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Two Tauberian theorems in Banach spaces

by

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In this note, we show how the classical Tauberian theorems concerning Abel and Borel summable real or complex sequences can be proved for sequences in a Banach space. The classical proofs are well-known and are due to Hardy and Littlewood [1]. A function-theoretic proof using results from complex analysis is due to W. Jurkat [2] and [3]. We follow the latter, giving details only when necessary. We shall use notations of [1].

Let E be a complex Banach space and E' its dual. A sequence $\{x_n\}$ in E is said to be *weakly Borel summable* to $0 \in E$ if for each $f \in E'$, the series

$$\sum_{n=0}^{\infty} f(x_n) \frac{\lambda^n}{n!}$$

converges for all complex λ and is $o(e^{\lambda})$ for real $\lambda \to \infty$.

If

$$\sum_{n=0}^{\infty} x_n \frac{\lambda^n}{n!}$$

converges for all complex λ in the norm topology and $o(1)$ for real $\lambda \to \infty$, then $\{x_n\}$ is said to be *strongly Borel summable* to 0.

Similarly, $\{x_n\}$ is said to be *weakly* (or *strongly*) *Abel summable* to 0, if for each $f \in E'$ the series $(1-\lambda)\sum_{n=0}^{\infty} f(x_n)\lambda^n$ converges for $|\lambda| < 1$ (or $(1-\lambda)\sum_{n=0}^{\infty} x_n\lambda^n$ converges in the norm topology for $|\lambda| < 1$) and is $o(1/1-\lambda)$ as real $\lambda \to 1^-$.

It is easy to see that a strongly Borel (resp. Abel) summable sequence is weakly Borel (resp. Abel) summable.

We consider Borel summable sequences first and prove the following:

THEOREM 1. Let E be a complex Banach space and E' its dual. Let $\{x_n\}$ be a sequence in E which is weakly Borel summable to 0. Suppose $||x_n - x_{n-1}|| = O(1/\sqrt{n})$ as $n \to \infty$. Then $\{x_n\}$ converges weakly to 0, i.e., for each $f \in E'$, the sequence $\{f(x_n)\}$ of complex numbers converges to 0.

We need the following lemmas.

LEMMA 1. Let

$$e^{\lambda} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!},$$

$\lambda = re^{i\theta}$, $r \geqq 0$, $i = \sqrt{-1}$. Then

$$\sum_{n=0}^{\infty} \left(\frac{r^n}{n!}\right)^2 = O\left(\frac{e^{2r}}{\sqrt{r}}\right).$$

PROOF: Since

$$|e^{\lambda}|^2 = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right) \overline{\left(\sum_{m=0}^{\infty} \frac{\lambda^m}{m!}\right)}$$

is absolutely and uniformly convergent inside any finite circle and since $\cos\theta \leqq 1-\delta\theta^2$ $(\delta > 0)$, by integrating termwise we have

$$\sum_{n=0}^{\infty} \left(\frac{r^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2r\cos\theta}\, d\theta = O(e^{2r})\int_0^{2\pi} e^{-2r\delta\theta^2}\, d\theta$$

$$= O(e^{2r})\int_0^{\pi} e^{-t}\frac{dt}{\sqrt{\delta r t}} = O\left(\frac{e^{2r}}{\sqrt{r}}\right).$$

LEMMA 2. If $\{x_n\}$ in E is a sequence which is weakly Borel summable to 0 and if $||x_n - x_{n-1}|| = O(1/\sqrt{n})$ as $n \to \infty$, then $\{||x_n||\}$ is a bounded sequence.

PROOF: As in § 3, [3], we see that for all k, $n \geqq 0$,

$$||x_k - x_n|| \leqq K\frac{|k-n|}{\sqrt{n+1}}$$

for some $K > 0$. Hence for $f \in E'$,

$$e^{-n}\sum_{k=0}^{\infty} f(x_k)\frac{n^k}{k!} - f(x_n) = e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!} f(x_k - x_n)$$

$$= O(e^{-n}||f||)\sum_{k=0}^{\infty}\frac{n^k}{k!}||x_k - x_n||$$

$$= O\left(\frac{||f||e^{-n}}{\sqrt{n}}\right)\sum_{k=0}^{\infty}\frac{n^k}{k!}|k-n|$$

$$= O(||f||) \qquad \text{(cf. [3], p. 280).}$$

Since $\{x_n\}$ is weakly Borel summable, $e^{-n}\sum_{k=0}^{\infty} f(x_k)\, n^k/k! = o(1)$ as $n \to \infty$ and therefore $\{f(x_n)\}$ is a bounded sequence of complex numbers. In other words, $\{x_n\}$ is a weakly bounded sequence in E. Since weakly and norm bounded sets in E are the same, $\{||x_n||\}$ is bounded.

LEMMA 3. If $\{x_n\}$ is a weakly Borel summable (to 0) sequence in E such that $||x_n - x_{n-1}|| = O(1/\sqrt{n})$, then for each $f \in E'$,

$$g(\lambda) = e^{-\lambda}\sum_{n=0}^{\infty} f(x_n)\frac{\lambda^n}{n!}$$

represents an analytic function in the whole complex plane. Moreover, $g(\lambda)$ is bounded in each parabola $|\lambda| - \mathscr{R}\lambda < M$, $M > 0$ ($\mathscr{R}\lambda$ denotes the real part of λ); and $g(\lambda) \to 0$ uniformly as $|\lambda| \to \infty$ in $|\lambda| - \mathscr{R}\lambda < M'$, $0 < M' < M$.

PROOF: It is clear that $g(\lambda)$ is analytic. By Lemma 2, $\{f(x_n)\}$ is bounded. Hence

$$|g(\lambda)| \leqq |e^{-\lambda}| \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} = e^{|\lambda| - \mathscr{R}\lambda}$$

shows that $g(\lambda)$ is bounded for $|\lambda| - \mathscr{R}\lambda < M$. The remainder follows from Hilfsatz 1, [3], where a well-known Montel theorem is used.

LEMMA 4. Let $g(\lambda)$ be as in Lemma 3. If $||x_n - x_{n-1}|| = O(1/\sqrt{n})$, then for each $f \in E'$, $\lambda = re^{i\theta}$,

$$\int_0^{2\pi} |\lambda g'(\lambda) e^{\lambda}|^2 d\theta = O(\sqrt{r}\, e^{2r}||f||),$$

where $g'(\lambda)$ is the derivative of $g(\lambda)$.

PROOF: By differentiating $g(\lambda)$, we obtain

$$e^{\lambda} g'(\lambda) = \sum_{n=1}^{\infty} f(x_n - x_{n-1}) \frac{\lambda^{n-1}}{(n-1)!}.$$

Hence

$$\lambda e^{\lambda} g'(\lambda) = \sum_{n=1}^{\infty} f(nx_n - nx_{n-1}) \frac{\lambda^n}{n!}.$$

Multiplying the last function by its conjugate and integrating termwise, we obtain

$$\begin{aligned}\int_0^{2\pi} |\lambda e^{\lambda} g'(\lambda)|^2 &= 2\pi \sum_{n=1}^{\infty} |f(nx_n - nx_{n-1})|^2 \left(\frac{r^n}{n!}\right)^2 \\ &= O(||f||) \sum_{n=1}^{\infty} n^2 ||x_n - x_{n-1}||^2 \left(\frac{r^n}{n!}\right)^2 \\ &= O(||f||) \sum_{n=1}^{\infty} n \left(\frac{r^n}{n!}\right)^2 \\ &= O(r||f||) \sum_{n=1}^{\infty} \frac{n}{r} \left(\frac{r^n}{n!}\right)^2 \\ &= O(r||f||) \left\{ \sum_{n/r \leqq 1} \frac{n}{r} \left(\frac{r^n}{n!}\right)^2 + \sum_{1 < n/r < n^2/r^2} \frac{n}{r} \left(\frac{r^n}{n!}\right)^2 \right\} \\ &= O(\sqrt{r}\, e^{2r} ||f||) \quad \text{by Lemma 1.}\end{aligned}$$

Lemma 5. Let $\alpha = \alpha_1 + i\alpha_2 = re^{i\theta}$, $\alpha \ngeqq 0$ and for $n \geqq 2$, let

$$F_n(\alpha) = \int_L \frac{e^{\lambda}}{\lambda^{n+1}} d\lambda,$$

where L is a half line given by either $\lambda = \alpha t$, $\pi \geqq |\theta| \geqq \pi/2$ or $\lambda = \alpha_1 + i\alpha_2 t$, $0 < |\theta| \leqq \pi/2$, t real $\geqq 1$.

Then, for some $K > 0$,

$$|F_n(\alpha)| \leqq K \left| \frac{e^{\alpha}}{n\alpha^n \theta} \right|.$$

Proof: Suppose $\lambda = \alpha t$, $|\theta| \geqq \pi/2$. Clearly

$$|e^{\alpha(t-1)}| = e^{r \cos \theta (t-1)} \leqq 1,$$

for $\cos \theta \leqq 0$ (because $|\theta| \geqq \pi/2$) and $t \geqq 1$. Hence

$$\begin{aligned}|F_n(\alpha)| &= \left| \frac{1}{\alpha^n} \int_1^{\infty} \frac{e^{\alpha t} dt}{t^{n+1}} \right| = \left| \frac{e^{\alpha}}{\alpha^n} \int_1^{\infty} \frac{e^{\alpha(t-1)}}{t^{n+1}} dt \right| \\ &\leqq \left| \frac{e^{\alpha}}{\alpha^n} \right| \int_1^{\infty} \frac{dt}{t^{n+1}} \\ &\leqq \left| \frac{e^{\alpha}}{n\alpha^n} \right| \leqq K \left| \frac{e^{\alpha}}{n\alpha^n \theta} \right|, \text{ since } |\theta| \leqq \pi.\end{aligned}$$

In the second case, suppose $\lambda = \alpha_1 + i\alpha_2 t$, $0 < |\theta| \leqq \pi/2$, t real $\geqq 1$. Then

$$
\begin{aligned}
|F_n(\alpha)| &= \left| \int_1^\infty \frac{e^{\alpha_1+i\alpha_2 t}}{(\alpha_1+i\alpha_2 t)^{n+1}} i\alpha_2 dt \right| \\
&= \left| \alpha_2 e^{\alpha} \int_1^\infty \frac{e^{i\alpha_2(t-1)}}{(\alpha_1+i\alpha_2 t)^{n+1}} dt \right| \\
&\leqq \left| \frac{e^\alpha}{\alpha_2} \right| \int_1^\infty \frac{\alpha_2^2 t dt}{(\alpha_1^2+\alpha_2^2 t^2)^{\frac{1}{2}(n+1)}} \\
&\leqq \left| \frac{e^\alpha}{\alpha_2} \right| \left[\frac{(\alpha_1^2+\alpha_2^2 t^2)^{\frac{1}{2}(-n+1)}}{-n+1} \right]_1^\infty \\
&\leqq K \left| \frac{e^\alpha}{n\alpha_2 \alpha^{n-1}} \right| \approx K \left| \frac{e^\alpha}{n\alpha^n \theta} \right|,
\end{aligned}
$$

because $\alpha_2 = |\alpha| \sin \theta \approx |\alpha|\,|\theta|$, if θ is small.

This establishes the lemma.

PROOF OF THEOREM 1. To show that for each $f \in E'$, $\{f(x_n)\}$ converges to 0, we consider

$$
e^\lambda g(\lambda) = \sum_{n=0}^{\infty} f(x_n) \frac{\lambda^n}{n!} = o(e^\lambda)
$$

for $|\lambda| \to \infty$ in $|\lambda| - \mathscr{R}\lambda < M$, $M > 0$ (cf. Lemma 3).

Put $\lambda = ne^{i\theta}$. Then by the Cauchy integral formula, we have

$$
\begin{aligned}
f(x_n) &= \frac{n!}{2\pi i} \int_{|\lambda|=n} \frac{e^\lambda g(\lambda) d\lambda}{\lambda^{n+1}} \\
&= \frac{n!}{2\pi i} \left[\int_{|\theta|<m/\sqrt{n}} + \int_{|\theta| \geqq m/\sqrt{n}} \right] \frac{e^\lambda g(\lambda)}{\lambda^{n+1}} d\lambda \\
&= \frac{n!}{2\pi i} [I_1 + I_2], \text{ say.}
\end{aligned}
$$

Integrating I_2 by parts and putting $G_n(\lambda) = \int (e^\lambda/\lambda^{n+1}) d\lambda$, we have

$$
\begin{aligned}
I_2 &= [g(\lambda)G_n(\lambda)]_{\theta=m/\sqrt{n}}^{\theta=2\pi-m/\sqrt{n}} - \int_{m/\sqrt{n}}^{2\pi-m/\sqrt{n}} G_n(\lambda) g'(\lambda) d\lambda \\
&= A - B, \text{ say.}
\end{aligned}
$$

By Lemma 5, (observe that $|G_n(\lambda)| \leqq |F_n(\lambda)|$),

$$B = O(1)\int_{m/\sqrt{n}}^{2\pi - m/\sqrt{n}} \left|\frac{\lambda e^{\lambda} g'(\lambda)}{n\lambda^n \theta}\right| d\theta$$

$$= O\left(\frac{1}{n^{n+1}}\right)\int_{m/\sqrt{n}}^{2\pi - m/\sqrt{n}} \left|\frac{\lambda e^{\lambda} g'(\lambda)}{\theta}\right| d\theta \quad (|\lambda| = n.)$$

$$= O\left(\frac{1}{n^{n+1}}\right)\left\{\int_{m/\sqrt{n}}^{2\pi - m/\sqrt{n}} |\lambda e^{\lambda} g'(\lambda)|^2\right\}^{\frac{1}{2}}\left\{\int_{m/\sqrt{n}}^{\pi} \frac{d\theta}{\theta^2}\right\}^{\frac{1}{2}} \quad \text{(by Parsevals' formula)}$$

$$= O\left(\frac{1}{n^{n+1}}\right)\{\sqrt{n}\, e^{2n}||f||\}^{\frac{1}{2}}\left\{\frac{\sqrt{n}}{m}\right\}^{\frac{1}{2}} \quad \text{(by Lemma 4)}$$

$$= O\left(\frac{||f||^{\frac{1}{2}}}{n^n e^{-n}\sqrt{n}}\right)\left(\frac{1}{\sqrt{m}}\right) \quad \text{uniformly in } n \text{ and } m.$$

Hence, by Stirling's formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$, we have

$$\frac{n!}{2\pi} B = O\left(\frac{||f||^{\frac{1}{2}}}{\sqrt{m}}\right).$$

Given $\varepsilon > 0$ and $f \in E'$ we can choose m large enough such that

$$\left|\frac{n!}{2\pi i} B\right| < \frac{\varepsilon}{3} \quad \text{for all } n.$$

Further, for λ's inside the parabola $|\lambda| - \mathscr{R}\lambda < M$, $M > 0$, we have $g(\lambda) = o(1)$ by Lemma 3. Hence for the fixed m chosen above and such that λ lies in the parabola $|\lambda| - \mathscr{R}\lambda < M$, we have (using Lemma 5)

$$\frac{n!}{2\pi i} A = o(1)(n!)\,[G_n(\lambda)]_{\theta = m\sqrt{n}}^{\theta = 2\pi - m/\sqrt{n}} = o(1)(n!)\left([F_n(\lambda)]_{\theta = m/\sqrt{n}}^{\theta = 2\pi - m/\sqrt{n}}\right)$$

$$= o(1)(n!)\frac{e^n\sqrt{n}}{n^{n+1}}\frac{1}{m} = o(1)\frac{n^n e^{-n}\sqrt{n}\, e^n\sqrt{n}}{n^{n+1}} = o(1),$$

as $n \to \infty$.

Also since $g(\lambda) = o(1)$ in $|\lambda| - \mathscr{R}(\lambda) < M$, and since m is fixed

$$\left|\frac{n!}{2\pi i} I_1\right| = o(1)(n!)\left(\frac{e^n n}{n^{n+1}}\right) O\left(\frac{1}{\sqrt{n}}\right) = o(1).$$

Thus for sufficiently large n, we have

$$\left|\frac{n!}{2\pi i} A\right| < \frac{\varepsilon}{3} \quad \text{and} \quad \left|\frac{n!}{2\pi i} I_1\right| < \frac{\varepsilon}{3}.$$

Hence $|f(x_n)| = |(n!/2\pi i)(I_1 + A - B)| < \varepsilon$ for sufficiently large n. This completes the proof.

THEOREM 2. Let E be a complex Banach space and E' its dual. Let $\{x_n\}$ be a sequence in E which is weakly Abel summable to 0. Suppose $||x_n - x_{n-1}|| = O(1/n)$ as $n \to \infty$. Then $\{x_n\}$ converges weakly to 0.

A simple way of proving this theorem is to show first that the Tauberian condition (T): $||x_n - x_{n-1}|| = O(1/n)$ implies that $||x_n|| = O(1)$ as $n \to \infty$. This follows exactly as in Lemma 2 by making appropriate changes. Next, we can show easily that if $\{x_n\}$ satisfies (T) and if for each $f \in E'$, $(1/n) \sum_{k=0}^{n} f(x_k) = o(1)$ as $n \to \infty$, then $\{x_n\}$ converges weakly to 0. Thus Theorem 2 would follow if we had showed that for any weakly Abel summable (to 0) sequence $\{x_n\}$ in E, $||x_n|| = O(1)$ as $n \to \infty$ implies that $\sum_{k=0}^{n} f(x_k) = o(n)$ as $n \to \infty$.

For this, as for Theorem 1, (cf. Lemma 3) we show that if $\{x_n\}$ is a bounded sequence $\{x_n\}$, then for each $f \in E'$,

$$h(\lambda) = (1-\lambda) \sum_{n=0}^{\infty} f(x_n)\lambda^n$$

is analytic for $|\lambda| < 1$, $h(\lambda) \to 0$ as $\lambda \to 1^-$ for λ inside $0 < |1-\lambda|/1-|\lambda|) < M$, $M > 0$.

Thus to prove Theorem 2, it is sufficient to prove the following so-called Abelian theorem.

THEOREM 3. If $\{x_n\}$ is a sequence in E such that $||x_n|| = O(1)$ as $n \to \infty$ and if for each $f \in E'$

$$h(\lambda) = \sum_{n=0}^{\infty} f(x_n)\lambda^n = o\left(\frac{1}{1-\lambda}\right)$$

for each $\lambda \to 1^-$ and $0 < (|1-\lambda|)/(1-|\lambda|) = O(1)$, then $f(x_n) = 0(n||f||)$.

The proof of Theorem 3 is exactly like that of Satz, 1[2] with appropriate changes and therefore omitted.

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