COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 18, nº 3 (1967), p. 189-200
<http://www.numdam.org/item?id=CM_1967__18_3_189_0>

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by

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1. Introduction

In Section 3 of [4] the author extended the topological methods developed by Porcelli and Connell [9] for handling isolated singularities of complex differentiable functions to the case of more complicated singularities. In particular he was able to resolve by topological methods the case when the singularity was a rectifiable arc. In this paper the results of [4] are generalized to the case of complex (Frechét) differentiable functions on a complex Euclidean space into itself. In particular the removable singularity problem, when the singularity is a "rectifiable interface" separating two adjacent cells, is resolved.

The topological index of Whyburn [12] is replaced by degree theoretic methods from algebraic topology [1, 2].

The topological analogue of our results, wherein the requirement of complex differentiability is replaced by the requirement of being light and locally sense preserving, may be found in the work of Titus and Young [10].

We are unable to generalize the algebra of difference quotients developed in [4, 6, 9, 12]; however, the need for such auxiliary functions is obviated by use of a geometric characterization of harmonic functions [Theorem 4.1].

Let B and C be Banach spaces, f a continuous function on an open set S in B into C, and $p \in S$. Then f is said to be (real Frechét) differentiable [3] at p if there exists a bounded linear operator A from B into C, such that for all $\varepsilon > 0$, there exists $\delta > 0$, such that $||x|| < \delta$, $x+p \in S$, implies $||f(x+p)-f(x)-A(x)|| \leqq \varepsilon ||x||$.

In Section 3 degree theoretic methods are employed to prove Maximum Modulus Theorems. In Section 4 the basic machinery for handling removable singularities is developed. In Section 5 this machinery is applied to the case when the singularity is a "rectifiable interface".

No use is made in this paper of Jacobian matrices or determinants, or of kernel integrals.

2. Notation and definitions

Let R denote the real numbers, K the complex numbers, and ω the positive integers. Throughout this paper E shall denote a fixed Euclidean space and m shall denote Lebesgue measure on E. For $x \in E, \delta > 0$, set

$$V_x(\delta) = \{y \in E; \|y-x\| < \delta\} \text{ and } B_x(\delta) = \bar{V}_x(\delta)-V_x(\delta).$$

If $A, B, C \subseteqq E$ and f and g are functions on A into Z and B into Z for $Z = R, K, E$, then we set $(f+g)(x) = f(x)+g(x)$ for $x \in A \cap B$, and we let $f|C$ be the function h on $A \cap C$ such that $h(x) = f(x)$ for $x \in A \cap C$. If f and g are functions we shall write fg for the composition of f and g.

$B[E, E]$ shall denote the Banach algebra of linear transformations of E into E. By I shall be meant the element of $B[E, E]$ such that $I(x) = x$. We shall denote by $G(E)$ the rotation group of E, that is the group of unitary transformations of $B[E, E]$. A subgroup G of $G(E)$ is said to be transitive if for all $x, y \in B_0(1)$, there exists $g \in G$, such that $g(x) = g(y)$.

Definition 2.1. Let F be a collection of continuous functions on open subsets of E into R, and G a subgroup of $G(E)$. Then F is called a TG family [7] if:

1. For $c \in R, f \in F, cf \in F$.
2. For $f, g \in F, f+g \in F$.
3. For $f \in F$, S an open set in E, $f|S$ lies in F.
4. For $f \in F$, and $x \in B$, the function f_x lies in F, where $g_x(y) = f(y-x)$ for $y \in E$ such that $y-x \in$ domain f.
5. For $f \in F, g \in G, fg \in F$.

A function $f \in F$ is said to maximum modular, if for all open sets S in E, such that $\bar{S} \subseteqq \operatorname{dom} f$, we have $f|(x)| \leqq \sup \{|f(t)|; t \in \bar{S}-S\}$ for all $x \in S$. A TG family F is called a TGM family if F contains the constant function $\bar{1}$, and if all elements of F are maximum modular.

Let S be an open set in E containing $V_0(1)$, and let f be a continuous function on S into E. If p is a point of $E-f(B)$, where $B = B_0(1)$, then the degree $\mu_B(f, p)$ of f with respect to B and p may be defined. If x is a point of S, such that there exists $\delta > 0$, such that $f(y) \neq f(x)$ for all $y \neq x$, $y \in V_x(\delta) \cap S$, then the local degree $\mu_x(f)$ of f at x may be defined. For formal definitions and development of degree theory one may refer to Alexandroff and Hopf [1] or Cronin [2]. Our notation is inspired by that used by Whyburn [12] in the two dimensional case.

We shall need the following facts from degree theory:

2.1. $\mu_B(f, p) = \mu_B(f-p, 0)$.

2.2. If $\mu_B(f, p) \neq 0$, then $p \in f[V_0(1)]$.

2.3. If $H = f^{-1}(p) \cap V_0(1)$ is finite, then $\mu_B(f, p) = \sum_{x \in H} \mu_x(f)$.

2.4. If p and q are points of the same component of $E-f(B)$, then $\mu_B(f, p) = \mu_B(f, q)$.

2.5. If M and N are elements of the same component of the set of invertible elements $I[E, E]$ of $B[E, E]$, then $\mu(M) = \mu_B(M, 0) = \mu_B(N, 0) = \mu(N)$.

2.6. If for some $p \in V_0(1)$, the derivative A of f at p exists and is invertible, then there exists $\delta > 0$, such that for $q = f(p)$, $V_p(\delta) \subseteqq V_0(1)$, $V_q(\delta) \subseteqq f[V_0(1)]$, $f(x) \neq f(p)$ for all $x \in V_p(\delta)$, $x \neq p$, and $\mu_x(f) = \mu_p(f) = \mu(A)$ for $x \in V_p(\mu)$.

We shall also need the following form of Sard's Theorem:

Theorem 2.1. *Let f be a (Frechét) differentiable function on an open set S in E into E, and let H be set of all points $x \in S$, such that the derivative A_x of f at x does not invert. Then* $m[f(H)] = 0$.

In the form found in Cronin [2, 32—36] the first partial derivatives of f are required to exist and be continuous. This implies the existence and *continuity* of the Frechét derivative of f. This continuity, however, is not assumed here. The argument for the case of complex differentiable functions from K to K may be found in Whyburn [12, 72—73].

Proof. Let $a > 0$, and let Q be a cube of side a in S. For $n \in \omega$, let F_n be the subdivision of Q into equal cubes of side $a/2^n$, where n is the dimension of E. Let $H_0 = H \cap Q$, and for $m \in \omega$, set $H_m = \{x \in H_0;\ ||A_x|| \leqq m\}$. Then $H_0 = U_1^\infty H_m$.

Let $\varepsilon > 0$, $\varepsilon < 1$, and let $m \in \omega$. Let $x \in H_m$. Then there exists $\delta > 0$, such that $V_x(\delta) \subseteqq S$, and such that

$$||f(x+h)-f(x)-A_x(h)|| \leqq \varepsilon ||h|| \text{ for all } h \in V_0(\delta).$$

Now there exists $k \in \omega$, and an the element E_x of F_k containing x which lies in $V_x(\delta)$. Let $D = \{E_x;\ x \in H_m\}$. For each $x \in H_m$, there exists $\tilde{x} \in H_m$, such that $E_x \subseteqq E_{\tilde{x}}$, and such that $E_{\tilde{x}}$ is not a proper subset of any element of D. Since $U_1^\infty F_i$ is countable, $D_0 = \{E_{\tilde{x}};\, x \in H_m\}$ is countable. Clearly $A = B$ or $A \cap B$ is a "face" cube of dimension $n-1$ for all $A, B \in D_0$.

Let $x \in H_m$, set $J = \{y-\tilde{x};\, y \in E_{\tilde{x}}\}$, and set $A = A_{\tilde{x}}$. Then there exists an $n-1$ dimensional subspace T of E such that $A(J) \subseteqq T$. For $y \in E$, set $\delta(y) = \inf\{||y-t||;\ t \in T\}$, and let $P(y)$ be the orthogonal projection of y into T. For $y \in J$,

$$||f(\tilde{x}+y)-f(\tilde{x})-A(y)|| \leqq \varepsilon||y|| \leqq \varepsilon\sqrt{n}s,$$

and

$$||A(y)|| \leqq m||y|| \leqq m\sqrt{n}s,$$

where s is the side of J. Thus for $y \in J$, $\delta[f(\tilde{x}+y)-f(\tilde{x})] \leqq \varepsilon\sqrt{n}s$, and $P[f(\tilde{x}+y)-f(\tilde{x})] \in V_0(m\sqrt{n}s+\varepsilon\sqrt{n}s) = V_0[(m+\varepsilon)\sqrt{n}s]$. Then

$$\begin{aligned} m[f(E_{\tilde{x}})] &\leqq C_{n-1}[(m+\varepsilon)\sqrt{n}s]^{n-1}[\varepsilon\sqrt{n}s] \\ &\leqq M\varepsilon s^n = M\varepsilon\cdot m(E_{\tilde{x}}), \end{aligned}$$

where C_{n-1} is a constant determined by $n-1$, and

$$M = C_{n-1}(m+1)^{n-1}n^{n/2}.$$

We note that $f(E_{\tilde{x}})$ lies in a "cylinder" with the subset

$$V_{\tilde{x}}[(m+\varepsilon)\sqrt{n}s]$$

of T as a base, and with altitude $\varepsilon\sqrt{n}s$. Then

$$m[f(H_m)] \leqq \sum_{J\in D_0} m[f(J)] \leqq \sum_{J\in D_0} M\varepsilon\cdot m(J) \leqq m\varepsilon\cdot m(Q).$$

Since ε is arbitrary, $m[f(H_m)] = 0$. Thus

$$m[f(H_0)] \leqq \sum_{i=1}^{\infty} m[f(H_i)] = 0.$$

3. Maximum modularity theory

Let f be a complex differentiable function on an open set S in K into K. Then f satisfies the Maximum Modulus Theorem. For $x \in S$, $t \in E_2 = K$, set $A_x(t) = f'(x)\cdot t$. Then A_x is the Frechét derivative of f at x, and A_x is an element of the family W of elements of $B[E_2, E_2]$ of the form rU, where $r \in R$, and U is a rotation of index 1. The maximum modularity of f can be deduced directly from the fact that W is a linear space all of whose elements which invert have index 1. The argument is independent of the dimension of the space in question, and does not involve K.

THEOREM 3.1. *Set $U = V_0(1)$ and $E^* = B[E, R]$ and let f be a continuous function on $\overline{U}$ into E, and H a nowhere dense subset of U, such that for $x \in U-H$, the derivative A_x of f at x exists. Then if*

1. *For all $x \in U-H$ and $r \in R$, such that A_x-rI is invertible, we have $\mu(A_x-rI) = 1$; and*

2. *For all $r \in R$, $m[f-rI)(H)] = 0$, then*

$$|Lf(x)| \leqq ||L|| \sup \{|Lf(t)|;\ t \in B_0(1)\},$$

and

$$||f(x)|| \leqq \sup \{||f(t)||;\ t \in B_0(1)\}$$

for all $x \in U$, $L \in E^*$.

PROOF. Let $L \in E^*$, and let $\varepsilon > 0$. There exists a countable subset X of $U-H$, dense in U. For $x \in X$, let C_x be the set of all $r \in R$, such that A_x-rI does not invert. Clearly, for $x \in X$, C_x is finite or empty, and thus $U_{x \in X} C_x$ is countable. Let r_0 be an element of R, such that $r_0 \geqq 0$, $r_0 < \varepsilon$, and $r_0 \notin U_{x \in X} C_x$. Then A_x-r_0I is invertible for all $x \in X$. Set $f_0 = f-r_0I$.

Let S be a component of $E-f_0[B_0(1)]$, such that $S \cap f(U) \neq \emptyset$. Then $P = f_0^{-1}(S)$ is an open set in U, and hence open in E. Thus there exists $x \in X$, such that $x \in P$. Since the derivative of f_0 at x is invertible, from Fact 2.6, we have that $S \supseteqq f_0(P)$ contains an open set Q. Let K be the set of all points $x \in U-H$, such that the derivative of f_0 at x does not invert. Then from Sard's Theorem, $m[f_0(K)] = 0$, and hence $f_0(K)$ is nowhere dense in E.

By hypothesis $f_0(H)$ is nowhere dense in E. Then

$$Q_0 = [f_0(H) \cup f_0(K)] \cap Q$$

is nowhere dense in Q. Let $p \in Q-Q_0 \subseteqq S$, and let $M = f^{-1}(p) \cap U$. Since M is compact, from Fact 2.6, M is finite. By hypothesis, the derivative of f_0 at x is of index 1 for all $x \in M$. Hence from Fact 2.6, for $x \in M$, $\mu_x(f_0) = 1$, and thus $\mu_B(f_0, p) = k > 0$, where k is the number of elements of M. Then from Fact 2.2, $S \subseteqq F(U)$.

Assume there exists $z \in U$, such that $|Lf_0(z)| > \sup \{|Lf_0(t)|;$ $t \in B_0(1)\}$. Then $f_0[B_0(1)] \subseteqq D = \{x \in E;\ |L(x)| < |Lf_0(z)|\}$, and $f_0(z) \in E-D$. Then $f_0(z)$ lies in the unbounded component P of $E-f_0[B_0(1)]$, and hence $P \subseteqq f_0(U)$. But $f_0(\overline{U})$ is compact and hence bounded. Similarly for $z \in U$, $||f_0(z)|| \leqq \sup\{||f_0(t)||; t \in B_0(1)\}$. Since ε is arbitrary the theorem follows.

THEOREM 3.2. *Let S be an open set in E, H and X subsets of S, and f a continuous function on S into E such that:*

1. *H is nowhere dense in S, and $m[f(H)] = 0$.*

2. *For* $x \in (S-H) \cup X$, *the derivative* A_x *of* f *at* x *exists, and is of index* 1 *if invertible.*

3. X *is dense in* S, *and for* $x \in X$, A_x *is invertible. Then if* f *is light,* f *is open* [12, 75—76].

PROOF. Let $x \in S$. Since f is light, from the Zoretti Theorem [12, 35], there exists an open set T containing x, such that $\bar{T} \subseteqq S$, $\bar{T}$ is homeomorphic to $\bar{V}_0(1)$, and such that $(\bar{T}-T) \cap f^{-1}f(x) = \emptyset$. Thus $f(x) \notin f(\bar{T}-T)$. Then from Fact 2.6, we see that the component V of $E-f(\bar{T}-T)$ containing $f(x)$ lies in $f(T)$, and thus $f(x)$ lies in the open subset V of $f(S)$.

4. Removable singularities

DEFINITION 4.1. Let f be a continuous function on an open set S in E into E, and A a subset of S. Then f is called a P_A function if for every $x \in A$, there exists $M_x > 0$, such that

$$||f(y)-f(x)|| \leqq M_x||y-x|| \quad \text{for all } \; y \in S.$$

It may be readily shown [4, Theorem 3.2] that if f is a P_A function and $m(A) = 0$, then $m[f(A)] = 0$.

THEOREM 4.1. *Let* G *be a compact transitive subgroup of* $G(E)$, *and* F *a TGM family of* E. *Then the elements of* F *are harmonic functions and hence continuously differentiable.*

PROOF. This lemma may be found in Lowdenslager [8, 468—469] and [7].

Set $U = V_0(1)$, and let W be the family of all continuous functions h on $\bar{U}$, such that $h|U$ lies in F. For $h \in W$ and $z \in \bar{U}$, set $L(h) = C \int_U h\,dm$, and $\tilde{h}(z) = \int_G hg(z)d\mu(g)$, where μ is normalized Haar measure on G, and $C^{-1} = m(U)$. Then for $h \in W$, $x \in \bar{U}$, and $g \in G$, $\tilde{h}g(x) = \tilde{h}(x)$. Hence for $0 < r \leqq 1$, since G is transitive, $\tilde{h}(x) = \tilde{h}(y)$ for all $x, y \in B_0(r)$.

Fix $h \in W$. Then $\tilde{h}$ lies in the closure of W and hence must be maximum modular, and thus must be a constant function. Thus

$$\begin{aligned} h(0) = \tilde{h}(0) = C\int_U \tilde{h}\,dm = L(\tilde{h}) = L\left[\int_G hg\,d\mu(g)\right] \\ = \int_G L(hg)d\mu(g) = \int_G L(h)d\mu = L(h). \end{aligned}$$

Thus the elements of W satisfy the volume mean characterization of harmonic functions.

THEOREM 4.2. *Let f be a bounded continuous function on an open set S in E into E, H a subset of S, and A an element of $B[E, E]$, $A \neq 0$, such that:*

1. *There exists a polynomial P, irreducible over R, such that $P(A) = 0$.*

2. *For $x \in S-H$, the derivative A_x of f at x exists, and is such that $A_x A = AA_x$.*

3. *Either:*

a. *$m(H) = 0$, and f is a P_H function; OR*

b. *H is countable.*

Then f is differentiable on S, and $A_x A = AA_x$ for all $x \in S$.

PROOF: Let Z be the subalgebra of $B[E, E]$ generated by A, and let T be the set of all elements B of $B[E, E]$ such that $AB = BA$. Since A is irreducible over R, Z is isomorphic to the complex field K. For $x \in E$, $c \in Z$, set $cx = c(x)$. Then E can be considered as a complex Hilbert space H over Z, with $T = B[H, H]$. Let $x \in S-H$. Then $A_x \in T$, and $P_x = \{c \in Z;\ A-cI$ does not invert$\}$ is finite. Thus there exists an arc W in Z with endpoints A_x and I, containing no points of P_x. Thus $W \subseteqq I[E, E]$, and from Fact 2.5 $\mu(A_x) = \mu(I) = 1$. Set $G = G(H) \subseteqq B[H, H] \subseteqq B[E, E]$. Then G is a compact transitive subgroup of $G(E)$.

Let V be the family of all continuous functions h on open subsets of E into E, such that there exists $H_h \subseteqq \text{dom}\, h$, such that h and H_h satisfy the hypothesis and conditions 1, 2 and 3.a of this theorem. Let $L \in E^*$, and set $V_L = \{Lh;\ h \in V\}$. Then $Lf \in V_L$.

For $h \in V$, since h is differentiable on $S-H_h$, where $S = \text{dom}\, h$, h is a P_S function. Let $h_1, \ldots, h_m \in V$, $r_1, \ldots, r_m \in R$, $m \in \omega$. Then, setting $H = U_1^m H_{h_i}$, $m(H) \leqq \sum_{i=1}^m m(H_{h_i}) = 0$. Clearly $h = \sum_{i=1}^m r_i h_i$ is a P_S function, and hence h is a P_H function. Then V_L is a TG family. For $r \in R$, and $h \in V$, since $h-rI$ and H_h satisfy condition 3.a, $m[(h-rI)(H_h)] = 0$, and hence from Theorem 3.1, Lh is maximum modular. Thus V_L is a TGM family.

From Theorem 4.1, the elements of V_L are harmonic functions and hence continuously differentiable. Since E is finite dimensional, we readily deduce [3] that the elements of V are continuously differentiable. For $h \in V$, since H_h is nowhere dense in E, and T is closed, A_x must lie in T for all $x \in H_h$.

The argument in the case that H satisfies condition 3.b. is similar.

REMARK 4.1. Theorem 4.1 asserts that the elements of a TGM family F are harmonic functions. Hence the elements of F are actually analytic. It is then easy to strenghten the conclusion of Theorem 4.2 to an assertion of analyticity.

REMARK 4.2. In the two dimensional theory [4, 6, 9, 12] strong use is made of auxilary functions of the form

$$\frac{f(x_n)-f(x)}{x_n-x}-\frac{f(x_m)-f(x)}{x_m-x};$$

however in the higher dimensional case, in general, no similar functions, are available. For example, let M be a closed subspace of dimension greater than one of a complex Banach space B. Assume there exists a complex differentiable function g on $M-\{0\}$ into B or into K, such that $||g(x)|| = ||x||^{-1}$ for $x \in M-\{0\}$. Now there exists a subhyperplane N of M, such that $0 \notin N$. Then $g|N$ is a bounded non-constant complex differentiable function on N, contradicting Liouville's Theorem [3, 5].

The only extension the author is aware of involves the space of quaterneons Q. Let H be a closed nowhere dense subset of $U = V_0(1)$, and let f be a continuous function on U into Q, such that for $x \in U-H$, the derivative A_x of f at x exists and is such that there exists $q_x \in Q$, such that $A_x(y) = q_x y$ for all $y \in Q$. Let x_1, $x_2 \in S-H$, and set, for $i = 1, 2$, $T_i(h) = [f(x_i+h)-f(x_i)]h^{-1}$ for all $h \in Q$, $h \neq 0$, such that $x_i+h \in H$, and set $T_i(h) = q_{x_i}$ for $h = 0$.

Then for $i = 1, 2$, T_i is continuous at 0, and the derivative H_x^i of T_i at x exists for all $x \in \operatorname{dom} T_i$, $x \neq 0$. Let $x \in \operatorname{dom} T_1 \cap \operatorname{dom} T_2$, $x \neq 0$. Then for $i = 1, 2$, by direct computation

$$\begin{aligned} H_x^i(t) &= A_x(t)x^{-1}-[f(x_i+x)-f(x_i)]x^{-1}tx^{-1} \\ &= q_x tx^{-1}-[f(x_i+x)-f(x_i)]x^{-1}tx^{-1} \\ &= p_x^i tx^{-1} \text{ for all } t \in Q, \end{aligned}$$

where $p_x^i = q_x-[f(x_i+x)-f(x_i)x^{-1}]$. Then $(H_x^1-H_x^2)(t) = (p_x^1-p_x^2)tx^{-1}$. Since Q is a skew field $H_x^1-H_x^2$ is identically 0 or inverts. In the latter case there exist continuous functions u and v on $[0, 1]$ into $Q-\{0\}$, such that $u(0) = v(0) = 1$, $u(1) = p_x^1-p_x^2$, and $v(1) = x^{-1}$. For $s \in [0, 1]$, $t \in Q$, set $g_s(t) = u(s)t \cdot v(s)$. Then g_s is invertible for all $s \in [0, 1]$, and hence from Section 2, $\mu(H_x^1-H_x^2) = \mu(g_1) = \mu(g_0) = \mu(I) = 1$. Then if $U-H$ has finitely many components, and $m(H) = 0$ and f is a P_H function, or in the terminology of [4] there exists an M family of partitions Σ of H and f is a Σ function, we conclude that T_1-T_2 satisfies the Maximum Modulus Theorem. Then working with sequences of difference quotients (cf. [9] or [4, Theorem 3.3]) we deduce that f is differentiable on U.

5. Rectifiable interfaces

DEFINITION 5.1. Throughout this section H shall denote a fixed finite dimensional complex Hilbert space with an involution $x \to x^*$, $x \in H$.

If f is a differentiable function on an open subset S of H into H, then f is said to be symmetrically differentiable if for all $x \in S$, $y, z \in H$, $[f'_x(y), z^*] = [f'_x(z), y^*]$. We let $f^{(n)}_x$ denote the n-th derivative of f at x for $x \in S$, $n \in \omega$.

We observe that if we replace H by a three dimensional real Euclidean space that the requirement of symmetric differentiability of a function g reduces to the requirement that the curl of g, $\nabla \times g$, vanishes identically.

THEOREM 5.1. *Let f be a symmetrically differentiable function on $U = V_0(1)$ into H. Then there exists a complex differentiable function h on U into K, such that for $x \in U$, $y \in H$, $h'_x(y) = [f(x), y^*]$.*

PROOF. Let $x \in U$, $s \in H$, and $\rho \in K$. Then for all $t \in H$, $[f'_x(\rho s), t^*] = [f'_x(t), (\rho s)^*] = \rho [f'_x(t), s^*] = \rho [f'_x(s), t^*]$, and thus $f'_x(\rho s) = \rho f'_x(s)$ and f is complex differentiable. Then (cf. Remark 4.1 and [3], [5]) $f^{(n)}$ exists for all $n \in \omega$, and the power series $\sum_0^\infty f_0^{(n)}(x, \ldots, x)/n!$ converges uniformly on compact subsets of U to $f(x)$. For $n \in \omega$, $x \in H$, set $k_n(x) = f_0^{(n)}(x, \ldots, x)/n!$ and $h_n(x) = [f_0^{(n)}(x, \ldots, x), x^*]$.

Let $\varepsilon > 0$, $x \in U$. Then there exists $0 < \delta < 1 - ||x||$, such that for $y \in U_x(\delta)$, $s, t \in H$, $||f'_{x+y} - f'_x - f''_x(y)|| \leqq \varepsilon ||y||/2$, and hence since f is symmetrically differentiable,

$$\begin{aligned} |[f''_x(y, s), t^*] - [f''_x(y, t), s^*]| &\leqq |[f'_{x+y}(s) - f'_x(s), t^*] \\ &\quad - [f''_x(y, s), t^*]| + |[f'_{x+y}(t) - f'_x(t), s^*] - [f''_x(y, t), s^*]| \\ &\leqq \varepsilon ||y|| \cdot ||s|| \cdot ||t||/2 + \varepsilon ||y|| \cdot ||t|| \cdot ||s||/2 = \varepsilon ||y|| \cdot ||s|| \cdot ||t|| \end{aligned}$$

and thus $[f''_x(y, s), t^*] = [f''_x(y, t), s^*]$ for all $x \in U$, $y, s, t \in H$. Continuing this process we deduce that

$$[f_0^{(n)}(x, \ldots, x, s), t^*] = [f_0^{(n)}(x, \ldots, x, t), s^*] \text{ for all } x, s, t \in H.$$

Now for $x, t \in H$,

$$\begin{aligned} (n+1)!\,(h_n)'_x(t) &= [nf_0^{(n)}(x, \ldots, x, t), x^*] + [f_0^{(n)}(x, \ldots, x), t^*] \\ &= [nf_0^{(n)}(x, \ldots, x), t^*] + [f_0^{(n)}(x, \ldots, x), t^*] \\ &= (n+1)[f^{(n)}(x, \ldots, x), t^*] = (n+1)!\,[k_n(x), t^*] \end{aligned}$$

and $||h_n(x)|| \leqq ||k_n(x)||/(n+1)$. Thus [3, 5], $\sum_0^\infty h_n$ converges uniformly on compact subsets of U to a differentiable limit function h, and for $x \in U$, $t \in H$,

$$h'_x(t) = \sum_0^\infty (h_n)'_x(t) = \sum_0^\infty [k_n(x), t^*] = [\sum_0^\infty k_n(x), t^*] = [f(x), t^*].$$

Now for

$$x \in U, y \in H, \rho \in K, h'_x(\rho y) = [f(x), (\rho y)^*] = \rho[f(x), y^*] = \rho h'_x(y)$$

and thus h is complex differentiable.

THEOREM 5.2. *Set $U = V_0(1)$, let $M > 0$, and let h be a homeomorphism of $\bar{U}$ onto $\bar{U}$, such that $||h(y)-h(x)|| \leqq M||y-x||$ and $||h^{-1}(y)-h^{-1}(x)|| \leqq M||y-x||$ for all $x, y \in \bar{U}$. Let $z \in E$, $||z|| = 1$, and set $A_0 = \{x \in U; [x, z] = 0\}$, and $S_0 = h(A_0)$, and let f be a continuous function on $\bar{U}$ into $\bar{U}$. Then if f is symmetrically differentiable on $U-S_0$, f is symmetrically differentiable on U.*

PROOF. Set $A_1 = \{x \in U; [x, z] < 0\}$, $A_2 = \{x \in U; [x, z] > 0\}$, and $S_1 = h(A_1)$, $S_2 = h(A_2)$. From Theorem 5.1, making use of the line integral analogue of [4] (or equivalently the monodromy theorem of analytic continuation theory), for $i = 1, 2$, a complex differentiable function g_i on S_i into K is found, such that for $x \in S_i$, $(g_i)'_x(y) = [f(x), y^*]$ for all $y \in H$.

We shall show that g_1 and g_2 can be continuously extended to $\bar{S}_1$ and $\bar{S}_2$ in such a way that they can be pieced together to form a single function g on $\bar{U}$ which satisfies a Lipschitz condition on $\bar{U}$. We will then apply Section 4 to g and deduce its differentiability, and hence that of f, everywhere on U.

If W is a rectifiable arc in H, let $L(W)$ denote the length of W. Let $i = 1, 2$, and let $x, y \in A_i$. Then the interval $[x, y] \subseteqq A_i$ and $h([x, y])$ is a rectifiable arc in S_i such that $L(h[x, y]) \leqq M||y-x||$. Then from the suitable form of the mean value theorem [3, 5], setting $N = \sup \{||f(t)||; t \in \bar{U}\}$,

$$\begin{aligned} |g_i h(y) - g_i h(x)| &\leqq L(h[x, y]) \cdot \sup \{||(g_i)'_t||; t \in h([x, y])\} \\ &= L(h[x, y]) \cdot \sup \{||f(t)||; t \in h([x, y])\} \\ &\leqq M||y-x|| \cdot N. \end{aligned}$$

Thus $g_i h$ and hence $g_i = (g_i h)h^{-1}$ is uniformly continuous on S_i, and thus g_i can be continuously extended to a continuous function $\bar{g}_i$ on $\bar{S}_i$.

Let $\varepsilon > 0$. Then there exists $\delta > 0$, such that $||y-x|| \leqq \delta$, $x, y \in \bar{U}$, implies $||f(y)-f(x)|| < \varepsilon$. Let $0 < \rho_0 \leqq \delta/2M$, and $0 < \rho < \rho_0$ and set for $x \in J = \{y \in A_0; y-\rho_0 z, y+\rho_0 z \in U\}$,

$$\theta_\rho(x) = g_2 h(x+\rho z) - g_1 h(x-\rho z).$$

Then for $x \in J$, $t = h(x)$, $a = h(x+\rho z)$, $b = h(x-\rho z)$,

$$||b-a|| \leqq M||(x+\rho z)-(x-\rho z)|| = 2M||\rho z|| = 2M\rho \leqq 2M\rho_0 \leqq \delta,$$

and

$$||(\theta_\rho h^{-1})'_t|| = ||(g_2)'_a - (g_1)'_b|| = ||f(a)-f(b)|| < \varepsilon.$$

Let $u, v \in h(J)$, and set $x = h^{-1}(u)$, $y = h^{-1}(v)$. Then

$$\begin{aligned} ||\theta_\rho(y)-\theta_\rho(x)|| &\leqq L([h(x), h(y)]) \cdot \sup \{||(\theta_\rho h^{-1})'_t||;\ t \in h([x, y])\} \\ &\leqq M||y-x||\varepsilon \leqq M^2||v-u||\varepsilon, \end{aligned}$$

and setting $c(t) = (\bar{g}_2 - \bar{g}_1)(t)$ for $t \in S_0$,

$$|c(v)-c(u)| = \lim_{\rho\to 0} |\theta_\rho(y)-\theta_\rho(x)| \leqq M^2||v-u||\varepsilon.$$

Since ε is arbitrary, $|c(v)-c(u)| = 0$, and c is a constant function $\bar{\sigma}$. Set $g(x) = \bar{g}_1(x)$ for $x \in \bar{S}_1$, and $g(x) = \bar{g}_2(x)+\sigma$ for $x \in \bar{S}_2 - S_0$. Then g is continuous on $\bar{U}$.

Clearly for

$$x, y \in \bar{U},\ |g(y)-g(x)| \leqq NM||h^{-1}(y)-h^{-1}(x)|| \leqq NM^2||y-x||,$$

and $m(S_0) = 0$. Hence from Theorem 4.2 and Remark 4.1, $g|U$ is at least twice continuously differentiable, and thus f is symmetrically differentiable on U.

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(Oblatum 9-11-1966) University of Virginia