqq 0,\ v \geqq 0. \tag{2.8}$$

From (2.1)—(2.4) it follows that φ is finite, nonnegative and nondecreasing. From (2.8)

$$u^{-1}\varphi(u) \leqq u^{-1}\{n\varphi(1)+\varphi(\vartheta)\}$$

for $u = n+\vartheta$, $n = 1, 2, \ldots, 0 \leqq \vartheta < 1$, so that

$$\varphi(u) \leqq ru, \qquad u \geqq 1,$$

which implies (2.6). Finally, by (2.4) and (2.6)

$$f(x) \leqq \lambda(x)f(1) \leqq cx^r, \qquad x \geqq 1.$$

Lemma 2. *If the function f satisfies* (2.1)—(2.4),

$$B_n \overset{df}{=} f(n) \sum_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{1}{f(k)},$$

with $0 < \alpha < 1$ and $\beta = 1-\alpha$, is bounded with respect to n.

Proof. By (2.4) and (2.6)

$$f(n)/f(k) \leqq c(n/k)^r \leqq c(n/k)^m, \quad k = 1, 2, \ldots, n.$$

where m is a natural number. Therefore

$$f(n)/f(k) \leqq \frac{c(m+1)!n^m}{(k+1)(k+2)\ldots k+m)},\ k = 1, 2, \ldots, n,$$

and

$$B_n \leqq c(m+1)!n^m \sum_{k=1}^{n} \frac{n!}{(k+m)!(n-k!)} \alpha^k \beta^{n-k},$$

$$B_n \leqq \frac{c(m+1)!\alpha^{-m} n^m}{(n+1)\ldots(n+m)} \sum_{h=m+1}^{n+m} \binom{n+m}{h} \alpha^h \beta^{n+m-h} \leqq \frac{c(m+1)!\alpha^{-m}n^m}{(n+1)\ldots(n+m)},$$

which is bounded with respect to n.

Under (2.1) and (2.2) the condition (2.4) is necessary in order that lemma 2 holds for all $\alpha \in (0, 1)$: If $n-1 < x \leqq n$, we have

$$f(n) \sum_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{1}{f(k)} \geqq \frac{f(x)}{f(\alpha x)} \sum_{k=1}^{[\alpha x]} \binom{n}{k} \alpha^k \beta^{n-k},$$

where

$$\lim_{x\to\infty} \sum_{k=1}^{[\alpha x]} \binom{n}{k} \alpha^k \beta^{n-k} = \tfrac{1}{2}$$

THEOREM. 1 *Let M be a finite signed measure on the Borel sets of the real line, and let $b_n = f(n)$, $n = 1, 2, \ldots$, where f satisfies (2.1)—(2.4). If P^m for some m contains P_1 and*

$$b_n||P_1^n M|| = O(1)$$

for $n \to \infty$, then

$$b_n||P^n M|| = O(1)$$

for $n \to \infty$. A similar conclusion holds if $O(1)$ is replaced by $o(1)$.

PROOF. Putting $P^m = Q$, we have by (1.1) and (1.2)

$$b_n||Q^n M|| = f(n)||\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} P_1^k P_2^{n-k} M||,$$

$$b_n||Q^n M|| \leqq f(n)\beta^n||M|| + \sum_{k=1}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \frac{f(n)}{f(k)} b_k||P_1^k M||.$$

If $b_k||P_1^k M|| \leqq c < \infty$, $k = 1, 2, \ldots$, then $b_n||Q^n M||$ is bounded in n by (2.7) and lemma 2. If $\lim_{k\to\infty} b_k ||P_1^k M|| = 0$, then $\lim_{n\to\infty} b_n||Q^n M|| = 0$ by (2.7), lemma 2 and the Toeplitz theorem (Loève [4], § 16.3, p. 238).

Now let $n = h_n m+d$, with h_n integer and $0 \leqq d < m$. Then by (2.4), (2.6) and (1.2)

$$f(n)\ ||P^n M|| \leqq (h_n m+d)^r h_n^{-r} f(h_n)||Q^{h_n} M||,$$

from which our assertions follow.

3. Convergence of order $n^{-\frac{1}{2}}$ in (1.4)

THEOREM 2. *If $a \in L_0$,*

$$||P^n - U_a P^n|| \leqq c(a, P) n^{-\frac{1}{2}}, \qquad n = 1, 2, \ldots.$$

PROOF. If $a \in L_0$, we have $||P^m - U_a P^m|| < 2$ for some m, so that by I, lemma 3

$$P^m = \alpha(\tfrac{1}{2}U_0 + \tfrac{1}{2}U_a)Q_0 + \beta Q_1$$

with $0 < \alpha \leqq 1$, $\beta = 1-\alpha$. Since

$$||(\tfrac{1}{2}U_0+\tfrac{1}{2}U_a)^n - U_a(\tfrac{1}{2}U_0+\tfrac{1}{2}U_a)^n|| \leqq cn^{-\frac{1}{2}}, \quad n = 1, 2, \ldots,$$

(see I, lemma 6), our assertion follows from (1.2) and theorem 1 with $M = I - U_a$.

If P has finite second moment, $||P^n - U_a P^n||$ cannot converge to zero faster than $n^{-\frac{1}{2}}$, since, if φ is the characteristic function of P

$$||P^n - U_a P^n|| \geqq \sup_u |(1-e^{iua})\varphi^n(u)|$$

and $\{1-\exp(iu_n a)\}\varphi^n(u_n) \sim cn^{-\frac{1}{2}}$ for $u_n = vn^{-\frac{1}{2}}$. (Cf. theorem 7 below).

To obtain sharper results, but of less general application than theorem 2, we need the following lemma.

Lemma 3. *Let P have finite absolute moment of order $2+\delta$ with $0 < \delta \leqq 1$, and let N be the gaussian probability distribution with the same first moment μ and variance σ^2 as P. Then*

$$||(P-N)N^n|| \leqq c(P, \vartheta)n^{-1-\vartheta}, \qquad n = 1, 2, \ldots,$$

for every $\vartheta \in (0, \frac{1}{2}\delta)$.

Proof. It is no restriction to take $\mu = 0$. Let p_n and q_n be the probability densities of PN^n and N^{n+1}, respectively. Then

$$(3.1)\qquad ||(P-N)N^n|| \leqq T_1(n)+T_2(n)+T_3(n),$$

$$T_1(n) = \int_{-\infty}^{A_n} p_n(x)dx + \int_{A_n}^{\infty} p_n(x)dx,$$

$$T_2(n) = \int_{-\infty}^{A_n} q_n(x)dx + \int_{A_n}^{\infty} q_n(x)dx,$$

$$T_3(n) = \int_{-A_n}^{A_n} |p_n(x)-q_n(x)|dx = \int_{-\infty}^{+\infty} g_n(x)\{p_n(x)-q_n(x)\}dx,$$

with

$$(3.2)\qquad g_n(x) = 0,\ |x| \geqq A_n,\ |g_n(x)| = 1,\ |x| < A_n.$$

Denoting by φ the characteristic function of P and by γ_n the Fourier transform of g_n, we have by Parseval's formula

$$T_3(n) = (2\pi)^{-1}\int \overline{\gamma_n(u)}\{\varphi(u)-\exp(-\tfrac{1}{2}\sigma^2u^2)\}\exp(-\tfrac{1}{2}n\sigma^2u^2)du.$$

Since $|\gamma_n(u)| \leqq 2A_n$ and there is $d > 0$ with

$$|\varphi(u)-\exp(-\tfrac{1}{2}\sigma^2u^2)| \leqq c_1 |u|^{2+\delta}, \qquad |u| \leqq d,$$

(see Loève [4], § 12.4, p. 199),

$$T_3(n) \leqq c_2A_n\lambda^n + c_3A_n\int_0^d u^{2+\delta}\exp(-\tfrac{1}{2}n\sigma^2u^2)du,$$

$$(3.3)\qquad T_3(n) \leqq c_2A_n\lambda^n + c_4A_n n^{-\frac{1}{2}(3+\delta)}, \qquad n = 1, 2, \ldots,$$

with $0 \leqq \lambda < 1$.

From the relation

(3.4) $\int_x^\infty (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}t^2)dt \leqq (2\pi)^{-\frac{1}{2}}x^{-1} \exp(-\frac{1}{2}x^2), \quad x > 0,$

given in Feller [1], section VII.1, we find

(3.5) $T_2(n) \leqq 2\sigma(2\pi)^{-\frac{1}{2}}(n+1)^{\frac{1}{2}}A_n^{-1} \exp\{-\frac{1}{2}\sigma^{-2}(n+1)^{-1}A_n^2\}, \quad n = 1, 2, \ldots.$

Since PN^n is the probability distribution of the sum of two independent random variables with distributions P and N^n, we have

$$T_1(n) \leqq \int_{|x| \geqq \frac{1}{2}A_n} dN^n(x) + \int_{|x| \geqq \frac{1}{2}A_n} dP(x).$$

So with (3.4) and Markov's inequality

$$T_1(n) \leqq 4\sigma(2\pi)^{-\frac{1}{2}}n^{\frac{1}{2}}A_n^{-1} \exp(-\tfrac{1}{8}\sigma^{-2}n^{-1}A_n^2) + 2^{2+\delta}A_n^{-2-\delta} \int |x|^{2+\delta}dP. \tag{3.6}$$

If we take $A_n = n^{\frac{1}{2}+\frac{1}{2}\delta-\vartheta}$, our assertion follows from (3.1), (3.3), (3.5) and (3.6).

THEOREM 3. *If P^m for some m has an absolutely continuous component and if the absolute moment of order $2+\delta$ of P is finite for some δ with $0 < \delta \leqq 1$,*

$$|\, ||P^n - U_a P^n|| - 2|a|\, (2\pi n\sigma^2)^{-\frac{1}{2}}| \leqq c(a, P, \vartheta)n^{-\frac{1}{2}-\vartheta}, \quad n = 1, 2, \ldots,$$

for every $\vartheta \in (0, \frac{1}{2}\delta)$.

PROOF. It is no restriction to assume that P has zero first moment. Let N be the gaussion probability distribution with zero first moment and the same variance σ^2 as P. We choose Q to be absolutely continuous with zero first moment, finite variance τ^2, characteristic function $\vartheta(u) \in L_1$ and probability density belonging to L_2. Then the densities p_n of P^nQ and q_n of N^nQ also belong to L_2. We have for $a > 0$,

$$||(I-U_a)P^nQ - (I-U_a)N^nQ|| \leqq T_1(n) + T_2(n) \tag{3.7}$$

with

$$T_1(n) = 2\int_{-\infty}^{-A_n+a} \{p_n(x)+q_n(x)\}dx + 2\int_{A_n-a}^{\infty} \{p_n(x)+q_n(x)\}dx,$$

$$T_2(n) = \int g_n(x)\{p_n(x) - p_n(x-a) - q_n(x) + q_n(x-a)\}dx,$$

where

$$g_n(x) = 0, \quad |x| \geqq A_n, \quad |g_n(x)| = 1, \quad |x| < A_n. \tag{3.8}$$

By Chebychev's inequality

$$T_1(n) \leqq 4(n\sigma^2+\tau^2)(A_n-a)^{-2}, \qquad n = 1, 2, \ldots. \tag{3.9}$$

By Parseval's formula

$$\begin{aligned} T_2(n) = (2\pi)^{-1} \int \overline{\gamma_n(u)}(1-e^{iua})\{\varphi^n(u) \\ - \exp(-\tfrac{1}{2}n\sigma^2u^2)\}\vartheta(u)du, \end{aligned} \tag{3.10}$$

where γ_n is the Fourier transform of g_n.

To estimate $T_2(n)$ the following identity between complex numbers is needed:

$$y^n = \sum_{j=0}^{k} \binom{n}{j} (y-x)^j x^{n-j} + (y-x)^{k+1} \sum_{l=0}^{n-k-1} \binom{k+l}{l} x^l y^{n-k-l-1}, \tag{3.11}$$
$$k = 0, 1, \ldots, n-1; \; n = 1, 2, \ldots$$

One proof of (3.11) makes use of the relation

$$f(y) = \sum_{j=0}^{k} \frac{(y-x)^j}{j!} f^{(j)}(x) + \int_0^{y-x} \frac{t^k}{k!} f^{(k+1)}(y-t)dt$$

with $f(x) = x^n$. By the substitution $t = (y-x)\tau$ in the integral, binomial expansion of the factor $\{x\tau+y(1-\tau)\}^{n-k-1}$ in the integrand and application of the formula for the beta function (3.11) follows.

Applying (3.11) to (3.10) we find

$$T_2(n) \leqq V_n+W_n \tag{3.12}$$

with

$$\begin{aligned} V_n = \sum_{j=1}^{k} \binom{n}{j} |(2\pi)^{-1} \int \overline{\gamma_n(u)}\, (1-e^{iua})\{\varphi(u)-\exp(-\tfrac{1}{2}\sigma^2u^2)\}^j \\ \cdot \exp\{-\tfrac{1}{2}(n-j)\sigma^2u^2\}\vartheta(u)du|, \end{aligned} \tag{3.13}$$

$$\begin{aligned} W_n = \sum_{l=0}^{n-k-1} \binom{k+l}{l} |(2\pi)^{-1} \int \overline{\gamma_n(u)}\, (1-e^{iua})\{\varphi(u) \\ - \exp(-\tfrac{1}{2}\sigma^2u^2)\}^{k+1} \\ \cdot \exp(-\tfrac{1}{2}l\sigma^2u^2)\{\varphi(u)\}^{n-k-l-1}\vartheta(u)du|. \end{aligned} \tag{3.14}$$

By Parseval's formula and (3.8)

$$V_n = \sum_{j=1}^{k} \binom{n}{j} | \int g_n(x)d\{(I-U_a)(P-N)^jN^{n-j}Q\}(x)|,$$

$$V_n \leqq \sum_{j=1}^{k} \binom{n}{j} ||(I-U_a)(P-N)^jN^{n-j}Q||.$$

So from (1.2), theorem 2 (cf. I, theorem 2) and lemma 3

$$(3.15)\quad V_n \leqq \sum_{j=1}^{k} \binom{n}{j} c(j) n^{-\frac{1}{2}-j-j\vartheta} \leqq c_1(k) n^{-\frac{1}{2}-\vartheta}, \qquad n > k.$$

To majorize W_n we need the inequality

$$|\gamma_n(u)| \leqq 2A_n,$$

which follows from (3.8), the relation

$$(3.16)\quad \sum_{l=0}^{n-k-1} \binom{k+l}{l} = \binom{n}{k+1}, \quad k = 0, 1, \ldots, n-1,\ n = 1, 2, \ldots.$$

(Feller [1], Ch. II. 12, no. 7; Netto [5], (§ 156, (11)), and the existence of $d > 0$ with

$$|(1-e^{iua})\{\varphi(u)-\exp(-\tfrac{1}{2}\sigma^2u^2)\}^{k+1}| \leqq c_2 u^{1+(k+1)(2+\delta)}, \quad |u| \leqq d,$$

(Loève [4], § 12.4, p. 199),

$$\exp(-\tfrac{1}{2}l\sigma^2u^2)|\varphi(u)|^{n-k-l-1} \leqq \exp\{-(n-k-{}^1)\alpha u^2\}, \quad |u| \leqq d,$$

for some $\alpha > 0$ (Loève [4], ibid.), and

$$\exp(-\tfrac{1}{2}l\sigma^2u^2)|\varphi(u)|^{n-k-l-1} \leqq c_3(k)\lambda^n, \qquad |u| > d,$$

with $0 < \lambda < 1$, since $|\varphi(u)| < 1$, $u \neq 0$ and

$$\limsup_{|u|\to\infty} |\varphi(u)| < 1$$

by the Riemann-Lebesgue lemma. So, since $\vartheta(u) \in L_1$, we have

$$W_n \leqq c_4(k) \binom{n}{k+1} A_n \lambda^n + \frac{c_2}{\pi} \binom{n}{k+1} A_n \int_{-d}^{d} u^{1+(k+1)(2+\delta)} e^{-(n-k-1)\alpha u^2} du,$$

$$(3.16a)\quad W_n \leqq c_5(k) n^{k+1} A_n \lambda^n + c_6(k) A_n n^{-1-\frac{1}{2}\delta(k+1)}, \qquad n \geqq k+2.$$

From (3.12), (3.15) and (3.16a)

$$(3.17)\quad T_2(n) \leqq c_1(k) n^{-\frac{1}{2}-\vartheta} + c_5(k) n^{k+1} A_n \lambda^n + c_6(k) A_n n^{-1-\frac{1}{2}\delta(k+1)}, \qquad n \geqq k+2.$$

We may take $A_n = n^\rho$ with ρ so large that by (3.9)

$$(3.18)\qquad T_1(n) \leqq c_7 n^{-2}, \qquad n = 1, 2, \ldots.$$

Then we may take k so large that $1+\frac{1}{2}\delta(k+1)-\rho \geqq 2$, say, so that from (3.17)

$$T_2(n) \leqq c_8 n^{-\frac{1}{2}-\vartheta}, \qquad n = 1, 2, \ldots. \tag{3.19}$$

From (3.7), (3.18) and (3.19) it follows now that

$$||(I-U_a)P^nQ-(I-U_a)N^nQ|| \leqq c_9 n^{-\frac{1}{2}-\vartheta}, \quad n = 1, 2, \ldots. \tag{3.20}$$

Since P^m has an absolutely continuous component, P^m contains P_1 with probability density belonging to L_2. By estimates analogous to (3.7), (3.9) and (3.10) with $A_n = n$, making use of the fact that $1-\vartheta(u) \sim \frac{1}{2}\tau^2 u^2$ for $u \to 0$, we may show that

$$||(I-U_a)P_1^nQ-(I-U_a)P_1^n|| = ||(I-U_a)(I-Q)P_1^n|| \leqq c_{10} n^{-1}, \qquad n = 1, 2, \ldots$$

so that by theorem 1 with $M = (I-U_a)(I-Q)$

$$||(I-U_a)P^nQ-(I-U_a)P^n|| \leqq c_{11} n^{-1}, \quad n = 1, 2, \ldots. \tag{3.21}$$

In the same way

$$||(I-U_a)N^nQ-(I-U_a)N^n|| \leqq c_{12} n^{-1}, \quad n = 1, 2, \ldots. \tag{3.22}$$

Finally

$$||(I-U_a)N^n|| = 2\int_{-a/2}^{a/2} (2\pi n\sigma^2)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}n^{-1}\sigma^{-2}x^2\right)dx,$$

$$\left|\,||(I-U_a)N^n||-2a(2\pi n\sigma^2)^{-\frac{1}{2}}\right| \leqq c_{13} n^{-\frac{3}{2}}, \quad n = 1, 2, \ldots. \tag{3.23}$$

Our assertion now follows from (1.1), (3.20), (3.21), (3.22) and (3.23).

The lattice versions of lemma 3 and theorem 3 are as follows.

Lemma 4. *Let P, Q and R be lattice distributions restricted to the integers, R being nondegenerate with span* 1. *If P and Q have the same first and second moments and have finite absolute moments of order $2+\delta$ for some δ with $0 < \delta \leqq 1$,*

$$||(P-Q)R^n|| \leqq c(P, Q, R, \vartheta)n^{-1-\vartheta}, \qquad n = 1, 2, \ldots$$

for every $\vartheta \in (0, \frac{1}{2}\delta)$,

Proof. First assume that $R = \frac{1}{4}U_{-1}+\frac{1}{2}U_0+\frac{1}{4}U_1$. Let $p_k^{(n)}$ and $q_k^{(n)}$ be the probabilities of PR^n and QR^n, respectively. Then

$$||(P-Q)R^n|| \leqq T_1(n)+T_2(n), \qquad n = 1, 2, \ldots. \tag{3.24}$$

$$T_1(n) = \sum_{|k| \geqq 2M_n} \{p_k^{(n)}+q_k^{(n)}\},$$

$$T_2(n) = \sum_{k=-\infty}^{+\infty} c_{nk}\{p_k^{(n)}-q_k^{(n)}\},$$

where

$$c_{nk} = 0,\ |k| \geqq 2M_n,\ |c_{nk}| = 1, \qquad k < 2M_n. \tag{3.25}$$

and

$$M_n - 1 < n^{\frac{1}{2}+\frac{1}{2}\delta-\vartheta} \leqq M_n, \qquad n = 1, 2, \ldots. \tag{3.26}$$

In the same way as (3.6) we derive

$$T_1(n) \leqq M_n^{-2-\delta}\left\{\int |x|^{2+\delta} dP + \int |x|^{2+\delta} dQ\right\} + 4 \sum_{k=n+M_n}^{2n} \binom{2n}{k} 2^{-2n},$$
$$n = 1, 2, \ldots.$$

By the estimate for the tail of a binomial distribution, given in Feller [1], Ch. VI.3:

$$\sum_{k=r}^{N} \binom{N}{k} p^k q^{n-k} \leqq \binom{N}{r} p^r q^{N-r} \frac{(r+1)q}{r+1-(N+1)p}, \quad r \geqq Np,$$

and by Stirling's formula and (3.26) it may be shown that

$$T_1(n) \leqq c_1 M_n^{-2-\delta} + c_2(\alpha) \frac{n^{\frac{1}{2}}(n+M_n+1)}{(M_n+\frac{1}{2})(n^2-M_n^2)^{\frac{1}{2}}} \exp(-\alpha M_n^2 n^{-1}), \tag{3.27}$$
$$n = 1, 2, \ldots.$$

for any $\alpha \in (0, 1)$.

By Parseval's relation

$$T_2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\gamma_n(u)}\, \{\varphi(u) - \psi(u)\} \chi^n(u) du,$$

where φ, ψ and χ are the characteristic functions of P, Q and R, respectively, and

$$\gamma_n(u) \overset{df}{=} \sum_k c_{nk}\, e^{iuk}. \tag{3.28}$$

From (3.25) and (3.28)

$$|\gamma_n(u)| \leqq 2M_n, \qquad n = 1, 2, \ldots.$$

Our assumptions on the moments of P and Q imply the existence of $d_1 > 0$ with

$$|\varphi(u) - \psi(u)| \leqq c_3 |u|^{2+\delta}, \qquad |u| \leqq d_1.$$

(Loève [4], § 12.4, p. 199). Since $\chi(u) = \cos^2 \frac{1}{2}u$,

$$|\chi(u)| \leqq \exp(-\alpha u^2), \qquad |u| \leqq d_2$$

with $\alpha > 0$, $d_2 > 0$, and

$$|\chi(u)| \leqq \lambda < 1, \qquad d_2 \leqq |u| \leqq \pi$$

Therefore, with $d = \min(d_1, d_2)$

$$T_2(n) \leqq c_4 M_n \lambda^n + c_5 M_n \int_0^d u^{2+\delta} \exp(-n\alpha u^2) du,$$

$$T_2(n) \leqq c_4 M_n \lambda^n + c_6 M_n n^{-\frac{3}{2}-\frac{1}{2}\delta}, \qquad n = 1, 2, \ldots. \tag{3.29}$$

For $R = \frac{1}{4}U_{-1} + \frac{1}{2}U_0 + \frac{1}{4}U_1$ the lemma now follows from (3.24), (3.27), (3.29) with (3.26). To prove the general case we note that, since R has span 1, there is m such that R^m contains $U_h(\frac{1}{4}U_{-1} + \frac{1}{2}U_0 + \frac{1}{4}U_1)$ for some integer h. The lemma then follows from theorem 1 and what was shown above.

Theorem 4. *If P is a lattice distribution with span c, and the absolute moment of order $2+\delta$ of P is finite for some $\delta \in (0, 1]$,*

$$|\, ||P^n - U_{jc} P^n|| - 2jc(2\pi n\sigma^2)^{-\frac{1}{2}}| \leqq b(P, \vartheta, j) n^{-\frac{1}{2}-\vartheta},$$
$$j = 1, 2, \ldots, \; n = 1, 2, \ldots.$$

Here σ^2 denotes the variance of P.

Proof. We assume $c = 1$, to which the general case is reduced easily. To find a suitable probability measure that plays the same rôle as N in the proof of theorem 3, we choose the integer m so that $m\sigma^2 \geqq 2$ and write $Q = U_b P^m$, where b is chosen so that Q is restricted to the integers and

$$0 < \mu_Q \stackrel{df}{=} \int x dQ(x) \leqq 1.$$

The parameters $\rho > 0$ and $p \in (0, 1)$ of a negative binomial distribution B now may be determined in such a way that B has the same first moment and variance as Q, the pertinent equations being

$$\rho p^{-1} q = \mu_Q, \quad \rho p^{-2} q = m\sigma^2, \tag{3.30}$$

where $q \stackrel{df}{=} 1 - p$.

Now, $a_k^{(h)}$ and $b_k^{(h)}$ denoting the probabilities of Q^h and B^h, respectively,

$$||(I - U_j)Q^h - (I - U_j)B^h|| \leqq T_1(h) + T_2(h), \tag{3.31}$$

$$T_1(h) = 2 \sum_{|k| \geqq M_h - j} \{a_k^{(h)} + b_k^{(h)}\},$$

$$T_2(h) = \sum_{k=-\infty}^{+\infty} \{a_k^{(h)} - a_{k-j}^{(h)} - b_k^{(h)} + b_{k-j}^{(h)}\} c_{hk},$$

with

$$c_{hk} = 0, \; k \geqq M_h, \; |c_{hk}| = 1, \; k < M_h. \tag{3.32}$$

By Chebychev's inequality

(3.33) $$T_1(h) \leqq 4mh\sigma^2(M_h-j)^{-2}, \quad M_h > j.$$

By Parseval's relation

$$T_2(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\gamma_h(u)}\,(1-e^{iuj})\{\psi^h(u)-\beta^h(u)\}du,$$

where $\gamma_h(u) \overset{df}{=} \sum_k c_{hk} \exp(iuk)$ and ψ and β are the characteristic functions of Q and B, respectively. In the same way as (3.17) was derived from (3.10), it may be shown that

(3.34) $$T_2(h) \leqq c_1(k)h^{-\frac{1}{2}-\vartheta}+c_2(k)h^{k+1}M_h\lambda^h+c_3(k)M_h h^{-1-\frac{1}{2}(k+1)\delta}, \quad h \geqq k+2,$$

with $0 \leqq \lambda < 1$. Instead of lemma 3 now lemma 4 should be used, whereas the relation $|\psi(u)| \leqq \lambda < 1$, $d \leqq |u| \leqq \pi$ follows from the fact that Q has span 1; see Gnedenko and Kolmogorov [2], § 14, corollary 2 to theorem 5. From (3.31), (3.33) and (3.34) by taking $M_h = [h^\alpha]$ with α sufficiently large, and then taking k sufficiently large,

(3.35) $$||(I-U_j)Q^h-(I-U_j)B^h|| \leqq c_4 h^{-\frac{1}{2}-\vartheta}, \qquad h = 1, 2, \ldots.$$

The probabilities

(3.36) $$b_k^{(h)} = \binom{-h\rho}{k} p^{h\rho}(-q)^k, \qquad k = 0, 1, 2, \ldots$$

of B^h satisfy

$$b_k^{(h)} \geqq b_{k-1}^{(h)}, \qquad 1 \leqq k \leqq h\rho q p^{-1}-pq^{-1},$$
$$b_k^{(h)} < b_{k-1}^{(h)}, \qquad k > h\rho q p^{-1}-qp^{-1}.$$

Therefore

(3.37) $$||B^h-U_jB^h|| = 2\{b_{k_0-j+1}^{(h)}+\ldots+b_{k_0}^{(h)}\}, \qquad h \geqq h_1,$$

with

(3.38) $$|k_0-h\rho q p^{-1}| \leqq qp^{-1}+j+1.$$

By evaluating the terms in the right-hand side of (3.37) with Stirling's formula, using (3.38), we may show that

$$\left|\,||B^h-U_jB^h||-2j(2\pi h\rho q p^{-2})^{-\frac{1}{2}}\right| \leqq c_5 h^{-\frac{3}{2}},$$

so with (3.30)

(3.39) $$\left|\,||B^h-U_jB^h||-2j(2\pi h m\sigma^2)^{-\frac{1}{2}}\right| \leqq c_5 h^{-\frac{3}{2}}, \quad h = 1, 2, \ldots.$$

From (3.35) and (3.39)

$$(3.40)\qquad \big|\,||Q^h-U_jQ^h||-2j(2\pi hm\sigma^2)^{-\frac{1}{2}}\big| \leqq c_6 h^{-\frac{1}{2}-\vartheta}, \quad h=1,2,\ldots.$$

Finally, let $n = hm+r$, h integer, $0 \leqq r < m$. Then

$$(3.41)\quad ||P^n-U_jP^n||=||U_{-hb}P^r(Q^h-U_jQ^h)||=||P^r(Q^h-U_jQ^h)||.$$

Denoting by $p_k^{(r)}$ the probabilities of P^r we have

$$||Q^h-P^rQ^h|| = \sum_k |\sum_i p_i^{(r)}\{a_k^{(h)}-a_{k-i}^{(h)}\}| \leqq \sum_i p_i^{(r)}||Q^h-U_iQ^h||.$$

Since Q has span 1, the set L_0 for Q contains 1 by I, theorem 3, so that $||Q^h-U_1Q^h|| \leqq \alpha h^{-\frac{1}{2}}$ by theorem 2 and $||Q^h-U_iQ^h||\leqq\alpha|i|h^{-\frac{1}{2}}$ by (1.1). Therefore

$$(3.42)\qquad ||Q^h-P^rQ^h|| \leqq c_7 h^{-\frac{1}{2}}, \qquad h=1,2,\ldots,$$

and with (1.2)

$$(3.43)\quad ||(I-U_j)P^rQ^h-(I-U_j)Q^h|| = ||(I-U_j)(I-P^r)Q^h|| \leqq c_8h^{-1}, \quad h=1,2,\ldots.$$

The theorem follows from (3.41), (3.40) and (3.43).

THEOREM 5. *If P is absolutely continuous with finite variance σ^2 and P^n is unimodal,* $n=1,2,\ldots,$

$$\lim_{n\to\infty} n^{\frac{1}{2}}\,||P^n-U_aP^n|| = 2a(2\pi\sigma^2)^{-\frac{1}{2}}, \qquad a>0.$$

PROOF. It is no restriction to assume that P has zero first moment. Let $p^{(n)}$ denote the density of P^n. The unimodality of P^n implies the existence of b_n with

$$p^{(n)}(x)-p^{(n)}(x-a) \geqq 0, \qquad x<b_n,$$
$$p^{(n)}(x)-p^{(n)}(x-a) \leqq 0, \qquad x>b_n,$$

so that

$$(3.44)\qquad ||P^n-U_aP^n|| = 2\int_{b_n-a}^{b_n} p^{(n)}(x)dx, \quad n=1,2,\ldots.$$

Let Q be any probability measure with finite absolute first moment and characteristic function belonging to L_1 and let $q^{(n)}$ be the density of P^nQ. Then by (3.44)

$$|\,||P^n-U_aP^n||-2\int_{b_n-a}^{b_n} q^{(n)}(x)dx\,| = 2|\int_{b_n-a}^{b_n}\{p^{(n)}(x)-q^{(n)}(x)\}dx|$$
$$= |\int_{-\infty}^{+\infty} f_n(x)\{p^{(n)}(x)-p^{(n)}(x-a)-q^{(n)}(x)+q^{(n)}(x-a)\}dx|,$$

where $f_n(x) = 1$, $x \leqq b_n$, $f_n(x) = -1$, $x > b_n$, so that

(3.45) $$\left| \|P^n - U_a P^n\| - 2\int_{b_n-a}^{b_n} q^{(n)}(x)dx \right| \leqq \|(I-U_a)(I-Q)P^n\|.$$

Now P contains a uniform distribution P_0. By (3.44) applied to P_0, since all P_0^n are unimodal (cf. Wintner [9], pp. 30, 32),

$$\|P_0^n - U_a P_0^n\| \leqq |a| \sup_x p_0^{(n)}(x),$$

$p_0^{(n)}(x)$ denoting the density of P_0^n. By writing $p_0^{(n)}(x)$ as a Fourier integral and noting that the characteristic function of P_0 for $u \to 0$ behaves as $\exp(-\alpha u^2)$ for some $\alpha > 0$, it may be shown that

(3.45a) $$\|P_0^n - U_a P_0^n\| \leqq c_0 |a| n^{-\frac{1}{2}}, \qquad n = 1, 2, \ldots.$$

Then by a derivation analogous to that leading to (3.42)

$$\|(I-Q)P_0^n\| \leqq c_2 n^{-\frac{1}{2}}, \qquad n = 1, 2, \ldots,$$

so that $\|(I-Q)P^n\| \leqq c_3 n^{-\frac{1}{2}}$ by theorem 1. Therefore, from (3.45) and (1.2)

(3.46) $$\left| \|P^n - U_a P^n\| - 2\int_{b_n-a}^{b_n} q^{(n)}(x)dx \right| \leqq c_4(a) n^{-1}, \quad n = 1, 2, \ldots.$$

We have

(3.47) $$2\int_{b_n-a}^{b_n} q^{(n)}(x)dx = \pi^{-1}\int_{-\infty}^{+\infty} e^{-iub_n} \frac{e^{iua}-1}{iu} \varphi^n(u)\psi(u)du,$$

where $\psi \in L_1$ and φ are the characteristic functions of Q and P. To every $\varepsilon \in (0, 1)$ there is $d = d(\varepsilon)$ with

$$|\varphi(u)| \leqq 1 - \tfrac{1}{2}(1-\varepsilon)\sigma^2 u^2 \leqq \exp\{-\tfrac{1}{2}(1-\varepsilon)\sigma^2 u^2\}, \quad |u| \leqq d,$$
$$|\varphi(u)| \leqq \lambda(\varepsilon) < 1, \qquad |u| > d.$$

Applying these relations to (3.47) we find

$$2\int_{b_n-a}^{b_n} q^{(n)}(x)dx \leqq 2a(1-\varepsilon)^{-\frac{1}{2}}(2\pi n\sigma^2)^{-\frac{1}{2}} + c_5\lambda^n(\varepsilon), \quad n = 1, 2, \ldots.$$

Since this holds for every $\varepsilon \in (0, 1)$, it follows from (3.46) that

(3.48) $$\limsup_{n\to\infty} n^{\frac{1}{2}} \|P^n - U_a P^n\| \leqq 2a(2\pi\sigma^2)^{-\frac{1}{2}}.$$

On the other hand

$$\|P^n - U_a P^n\| \geqq \left| \int g(x)\{p^{(n)}(x) - p^{(n)}(x-a)\}dx \right|,$$

with $g_n(x) = 1$, $x \leqq 0$, $g_n(x) = -1$, $x > 0$,

$$||P^n - U_a P^n|| \geqq 2 \int_{-a}^{0} p^{(n)}(x)dx, \qquad n = 1, 2, \ldots.$$

It was shown by Shepp [6], that

$$\lim_{n\to\infty} n^{\frac{1}{2}} \int_{-a}^{0} p^{(n)}(x)dx = a(2\pi\sigma^2)^{-\frac{1}{2}}, \tag{3.49}$$

so that

$$\liminf_{n\to\infty} n^{\frac{1}{2}} \, ||P^n - U_a P^n|| \geqq 2a(2\pi\sigma^2)^{-\frac{1}{2}}. \tag{3.50}$$

The theorem follows from (3.48) and (3.50).

THEOREM 6. *If P is a lattice distribution with span c and finite variance σ^2, and P^n is unimodal, $n = 1, 2, \ldots$,*

$$\lim_{n\to\infty} n^{\frac{1}{2}} \, ||P^n - U_{jc} P^n|| = 2jc(2\pi\sigma^2)^{-\frac{1}{2}}, \quad j = 1, 2, \ldots.$$

PROOF. The proof is analogous to that of theorem 5. Taking $c = 1$ we have for some l_n

$$\begin{aligned} ||P^n - U_j P^n|| &= 2\{p_{l_n}^{(n)} + p_{l_n-1}^{(n)} + \ldots + p_{l_n-j+1}^{(n)}\} \\ &= \pi^{-1} \sum_{h=0}^{j-1} \int_{-\pi}^{\pi} \varphi^n(u) \exp\{-iu(l_n - h)\} du, \end{aligned} \tag{3.51}$$

φ being the characteristic function of P and $p_k^{(n)}$, $k = \ldots -1, 0, 1, \ldots$ the probabilities of P^n. The right-hand side now may be majorized without the intervention of an extra distribution Q. The lattice analogon of (3.49) also is due to Shepp [6].

4. Convergence faster than $n^{-\frac{1}{2}}$.

THEOREM 7. *Let h be a continuous and strictly increasing function on $[1, \infty]$ with*

$$h(1) = 1, \tag{4.1}$$

$$\lim_{x\to\infty} h(x) = +\infty \tag{4.2}$$

$$\lim_{x\to\infty} h(x)/h(x+1) = 1. \tag{4.3}$$

whereas the inverse function h^{-1} on $[1, \infty)$ satisfies

$$\sup_x h^{-1}(\alpha x)/h^{-1}(x) = \lambda(\alpha) < \infty, \quad \alpha \geqq 1. \tag{4.4}$$

Then a necessary condition that

$$h(n)\|RP^n - U_a RP^n\| \leqq c < \infty, \quad n = 1, 2, \ldots, \tag{4.5}$$

for some R *and some* $a > 0$, *is the existence of* $\gamma > 0$, $\delta > 0$ *with*

$$|\varphi(u)| \leqq 1 - \gamma g(|u|), \quad -\delta \leqq u < \delta. \tag{4.6}$$

Here φ *is the characteristic function of* P *and*

$$g(x) \overset{df}{=} 1/h^{-1}\left(\frac{1}{x}\right), \quad 0 < x \leqq 1. \tag{4.7}$$

PROOF. We have to show that under (4.5)

$$\liminf_{u \to 0} \{1 - |\varphi(u)|\}/g(|u|) > 0.$$

If this were not true, there would be a sequence $u_k \downarrow 0$ with

$$1 - |\varphi(u_k)| = \varepsilon_k g(u_k) \tag{4.8}$$

and $\varepsilon_k \to 0$, since $|\varphi(u)|$ is even. We may assume $0 < u_k \leqq 1$, $0 \leqq \varepsilon_k < 1$, $k = 1, 2, \ldots$.

Take A and B fixed with

$$c/a < A < B < \infty, \quad A \geqq 1. \tag{4.9}$$

There is k_0 and a sequence $\{n_k\}$ of integers, such that $n_k \to +\infty$ and

$$u_k = A_k/h(n_k), \quad k \geqq k_0, \tag{4.10a}$$

$$A \leqq A_k \leqq B, \quad k \geqq k_0. \tag{4.10b}$$

To prove this, we note that there is k_0' with

$$h(1)u_k < A, \qquad k \geqq k_0'.$$

To every $k \geqq k_0'$ there is a single integer $n_k \geqq 2$ with

$$h(n_k - 1)u_k < A \leqq h(n_k)u_k. \tag{4.11}$$

Since $u_k \downarrow 0$, we have $n_k \uparrow \infty$. From (4.11)

$$h(n_k)u_k - A < \{h(n_k) - h(n_k - 1)\}u_k < \left\{\frac{h(n_k)}{h(n_k - 1)} - 1\right\} A. \tag{4.12}$$

By (4.3), as $n_k \to \infty$, there is k_0'' with

$$\left\{\frac{h(n_k)}{h(n_k - 1)} - 1\right\} A \leqq B - A, \quad k \geqq k_0''. \tag{4.13}$$

So from (4.11), (4.12) and (4.13)

$$A \leqq h(n_k)u_k \leqq B, \qquad k \geqq k_0,$$

with $k_0 = \max(k_0', k_0'')$, wich proves (4.10).

Now, if ψ denotes the characteristic function of R, it follows from (4.8) and (4.10a) that for $k \geqq k_0$

$$h(n_k)\,||RP^{n_k}-U_aRP^{n_k}|| \geqq h(n_k)|1-e^{iu_ka}|\;|\varphi(u_k)|^{n_k}|\psi(u_k)|$$
$$= h(n_k)\,|1-\exp\{ia\,A_k/h(n_k)\}\,|[1-\varepsilon_k g\{A_k/h(n_k)\}]^{n_k}\,|\psi(u_k)|.$$

From (4.7) and (4.4), since $A_k \geqq 1$,

$$h(n_k)\,||P^{n_k}-U_aP^{n_k}|| \geqq h(n_k)\,|1-\exp\{iaA_k/h(n_k)|$$
$$\{1-\varepsilon_k\lambda(A_k)n_k^{-1}\}^{n_k}|\psi(u_k)|.$$

So, since $A_k \geqq A$, $\psi(u_k) \to 1$ and $\lambda(A_k) \leqq c_0A_k^\rho \leqq c_0B^\rho$ by lemma 1, we have, taking into account (4.9),

$$\limsup_{k\to\infty} h(n_k)\,||P^{n_k}-U_aP^{n_k}|| \geqq aA > c,$$

in contradiction with (4.5).

An important special case is $h(x) = x^{1/\alpha}$ with $0 < \alpha \leqq 2$. It follows from theorem 7 that

$$(4.14) \qquad |\varphi(u)| \leqq 1-\gamma|u|^\alpha, \qquad |u| \leqq \delta,$$

with $\gamma > 0$, $\delta > 0$ is a necessary condition that

$$||RP^n-U_aRP^n|| \leqq cn^{-1/\alpha}.$$

So if P belongs to the domain of normal attraction of a stable law of order α or if P has a finite absolute moment of order α, the convergence in (1.4) and (1.5) can be no faster than of order $n^{-1/\alpha}$.

We note that the convergence may be faster than any power of n^{-1}. As an example we take $P = NQ_0$, where N is any gaussian probability distribution and Q_0 has characteristic function f with

$$\log f(u) = \int_{-\infty}^{+\infty}(e^{iux}-1)q(x)dx,$$
$$q(x) = \frac{1}{4e},\ |x| \leqq e,\ q(x)=\{4\,|x|\log^2|x|\}^{-1},\ |x| > e.$$

Q_0 is a variant of the probability distribution belonging to no domain of proper partial attraction, given in § 37 of Gnedenko and Kolmogorov [2]. Their estimate of the characteristic function can be taken over immediately, giving

$$(4.15) \qquad |f(u)| \leqq \exp\left\{\frac{1}{10\log|u|}\right\}, \ |u| \leqq \delta.$$

Since the measure with density function q is symmetric and unimodal, so are its convolutions (Wintner [9], pp. 30.32) and so are all P^n. By relations analogous to (3.44) and (3.47), using (4.15), one sees that $||P^n - U_a P^n||$ converges to zero at least of order $\exp(-cn^{\frac{1}{2}})$ with $c > 0$.

The author conjectures that (4.14) in many cases is sufficient to have convergence of order $n^{-1/\alpha}$ in (1.4) but he succeeded only in obtaining weaker results, theorems 8 and 9 below. The order of convergence in (1.5) is considered in section 5.

For the proofs of theorem 8 and 9 the following lemma is needed.

LEMMA 5. *Let Q be a symmetric stable probability distribution with characteristic function* $\exp(-\beta|u|^\alpha)$, *where* $\beta > 0$, $0 < \alpha < 2$. Then

$$(4.16) \quad ||Q^n - U_a Q^n|| \leqq c_1 |a| n^{-1/\alpha}, \quad -\infty < a < \infty, \quad n = 1, 2, \ldots,$$

$$(4.17) \qquad ||(I-Q)Q^n|| \leqq c_2 n^{-1}, \qquad n = 1, 2, \ldots.$$

PROOF. Let q and $q^{(n)}$ be the densities of Q and Q^n, respectively. We have

$$(4.18) \qquad q^{(n)}(x) = n^{-1/\alpha} q(n^{-1/\alpha} x).$$

A symmetric stable distribution is unimodal. We refer to Gnedenko and Kolmogorov [2], appendix 2. The unimodality also follows from the relation

$$q(x) = \int (2\pi\sigma^2 t)^{-\frac{1}{2}} \exp(-x^2/2\sigma^2 t) g(t) dt$$

where g is a probability density (see Bochner [10], section 4.3; Stam [11] section 7). So

$$||Q^n - U_a Q^n|| = 2\int_{-\frac{1}{2}|a|}^{\frac{1}{2}|a|} q^{(n)}(x) dx \leqq 2\,|a| q^{(n)}(0) = 2\,|a| n^{-1/\alpha} q(0).$$

To prove (4.17) we start from (4.18):

$$\begin{aligned} ||Q^n - Q^{n+1}|| &= \int |n^{-1/\alpha} q(xn^{-1/\alpha}) - (n+1)^{-1/\alpha} q\{x(n+1)^{-1/\alpha}\}| dx \\ &\leqq \int |r_n - 1| q(r_n y) dy + \int |q(r_n y) - q(y)| dy, \end{aligned}$$

where $r_n = (1+n^{-1})^{1/\alpha}$. Since Q is unimodal and symmetric,

$$|q(r_n y) - q(y)| = q(y) - q(r_n y),$$
$$||Q^n - Q^{n+1}|| \leqq 2 - 2r_n^{-1},$$

which proves (4.17).

THEOREM 8. *Let P^m for some m have an absolutely continous component and let the absolute moment of P of some positive order ρ be finite. Moreover, let the characteristic function φ of P satisfy*

$$|\varphi(u)| \leqq 1-\gamma|u|^\delta, \qquad -\varepsilon < u < \varepsilon, \tag{4.19}$$

with $\gamma > 0$, $\varepsilon > 0$ and $0 < \delta < 2$. Then

$$||P^n-U_aP^n|| \leqq c(a, P, r)n^{-r}, \quad n = 1, 2, \ldots, \quad a > 0, \tag{4.20}$$

for every $r < \delta^{-1}$.

PROOF. First assume that P is absolutely continuous with density in L_2. Then the density $p^{(n)}$ of P^n also belongs to L_2. We have

$$\begin{aligned} ||P^n-U_aP^n|| &\leqq T_1(n)+T_2(n), \\ T_1(n) &= 2\int_{|x|\geqq A_n-a} p^{(n)}(x)dx, \\ T_2(n) &= \int_{-\infty}^{+\infty} g_n(x)\{p^{(n)}(x)-p^{(n)}(x-a)\}dx, \end{aligned} \tag{4.21}$$

with

$$g_n(x) = 0, \ |x| > A_n, \ |g_n(x)| = 1, \ |x| \leqq A_n. \tag{4.22}$$

By Markov's inequality, since it is no restriction to assume $\rho \leqq 1$, so that

$$\int |x|^\rho dP^n(x) \leqq n\int |x|^\rho dP(x),$$

$$T_1(n) \leqq c_1 n(A_n-a)^{-\rho}, \qquad n = 1, 2, \ldots. \tag{4.23}$$

With Parseval's formula

$$T_2(n)=(2\pi)^{-1}\int \overline{\gamma_n(u)}(1-e^{iua})\varphi^n(u)du,$$

where γ_n is the Fourier transform of g_n, so that from (4.22)

$$|\gamma_n(u)| \leqq 2A_n. \tag{4.24}$$

From (3.11) with $y = \varphi(u)$, $x = \varphi(u)s(u)$, where $s(u)=\exp(-\beta|u|^\alpha)$ is the characteristic function of a symmetric stable distribution Q of order $\alpha > \delta$,

$$T_2(n) \leqq V(n)+W(n), \qquad n \geqq k+2, \tag{4.25}$$

$$V(n)= (2\pi)^{-1}\sum_{j=0}^{k} \binom{n}{j}\left|\int \overline{\gamma_n(u)}(1-e^{iua})\{1-s(u)\}^j\varphi^n(u)s^{n-j}(u)du\right|,$$

$$W(n)=(2\pi)^{-1}\sum_{l=0}^{n-k-1}\binom{k+l}{l}\left|\int\overline{\gamma_n(u)}(1-e^{iua})\{1-s(u)\}^{k+1}\varphi^n(u)s^l(u)du\right|.$$

From Parseval's formula and (4.22)

$$V(n) \leqq \sum_{j=0}^{k} \binom{n}{j} \,||(I-U_a)(I-Q)^j P^n Q^{n-j}||,$$

so that by (1.2), (4.16) and (4.17)

$$V(n) \leqq c_2 n^{-1/\alpha}, \qquad n = 1, 2, \ldots. \tag{4.26}$$

With (4.24), (3.16), the estimates

$$|1-s(u)| = |1-\exp(-\beta|u|^\alpha)| \leqq \beta|u|^\alpha,$$
$$|\varphi(u)| \leqq 1-\gamma|u|^\delta \leqq \exp(-\gamma|u|^\delta), \ |u| \leqq \varepsilon,$$

and the fact that $\varphi \in L_2$ and $\lim_{|u|\to\infty}|\varphi(u)| = 0$,

$$W(n) \leqq c_3(k)A_n n^{k+1}\lambda^n + c_4(k)A_n n^{k+1-(2+\alpha k+\alpha)/\delta}, \tag{4.27}$$

with $0 \leqq \lambda < 1$.

Putting $\alpha = 1/r = \delta(1+\eta)$, where $\eta > 0$, we have from (4.21), (4.23), (4.25), (4.26) and (4.27)

$$||P^n - U_a P^n|| \leqq c_1 n(A_n-a)^{-\rho} + c_2 n^{-r} + c_3(k)A_n n^{k+1}\lambda^n$$
$$+ c_4(k)A_n n^{-2/\delta-\eta(k+1)}, \qquad n = 1, 2, \ldots.$$

By taking $A_n = n^t$ with $1-\rho t \leqq -r$ and then k so large that $t-2/\delta-\eta(k+1) \leqq -r$, the relation (4.20) follows.

Finally, if P^m has an absolutely continuous component, P^m contains P_0 absolutely continuous with density in L_2. Then P^{2m} contains P_0P^m, the density of P_0P^m belongs to L_2 and the characteristic function of P_0P^m satisfies (4.19). The relation (4.20) now follows with theorem 1.

THEOREM 9. *Let P be a lattice distribution with span c, having finite absolute moment of some positive order ρ. Moreover, let the characteristic function φ of P satisfy* (4.19). *Then*

$$||P^n - U_{hc}P^n|| \leqq hc(P, r)n^{-r}, \ h = 1, 2, \ldots, \ n = 1, 2, \ldots, \tag{4.28}$$

for every $r < \delta^{-1}$.

PROOF. It is no restriction to assume that P is restricted to the integers and has span 1. By (1.1) it is sufficient to prove (4.28) for $h = 1$.

If the probabilities of P^n are denoted by $p_k^{(n)}$,

$$||P^n - U_1P^n|| \leqq T_1(n) + T_2(n), \tag{4.29}$$
$$T_1(n) = 2 \sum_{|k|\geqq M_n} p_k^{(n)},$$
$$T_2(n) = \sum_{k=-\infty} c_{nk}(p_k^{(n)} - p_{k-1}^{(n)}),$$

with

(4.30) $c_{nk} = 0,\ k \geqq M_n,\ |c_{nk}| = 1,\ |k| < M_n.$

From Markov's inequality, in the same way as (4.23),

(4.31) $T_1(n) \leqq c_1 n M_n^{-\rho}, \qquad n = 1, 2, \ldots.$

With Parseval's formula

$$T_2(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} \overline{\chi_n(u)}\,(1-e^{iu})\varphi^n(u)du$$

where

$$\chi_n(u) = \sum_k c_{nk} e^{iuk}.$$

From (3.11) with $y = \varphi(u)$, $x = \varphi(u)s(u)$, where $s(u) = \exp(-\beta|u|^\alpha)$ is the characteristic function of a stable distribution Q of order $\alpha = r^{-1}$,

(4.32) $T_2(n) \leqq V(n) + W(n), \qquad n \geqq k+2,$

$$V(n) = (2\pi)^{-1} \sum_{j=0}^{k} \binom{n}{j} \left| \int_{-\pi}^{\pi} \overline{\chi_n(u)}\,(1-e^{iu})\{1-s(u)\}^j \varphi^n(u) s^{n-j}(u)du \right|,$$

$$W(n) = (2\pi)^{-1} \sum_{l=0}^{n-k-1} \binom{k+l}{l}$$
$$\left| \int_{-\pi}^{\pi} \overline{\chi_n(u)}\,(1-e^{iu})\{1-s(u)\}^{k+1} \varphi^n(u) s^l(u)du \right|.$$

Since $|\chi_n(u)| \leqq 2M_n$ by (4.30),

$$V(n) \leqq c_2(k) n^k M_n \lambda^n$$
$$+\ (2\pi)^{-1} \sum_{j=0}^{k} \binom{n}{j} \left| \int_{-\infty}^{\infty} \overline{\chi_n(u)\frac{e^{iu}-1}{iu}}\,\{1-s(u)\}^j \varphi^n(u)(-iu)s^{n-j}(u)du \right|,$$

with $0 \leqq \lambda < 1$.

Now $\chi_n(u)(1-e^{iu})/iu$ is the Fourier transform of the function $g_n(x) = \sum_k c_{nk} f(x-k)$, where f is the indicator function of the interval $[-1, 0]$. So with Parseval's formula, since $|g_n(x)| \leqq 1$ by (4.30)

$$V(n) \leqq c_2(k) n^k M_n \lambda^n + \sum_{j=0}^{k} \binom{n}{j} ||M_{n,j}(I-Q)^j P^n Q^{n-j-m(j)}||,$$

where

$$m(j) = [n/(j+1)],\ j \neq 0,\ m(0) = n,$$

and $M_{n,j}$ is the signed measure with density function $f_{n,j}$ equal to the derivative of the probability density $q_{n,j}$ of $Q^{m(j)}$. From the unimodality and symmetry of the Q^n (cf. the proof of lemma 5) and from (4.18)

$$(4.33) \qquad ||M_{n,j}|| = \int |f_{n,j}(x)|dx = -2\int_0^\infty f_{n,j}(x)dx$$
$$= 2q_{n,j}(0) \leqq c_3(k)n^{-1/\alpha}.$$

So from (1.2), (4.33) and (4.17), with $\alpha = r^{-1}$

$$(4.34) \qquad V(n) \leqq c_2(k)n^k M_n \lambda^n + c_4(k)n^{-r}, \qquad n = 1, 2, \ldots.$$

Furthermore with $0 \leqq \vartheta < 1$,

(4.35)

$$W(n) \leqq c_5(k)n^{k+1}M_n\vartheta^n + c_6(k)A_n n^{k+1-(2+\alpha k+\alpha)/\delta}, \quad n = 1, 2, \ldots.$$

This is proved in the same way as (4.27), except that the inequality $|\varphi(u)| \leqq \vartheta$, $\varepsilon \leqq |u| \leqq \pi$, now follows from the fact that P has span 1. (See Gnedenko and Kolmogorov [2], § 14, corollary 2 to theorem 5.)

The relation (4.28) follows from (4.29), (4.31), (4.32), (4.34), (4.35) and the relation $\alpha = r^{-1} > \delta$, by taking $M_n \sim n^t$ with $1-t\rho < -r$ and then k sufficiently large.

If P^n is unimodal, $n = 1, 2, \ldots$, theorems 8 (with absolutely continuous P) and 9 can be sharpened to $r = \delta^{-1}$. The proofs start from relations analogous to (3.44), (3.47) and (3.51). If P has a density not belonging to L_2, theorem 1 may be applied. We note that P contains a truncated gaussian distribution P_1. Then P^2 contains PP_1, the characteristic function of PP_1 satisfies (4.19) and belongs to L_2, and the $P^nP_1^n$ are unimodal by Ibragimov's theorem [3].

5. The relation (1.5)

First we intend to show that the convergence in (1.5) cannot be uniform with respect to absolutely continuous Q, unless $a \in L_0$, and that a similar conclusion holds for the order of convergence in (1.5).

THEOREM 10. *If the sequence Q_k converges completely (Loève [4], § 11.2) to U_0 and*

$$(5.1) \qquad \lim_{n\to\infty} ||P^nQ_k - U_a P^n Q_k|| = 0,$$

uniformly in k, then $\lim_{n\to\infty} ||P^n - U_a P^n|| = 0$.

PROOF. For any finite signed measure M on the Borel sets of the real line

$$||M|| = \sup |\int f(x)dM(x)|,$$

the supremun being taken over all uniformly continuous f on $(-\infty, +\infty)$ with $|f(x)| \leqq 1$, $-\infty < x < \infty$. So from (5.1), putting $P^n - U_a P^n = M_n$, we have for $n \geqq n(\varepsilon)$

$$(5.2) \qquad \int |f(x)dM_n Q_k(x)| = |\int \{f(x+y)dQ_k(y)\}dM_n(x)| < \varepsilon$$

for all k and all uniformly continuous f bounded by 1.

If any such f is kept fixed, then since

$$|f(x) - \int f(x+y)dQ_k(y)| \leqq \int_{-\delta}^{\delta} |f(x)-f(x+y)|dQ_k(y) + 2\int_{|x|>\delta} dQ_k(y),$$

we may choose δ so small and then $k = k_0$ so large that

$$(5.3) \qquad |f(x) - f(x+y)dQ_{k_0}| < \varepsilon, \qquad -\infty < x < \infty$$

So from (5.2) and (5.3) for $n \geqq n(\varepsilon)$

$$|\int f(x)dM_n(x)| < \varepsilon + \varepsilon ||M_n|| \leqq 3\varepsilon,$$

for every uniformly continuous f bounded by 1, and therefore also $||M_n|| < 3\varepsilon$ for $n \geqq n(\varepsilon)$.

THEOREM 11. *Let* $\{b_n\}$ *be a sequence of positive numbers with* $b_n \to \infty$. *If*

$$(5.4) \qquad b_n ||P^n Q - U_a P^n Q|| \leqq c(Q) < \infty, \qquad n = 1, 2, \ldots,$$

for every absolutely continuous Q, *then*

$$(5.5) \qquad b_n ||P^n - U_a P^n|| \leqq c < \infty, \qquad n = 1, 2, \ldots,$$

so that $a \in L_0$.

PROOF. Consider the Banach space χ of finite complex absolutely continuous measures on the Borel sets of the real line, with norm defined as total absolute variation. We may identify χ with L_1. Define the bounded linear operators T_n on χ into χ by

$$T_n(M) \stackrel{df}{=} b_n(P^n - U_a P^n)M, \qquad M \in \chi.$$

From (5.4), by decomposing M into positive and negative real and imaginary parts,

$$||T_n(M)|| \leqq c(M) < \infty, \quad n = 1, 2, \ldots, \quad M \in \chi.$$

So by the principle of uniform boundedness (Taylor [8], theorem 4.4-E)

$$||T_n|| \leqq c_0 < \infty, \qquad n = 1, 2, \ldots,$$

so that

$$(5.6) \qquad b_n||P^nQ - U_a P^nQ|| \leqq c_0||Q|| = c_0 < \infty, \quad n = 1, 2, \ldots,$$

for all absolutely continuous probability measures Q.

The relation (5.5) is derived from (5.6) in the same way as theorem 10 is proved, the principal difference being that ε in the counterpart of (5.3) should be replaced by $c_1 b_n^{-1}$, so that k_0 comes to depend on n.

Theorem 11 shows that for (1.5) there are no counterparts of theorems 3 and 8 if $a \notin L_0$, not even if a proportionality constant depending on Q is inserted. The following, however, can be said.

THEOREM 12. *Let P be nonlattice with finite absolute moment of order $2+\delta$ for some $\delta \in (0, 1]$ and with variance σ^2, and let $\mathscr{L}$ (a) denote the class of all absolutely continuous Q with*

$$\left| ||P^nQ - U_a P^nQ|| - 2|a|(2\pi n\sigma^2)^{-\frac{1}{2}} \right| \leqq c(P, Q, \vartheta) n^{-\frac{1}{2}-\vartheta}, \qquad n = 1, 2, \ldots,$$

for every $\vartheta(0, \frac{1}{2}\delta)$. Then $\mathscr{L}(a)$ contains a sequence converging completely (Loève [4], § 11.2) to U_0; in fact, $\mathscr{L}(a)$ contains all Q with characteristic functions vanishing outside finite intervals. If the characteristic function φ of P satisfies

$$\limsup_{|u| \to \infty} |\varphi(u)| < 1,$$

$\mathscr{L}(a)$ contains all Q with characteristic functions belonging to L_1.

PROOF. The proof is similar to that of theorem 3, except that (3.21) is not needed. Our conditions on Q are sufficient that (3.16a) continues to hold with $0 \leqq \lambda < 1$.

Theorem 8 may be extended in the same way as theorem 3 is extended by theorem 12.

6. Dependence on *a*

We intend to prove the following theorem.

THEOREM 13. *Let $\{b_n\}$ be a sequence of positive numbers, such that for some $r_0 \in [\frac{1}{2}, \infty)$*

$$(6.1a) \qquad \lim_{n \to \infty} b_n n^{-\xi} = +\infty, \qquad \xi < r_0,$$

$$(6.1b) \qquad \lim_{n \to \infty} b_n n^{-\eta} = 0, \qquad \eta > r_0.$$

Let P have a finite absolute moment of some positive order, let P^m for some m have an absolutely continuous component and let

$$(6.2) \qquad b_n||P^n - U_\beta P^n|| \leqq c < \infty, \qquad n = 1, 2, \ldots,$$

for some $\beta \neq 0$. Then

$$(6.3) \qquad \limsup_{n\to\infty} b_n||P^n - U_a P^n|| = c'|a| < \infty, \quad -\infty < a < \infty,$$

and if $\lim_{k\to\infty} b_{n_k}||P^{n_k} - U_a P^{n_k}||$ exists for a single value of a,

$$(6.4) \qquad \lim_{n\to\infty} b_{n_k}||P^{n_k} - U_a P^{n_k}|| = c_1|a|, \qquad -\infty < a < \infty.$$

It is noted that if no P^m has an absolutely continuous component, a theorem of this type does not hold for the relation (1.5), at least not if $L_0 \neq \{0\}$. For then $n^{\frac{1}{2}}||P^n Q - U_\alpha P^n Q|| \leqq c < \infty$ for all Q and some $\alpha \neq 0$ by theorem 2. If this would imply $\lim_{n\to\infty} \sup n^{\frac{1}{2}}||P^n Q - U_a P^n Q|| < \infty$ for all a and every absolutely continuous Q, we would have $L_0 = (-\infty, +\infty)$ by theorem 11, in contradiction with I, theorem 2.

The following lemma will be used in proving theorem 13.

Lemma 6. *Let S be a stable probability measure of order α, symmetric about zero.*

Then

$$(6.5) \quad ||(I - U_{a\lambda} - \lambda I + \lambda U_a) S^n|| \leqq c\lambda(1-\lambda) a^2 n^{-2/\alpha},$$
$$n = 1, 2, \ldots, \; 0 \leqq \lambda \leqq 1, \; -\infty < a < \infty,$$

where c does not depend on λ, a and n.

Proof. For $\lambda = 0$ and $\lambda = 1$ the assertion is trivial. First assume then, that λ is rational, $\lambda = k/m$, $3 \leqq k \leqq m-3$. Putting $U_{a/m} = V$, we have

$$I - U_{\lambda a} - \lambda I + \lambda U_a = m^{-1}\{m(I - V^k) - k(I - V^m)\}$$
$$= m^{-1}(I - V)\{m \sum_{j=0}^{k-1} V^j - k \sum_{i=0}^{m-1} V^i\}$$
$$= m^{-1}(I - V)\{m \sum_{j=1}^{k-1} (V^j - I) - k \sum_{i=1}^{m-1} (V^i - I)\}$$
$$= m^{-1}(I - V)^2\{k \sum_{r=0}^{m-2} (m - r - 1) V^r - m \sum_{s=0}^{k-2} (k - s - 1) V^s\}$$
$$= m^{-1}(I - V)^2\{k \sum_{r=k-1}^{m-2} (m - r - 1) V^r + (m-k) \sum_{r=0}^{k-2} (r+1) V^r\}.$$

With (1.1) and (1.2) for $\lambda = k/m$

$$||(I-U_{\lambda a}-\lambda I+\lambda U_a)S^n|| \leqq \tfrac{1}{2}k(m-k)||(I-V)^2S^n||.$$

Since $S = S_1^2$, with S_1 symmetric stable of order α, we have by (1.2) and (4.16)

$$||(I-U_{\lambda a}-\lambda I+\lambda U_a)S^n|| \leqq c\lambda(1-\lambda)a^2n^{-2/\alpha}, \quad n = 1, 2, \ldots,$$

for rational λ. For λ irrational (6.5) follows from the fact that $||(I-U_{\lambda a}-\lambda I+\lambda U_a)S^n||$ for fixed n is a continuous function of λ.

Proof of Theorem 13. *First assume* $r_0 > \frac{1}{2}$. *From* (6.1a) *and* (6.2)

$$\lim_{n\to\infty} n^\xi||P^n-U_\beta P^n|| = 0, \qquad \tfrac{1}{2} \leqq \xi < r_0.$$

Therefore, by theorem 7, the characteristic function φ of P satisfies

$$(6.6) \qquad |\varphi(u)| \leqq 1-\gamma|u|^{1/\xi}, \qquad -\delta \leqq u \leqq \delta,$$

with $\gamma = \gamma(\xi) > 0$, $\delta = \delta(\xi) > 0$. From (6.6) it may be shown that

$$(6.7) \qquad ||(I-U_{\lambda\beta})P^n-\lambda(I-U_\beta)P^n|| \leqq c(\lambda, r, \beta, P)n^{-2r}, \quad n = 1, 2, \ldots,$$

for every $r < \xi < r_0$. The proof is similar to the proof of theorem 8, the principal difference being that (4.26) is replaced by $V(n) \leqq c_2n^{-2/\alpha}$, which is obtained by (4.17) and (6.5).

From (6.7) and (6.1b), by taking r sufficiently close to r_0,

$$(6.8) \quad \big|b_n||P^n-U_{\lambda\beta}P^n||-\lambda b_n||P^n-U_\beta P^n||\big| \leqq c(\lambda, \vartheta, \beta, P)n^{-\vartheta}, \quad n = 1, 2, \ldots,$$

for some $\vartheta > 0$. The theorem follows immediately from (6.8).

If $r_0 = \frac{1}{2}$, we use the relation

$$(6.9) \quad ||(I-U_{\lambda\beta})P^n-\lambda(I-U_\beta)P^n|| \leqq c(\lambda, \beta, P)n^{-1}, \quad n = 1, 2, \ldots,$$

that may be derived as follows. Since P^m has an absolutely continuous component, it contains P_1 having a bounded probability density. Then P^{2m} contains P_1^2 with continuous density and therefore contains P_0^2, where P_0 is the uniform probability distribution on some interval. Now P_0^2 satisfies (6.9). This is proved in the same way as lemma 6, but now (3.45a) is used. Theorem 1 implies that P satisfies (6.9).

REFERENCES

FELLER, W.,

[1] An Introduction to Probability Theory and its Applications, Vol. I, Sec. Ed. Wiley, 1957.

GNEDENKO, B. V. and A. N. KOLMOGOROV,

[2] Limit Distributions for Sums of Independent Random Variables, Addison-Wesley, 1954.

IBRAGIMOV, I. A.,

[3] On the Compositions of Unimodal Distributions. Theory of Probability and its Applications 1 (1956), 255—260. (Transl. Teoriya Veroyatnostei i ee Primeneniya.)

LOÈVE, M.,

[4] Probability Theory, Sec. Ed. Van Nostrand, 1960.

NETTO, E.,

[5] Lehrbuch der Combinatorik. Teubner, 1927.

SHEPP, L. A.,

[6] A Local Limit Theorem. Ann. Math. Stat. 35 (1964), 419—423.

STAM, A. J.,

[7] On shifting Iterated Convolutions I. Compositio Mathematica 17 (1967), 268—280.

TAYLOR, A. E.,

[8] Introduction to Functional Analysis. Wiley, 1958.

WINTNER, A.,

[9] Asymptotic Distributions and Infinite Convolutions. Edwards Brothers, Ann Arbor, Michigan, 1938.

BOCHNER, S.,

[10] Harmonic Analysis and the Theory of Probability. Univ. of Calif. Press, 1955.

STAM, A. J.,

[11] Derived Stochastic Processes. Compositio Mathematica 17 (1965), 102—140.

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