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On an integral equation in diffraction theory

by

Albert E. Heins

The following remarks are intended to shed further light on representation theorems for partial differential equations of a special form, in particular the two-dimensional wave equation. It has already been shown in the past decade that axially-symmetric boundary value problems of the Dirichlet or Neumann type for the three dimensional-wave equation for the disk or a disk between parallel planes may be formulated as a regular integral equation of the second kind. The advantage to such a formulation requires no further comment. These Fredholm integral equations of the second kind are derived from the Poisson representation for solutions of the wave equation and the analytically continued axis data of the Helmholtz representation. In a recent paper, J. Boersma [1] has given a similar representation for the two dimensional wave equation by accepting the three dimensional representation as a guide to construct the two dimensional one. We show here that this is again a result of the analytically continued data of the Helmholtz representation in two dimensions and the corresponding version of the Poisson representation theorem.

The Poisson representation theorem [3] for the equation

$$\phi_{xx}+\phi_{yy}+k^2\phi = 0 \tag{1}$$

has the form

$$\begin{aligned}\phi(x, y) = {} & \tfrac{1}{2}[\phi(x+iy, 0)+\phi(x-iy, 0)] \\ & -\frac{ky}{2}\int_0^{\pi} \phi(x+iy\cos\theta, 0)J_1(ky\sin\theta)d\theta \\ & +\frac{y}{2}\int_0^{\pi} \psi(x+iy\cos\theta)\sin\theta J_0(ky\sin\theta)d\theta\end{aligned} \tag{2}$$

where the term

$$\phi(x+iy, 0)+\phi(x-iy, 0)-\frac{ky}{2}\int_0^{\pi} \phi(x+iy\cos\theta, 0)J_1(ky\sin\theta)d\theta \tag{3}$$

is even in y and the term

$$(4) \qquad \frac{y}{2}\int_0^{\pi} \sin\,\theta J_0(ky\sin\,\theta)\psi(x+iy\cos\,\theta)d\theta$$

is odd in y. The symbols J_0 and J_1 refer to the usual notation for the regular and real Bessel functions of order zero and one respectively. Furthermore, (3) in addition to satisfying the equation (1) for $\phi(x, 0) \in C^2$, reduces to $\phi(x, 0)$ when $y = 0$ while $\partial\phi/\partial y$ vanishes at $y = 0$. On the other hand (4) vanishes at $y = 0$ while its y derivative reduces to $\psi(x)$ at $y = 0$. Hence (2) describes a regular initial-value problem with respect to the line $y = 0$. In particular if $\phi(0, y)$ vanishes for $|y| < b$, (2) implies that

$$\phi(iy, 0)+\phi(-iy, 0) = 0 \qquad |y| < b$$

and

$$\psi(iy)+\psi(-iy) = 0 \qquad |y| < b.$$

We shall make use of these remarks to provide integral equations of the type which Boersma described.

Let us now recall that the conventional Helmholtz representation for a plane wave incident upon a strip on which a Dirichlet condition is satisfied, that is, $\phi(0, y) = 0$, $|y| < b$ is given by

$$(5) \quad \phi(x, y) = \exp(ixk_x+iyk_y)+\frac{i}{4}\int_{-b}^{b} A(t)H_0^{(1)}[k\{x^2+(y-t)^2\}^{\frac{1}{2}}]dt$$

where $A(t)$ is the discontinuity of the x derivative of $\phi(x, y)$ on $x = 0$, $|y| < b$. $H_0^{(1)}$ is the customary symbol for the Hankel function of order zero and of the first kind, while the term $\exp(ixk_x+iyk_y)$ represents the incident plane wave. The boundary condition $\phi(0, y) = 0$, $|y| < b$ produces an integral equation of the Fredholm type and of the first kind. It is our goal to show that because of the symmetry of the strip about the line $y = 0$, that another integral equation may be found, although it does not appear to possess the simple form which was found in the axially-symmetric case [2]. Nevertheless, as Boersma has shown, it turns out to be useful.

We observe that (5) may be decomposed into the form $\phi(x, y) = \phi^e(x, y)+\phi^0(x, y)$ where $\phi^e(x, y)$ is even in y and $\phi^0(x, y)$ is odd in y. Here we have from (5) that

$$(6) \quad \begin{aligned} \phi^e(x, y) &= \exp(ixk_x)\cos(yk_y) \\ &+\frac{i}{8}\int_{-b}^{b} A(t)\{H_0^{(1)}[k(x^2+(y-t)^2)^{\frac{1}{2}}]+H_0^{(1)}[k(x^2+(y+t)^2)^{\frac{1}{2}}]\}dt \end{aligned}$$

and

$$\phi^0(x, y) = i \exp(ixk_x) \sin(yk_y)$$

(7)
$$+\frac{i}{8}\int_{-b}^{b} A(t)\{H_0^{(1)}[k(x^2+(y-t)^2)^{\frac{1}{2}}]-H_0^{(1)}[k(x^2+(y+t)^2)^{\frac{1}{2}}]\}dt.$$

In particular

(6a) $$\phi^e(x, 0) = \exp(ixk_x)+\frac{i}{4}\int_0^b [A(t)+A(-t)]H_0^{(1)}[k(x^2+t^2)^{\frac{1}{2}}]dt$$

and this is an analytic function of the complex variable $z = x+iy$ in the complex plane cut along the line $x = 0$. Indeed for $0 < y < b$, we have, if we take $(x^2+t^2)^{\frac{1}{2}}$ to be real and positive for x real, that $(z^2+t^2)^{\frac{1}{2}} = (t^2-y^2)^{\frac{1}{2}}$, $0 < y < t < b$, $x \to 0^{\pm}$. On the other hand $(z^2+t^2)^{\frac{1}{2}} = \exp(-i\pi/2)(y^2-t^2)^{\frac{1}{2}}$, $0 < t < y < b$ for $x \to 0^+$ and it is equal to $\exp(i\pi/2)(y^2-t^2)^{\frac{1}{2}}$, $0 < t < y < b$ for $x \to 0^-$. Furthermore if $y < 0$, we have that $(z^2+t^2)^{\frac{1}{2}} = (t^2-y^2)^{\frac{1}{2}}$, $0 < -y < t < b$, $x \to 0^{\pm}$ and it is equal to $\exp(i\pi/2)(y^2-t^2)^{\frac{1}{2}}$, $0 < t < -y < b$, $x \to 0^+$ and to $\exp(-i\pi/2)(y^2-t^2)^{\frac{1}{2}}$, $0 < t < -y < b$, $x \to 0^-$. Hence for $x \to 0^+$, $0 < y < b$, we get from (6a)

$$\phi^e(iy, 0) = \exp(-yk_x)+\frac{i}{4}\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]B(t)dt$$
$$+\frac{i}{4}\int_0^y \{2I_0[k(y^2-t^2)^{\frac{1}{2}}]+\frac{2}{i\pi}K_0[k(y^2-t^2)^{\frac{1}{2}}]\}B(t)dt$$

where $B(t) = A(t)+A(-t)$, $I_0(k\lambda)$ is the regular imaginary Bessel function of order zero and $K_0(k\lambda)$ is the MacDonald function of order zero. For $x \to 0^+$, $0 < -y < b$, we have

$$\phi^e(iy, 0) = \exp(-yk_x)+\frac{1}{2\pi}\int_0^{-y} K_0[k(y^2-t^2)^{\frac{1}{2}}]B(t)dt$$
$$+\frac{i}{4}\int_{-y}^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]B(t)dt.$$

Hence

$$\phi^e(iy, 0)+\phi^e(-iy, 0) = 2\cosh yk_x+\frac{i}{2}\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]B(t)dt$$
$$+\frac{i}{2}\int_0^y \{I_0[k(y^2-t^2)^{\frac{1}{2}}]+\frac{2}{\pi i}K_0[k(y^2-t^2)^{\frac{1}{2}}]\}B(t)dt.$$

If we were to repeat this calculation for $x \to 0^-$, $|y| < b$, we would get the same result. Observe now that $\phi^e(iy, 0)+\phi^e(-iy, 0) = 0$ from the Poisson representation and hence we have the type of integral equation that Boersma gave, that is

$$4\cosh yk_x+\int_0^y \{iI_0[k(y^2-t^2)^{\frac{1}{2}}]+\frac{2}{\pi}K_0[k(y^2-t^2)^{\frac{1}{2}}]\}B(t)dt$$
$$+i\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]B(t)dt = 0.$$

Unlike Boersma's result, $B(t)$ can be identified as the even part of the normal derivative of ϕ on $x = 0$, $|y| < b$.

Now we turn to the odd component. First it is noted that

$$2\left.\frac{\partial\phi^0}{\partial y}\right|_{y=0} = 2\psi(x) = 2i\exp(ixk_x)k_y - \frac{i}{2}\int_{-b}^b A(t)\frac{\partial H_0^{(1)}}{\partial t}[k(x^2+t^2)^{\frac{1}{2}}]dt$$

$$= 2ik_y\exp(ixk_x) - \frac{i}{2}\int_0^b C(t)\frac{\partial H_0^{(1)}}{\partial t}[k(x^2+t^2)^{\frac{1}{2}}]dt \tag{7a}$$

where $C(t) = A(t)-A(-t)$. Now (7a) may be rewritten as

$$2\psi(x) = 2ik_y\exp(ixk_x) - \frac{i}{2x}\frac{\partial}{\partial x}\int_0^b tC(t)H_0^{(1)}[k(x^2+t^2)^{\frac{1}{2}}]dt$$

or

$$\begin{aligned}2\int_0^x \lambda\psi(\lambda)d\lambda &= 2ik_y\exp(ixk_x)(x/ik_x+1/k_x^2)+\alpha\\ &\quad -\frac{i}{2}\int_0^b tC(t)H_0^{(1)}[k(x^2+t^2)^{\frac{1}{2}}]dt\end{aligned} \tag{8}$$

where α is a constant of integration. The constant can be evaluated in terms of $C(t)$ by noting that the left side of (8) vanishes when $x = 0$ and hence

$$\alpha = \frac{i}{2}\int_0^b tC(t)H_0^{(1)}(kt)dt - 2ik_y/k_x^2.$$

Now we continue (8) analytically into the domain of the complex variable $z = x+iy$ as we did in the even case. The new term is the integral on the left side of (8). For $0 < y < b$, we have

$$\int_0^{iy}\lambda\psi(\lambda)d\lambda = -\int_0^y \gamma\psi(i\gamma)d\gamma$$

so that for $x\to 0^+$, $0 < y < b$, we have

$$-2\int_0^y t\psi(it)dt = 2ik_y \exp(-yk_x)(y/k_x+1/k_x^2)+\alpha$$
$$-\frac{i}{2}\int_0^y \{2I_0[k(y^2-t^2)^{\frac{1}{2}}]+\frac{2}{\pi i}K_0[k(y^2-t^2)^{\frac{1}{2}}]\}tC(t)dt$$
$$-\frac{i}{2}\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]tC(t)dt$$

while for $x \to 0^+$, $0 < -y < b$, we have

$$\begin{aligned}-2\int_0^y t\psi(it)dt &= 2ik_y \exp(-yk_x)(y/k_x+1/k_x^2)+\alpha\\ &-\frac{i}{2}\int_{-y}^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]tC(t)dt\\ &-\frac{i}{2}\int_0^{-y}\frac{2}{\pi i}K_0[k(y^2-t^2)^{\frac{1}{2}}]tC(t)dt.\end{aligned}\tag{10}$$

Upon replacing $-y$ by y in equation (10) we get

$$\begin{aligned}-2\int_0^y t\psi(-it)dt &= 2ik_y \exp(yk_x)(-y/k_x+1/k_x^2)+\alpha\\ &-\frac{i}{2}\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]tC(t)dt\\ &-\frac{1}{\pi}\int_0^y K_0[k(y^2-t^2)^{\frac{1}{2}}]tC(t)dt.\end{aligned}\tag{11}$$

Hence upon addition of (10) and (11) we get

$$-2\int_0^y t[\psi(it)+\psi(-it)]dt = 2ik_y[-2y(\sinh yk_x)/k_x+2(\cosh yk_x)/k_x^2]$$
$$+2\alpha-i\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]tC(t)dt$$
$$-i\int_0^y \{I_0[k(y^2-t^2)^{\frac{1}{2}}]+\frac{2}{\pi i}K_0[k(y^2-t^2)^{\frac{1}{2}}]\}tC(t)dt.$$

Now we know from the odd part of the Poisson representation that $\psi(it)+\psi(-it) = 0$ and hence we are left with the following integral equation

$$4ik_y[-y(\sinh yk_x)/k_x+(\cosh yk_x)/k_x^2]$$
$$+2\alpha-i\int_y^b H_0^{(1)}[k(t^2-y^2)^{\frac{1}{2}}]tC(t)dt$$
$$-i\int_0^y \{I_0[k(y^2-t^2)^{\frac{1}{2}}]+\frac{2}{\pi i}K_0[k(y^2-t^2)^{\frac{1}{2}}]\}tC(t)dt = 0.$$

Neumann boundary conditions may be dealt with in a similar fashion.

If instead of a plane wave term in (5), we had a term which could not be continued into the domain of the complex variable $z = x+iy$, for example a line source term, we could redefine $\phi(x, y)$ and proceed as before. Suppose that the source term is arbitrary and has the form $\phi(x, y) = \phi_i^e(x, y)+\phi_i^e(x, y)$ where $\phi_i^e(x, y)$ and $\phi_i^e(x, y)$ are the even and odd parts of $\phi_i(x, y)$. Then if we put $\phi(x, y)-\phi_i(x, y) = \Phi(x, y)$, we have that $\Phi(0, y) = \phi_i(0, y)$, $|y| < b$. The Poisson representation is now inhomogeneous and may be solved as an integral equation to produce $\phi(iy, 0)+\phi(-iy, 0)$ or $\psi(iy)+\psi(-iy)$. The analytic continuation of the modified Helmholtz representation proceeds as before.

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REFERENCES

J. Boersma

[1] Boundary value problems in diffraction theory and lifting surface theory, Compositio Mathematica, 16, 205–293 (1964).

A. E. Heins and R. C. MacCamy

[2] On the scattering of waves by a disk, Z. Angew. Math. Phys. 11, 249–264 (1960).

P. Henrici

[3] Zur Funktionentheorie der Wellengleichung, Comment. Math. Helv. 27, 235–293 (1953).

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