A. F. RUSTON

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by

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1. Introduction

In a well-known paper [31], F. Riesz studied the behaviour of compact (= completely continuous) linear operators on the Banach space of continuous functions on a closed interval. His arguments can readily be applied in any Banach space, real or complex, and the Riesz theory has been discussed by many writers (see, for instance, [1] pp. 151—157, [3, 5, 15, 23, 30], [32] pp. 178—182, [36, 51, 52, 53, 54, 55] and [56] pp. 330—344; see also [13] pp. 577—580).

The classical Fredholm theory, as expounded for instance in [14][2], expresses the solution of an integral equation such as

$$x(s) = y(s) + \lambda \int_a^b k(s, t)x(t)dt \qquad (a \leqq s \leqq b),$$

where y and k are given continuous functions and the continuous function x is to be determined, in terms of the Fredholm minors

$$d \begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} \lambda \Bigg),$$

which (for fixed $s_1, s_2, \ldots, s_n$; $t_1, t_2, \ldots, t_n$) are integral functions of λ. This theory, like the Riesz theory, has been placed in a more general setting by a number of writers (see, for instance, [17, 22, 24, 25, 29, 33, 34, 35, 40, 41, 42, 43, 44, 45, 46, 47], [48] pp. 79—105 and [56] pp. 261—278).

Use has already been made (cf. [35] and [36]) of the Riesz theory in discussing the Fredholm theory. The purpose of the present paper is to delve more deeply into the relation between

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[1] A sketch of part of the theory given in this paper, for the space of continuous functions on a closed interval, was presented (under the same title) before the International Congress of Mathematicians 1954 at Amsterdam (cf. [37]).

[2] Other references are given in [34]. See also [48].

the two theories. In particular, we shall establish a relation [3] between the dimension numbers of certain subspaces occurring in the Riesz theory and the orders of the corresponding zero of certain Fredholm formulae (Theorem 3.4; cf. [37]), and we shall identify the null space and nucleus space of the Riesz theory with the range and kernel of a certain operator in the Fredholm theory (Theorem 4.2).

Throughout this paper, the Banach space $\mathfrak{B}$ under consideration can be either real or complex.

2. Fredholm formulae

As in previous papers, we shall make constant use of the theory of n-operators developed in [34] (which was based on Schatten's notion of direct product — cf. [39]). An n-operator on a Banach space $\mathfrak{B}$ was defined ([34], p. 352) to be a continuous linear [4] operator on $\mathfrak{B}_\gamma^n$ into $\mathfrak{B}_\lambda^n$. This definition was chosen because it led to a simple formula for the bound-norm ([34] Theorem 3.1.1, p. 352), which enabled us to obtain some important inequalities ([34] § 3.7, pp. 370—376) used in proving the convergence of certain series ([34] Theorem 3.8.1, p. 376). In later work ([35] and [36]) a different method was used to prove the convergence of the series concerned. For this later work, we could [5] have used (for instance) continuous linear operators on $\mathfrak{B}_\gamma^n$ into $\mathfrak{B}_\gamma^n$, or continuous linear operators on $\mathfrak{B}_\lambda^n$ into $\mathfrak{B}_\lambda^n$. To avoid confusion, however, I shall continue to use n-operators as defined above. For further information on the theory of n-operators, the reader is referred to [34] (see also [38]).

DEFINITION 2.1 [6]. *A continuous linear operator K on a Banach space $\mathfrak{B}$ into itself will be called a Fredholm operator on $\mathfrak{B}$ iff* [7] *there is a scalar integral function $\Delta_0(\lambda) = \sum_{r=0}^{\infty} \Delta_0^r \lambda^r$ of the scalar*

[3]) I am indirectly indebted to Professor D. E. Littlewood in this connection. It was a (somewhat hazy) recollection of his Part III lectures at Cambridge which first put me on to the nature of this relation.

[4]) In my earlier papers (e.g. [33—38]) I used the word "linear" (following Banach, cf. [1] p. 23, [32] p. 149) to imply continuity. More recently I have come into line with most modern writers, and use it in a purely algebraic sense (cf. [13] pp. 36—37, [21] p. 16, [49] p. 18, [56] p. 134).

[5]) I am indebted to A. M. Deprit for drawing my attention to this fact.

[6]) Cf. [36] Definition 2.1, p. 319.

[7]) Following Halmos, I use "iff" in a definition where the meaning is "if and only if".

λ, *not identically zero, such that* $\sum_{r=0}^{\infty} \Delta_n^r \lambda^r$ *is an absolutely convergent series of n-operators for every scalar* λ *and every non-negative integer* n, *where* [8] $\Delta_n^r = \sum_{s=0}^{r} \Delta_0^{r-s} K_n^{s+1}$. *The integral function* $\Delta_0(\lambda)$ *will be called a Fredholm determinant for K, and the n-operators* $\{\Delta_n^r\}$ *will be called the Fredholm coefficients corresponding to* $\Delta_0(\lambda)$.

REMARK. If $\mathfrak{B}$ is n-dimensional, and K is represented by the $n \times n$ matrix κ, we can take $\Delta_0(\lambda) = \det (I_n - \lambda\kappa)$ (cf. [34] p. 365). The Cayley-Hamilton theorem [9] then tells us that $\sum_{r=0}^{n} \Delta_0^r \kappa^{n-r} = 0$, from which it follows that $\sum_{r=0}^{n} \Delta_0^r K^{n-r} = \Theta$ (the zero operator). For a Fredholm operator K with Fredholm determinant $\Delta_0(\lambda)$ on a general Banach space $\mathfrak{B}$, we have (by Cauchy's test) a generalization of the Cayley-Hamilton theorem, namely that [10]

$$|| \sum_{r=0}^{n} \Delta_0^r K^{n-r}||^{1/n} \to 0 \text{ as } n \to \infty.$$

From the absolute convergence of $\sum_{r=0}^{\infty} \Delta_n^r \lambda^r$ for any scalar λ follows at once (by the comparison test) the absolute convergence of the series

$$\sum_{s=0}^{\infty} \frac{(r+s)!}{r!s!} \Delta_n^{r+s} \lambda^s$$

for any non-negative integer r and any scalar λ.

DEFINITION 2.2. *Let K be a Fredholm operator on a Banach space* $\mathfrak{B}$, *and let* $\Delta_0(\lambda)$ *be a Fredholm determinant for K. Then we define*

$$\Delta_n^r(\lambda) = \sum_{s=0}^{\infty} \frac{(r+s)!}{r!s!} \Delta_n^{r+s} \lambda^s,$$

where $\{\Delta_n^r\}$ *are the Fredholm coefficients corresponding to* $\Delta_0(\lambda)$. *We call* $\{\Delta_n^r(\lambda)\}$ *the Fredholm formulae corresponding to* $\Delta_0(\lambda)$.

Clearly $\Delta_n^r(\lambda)$ is a skew n-operator ([34] Definition 3.1.1, p. 353) for any scalar λ, $\Delta_n^r(0) = \Delta_n^r$, and $\Delta_0^0(\lambda) = \Delta_0(\lambda)$ for any scalar λ.

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[8]) We recall that

$(f^{(1)} \otimes f^{(2)} \otimes \ldots \otimes f^{(n)}) K_n^{s+1} (x^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(n)}) = \sum \det [f^{(i)}(K^{\mu_i} x^{(j)})]$,

where summation is over all sets of positive integers $\mu_1, \mu_2, \ldots, \mu_n$ with $\mu_1 + \mu_2 + \ldots + \mu_n = n+s$ (cf. [34] Theorem 3.1.4, p. 356).

[9]) Cf. [2] p. 320, [13] p. 562, [18] p. 169, [19] p. 106, [26] p. 105, [27] p. 18 and [28] p. 206.

[10]) Here I use $||.||$ to denote the bound-norm (of a continuous linear operator), denoted in some of my earlier papers by $\beta(.)$.

Elsewhere we have called $\varDelta_n^0(\lambda)$ a *Fredholm minor*, and have denoted it by $\varDelta_n(\lambda)$ (cf. [35] Definition 2.3, p. 372). In this paper we shall frequently have occasion to consider the Fredholm formulae for other values of r, and the value $r = 0$ will not in general be singled out for special treatment.

NOTE. In an alternative treatment, one could consider $\{r!\varDelta_n^r\}$ in place of the Fredholm coefficients, and $\{r!\varDelta_n^r(\lambda)\}$ in place of the Fredholm formulae. The above equation would then be replaced by a slightly simpler one (say $\mathfrak{D}_n^r(\lambda) = \sum_{s=0}^{\infty} \mathfrak{D}_n^{r+s}\lambda^s/s!$). To avoid confusion, however, I shall continue to use the notation I have used elsewhere.

The fundamental properties of the Fredholm formulae on which the calculations in this paper are based are given in the next two theorems, which are immediate consequences of the definitions (cf. [34] Theorems 3.8.2 and 3.8.5, pp. 377 and 379).

THEOREM 2.1.

$$\varDelta_n^0(\lambda) = \sum_{r=0}^{\infty} (\lambda-\lambda_0)^r \varDelta_n^r(\lambda_0)$$

for any scalar λ_0, the series on the right being absolutely convergent in bound-norm for every scalar λ.

THEOREM 2.2. *If $n \geqq 1$ and $r \geqq 1$, then*

(i) $(I-\lambda K)\{1\}\varDelta_n^0(\lambda) = K \wedge \varDelta_{n-1}^0(\lambda)$;

(ii) $\varDelta_n^0(\lambda)\{1\}(I-\lambda K) = K \vee \varDelta_{n-1}^0(\lambda)$;

(iii) $(I-\lambda K)\{1\}\varDelta_n^r(\lambda) = K\{1\}\varDelta_n^{r-1}(\lambda)+K \wedge \varDelta_{n-1}^r(\lambda)$;

(iv) $\varDelta_n^r(\lambda)\{1\}(I-\lambda K) = \varDelta_n^{r-1}(\lambda)\{1\}K+K \vee \varDelta_{n-1}^r(\lambda)$.

We may note, in passing, that (i) and (ii) of this theorem can be expressed in the form (iii) and (iv) by putting, conventionally, $\varDelta_n^r(\lambda) = \varTheta$ when $r < 0$.

We shall be concerned with relations between the Fredholm formulae with particular reference to a fixed (but arbitrary) value λ_0 of the parameter. Considerable light will be thrown on these relations by our discussion of the Riesz theory, but before starting that discussion we prepare the ground.

We shall be interested, in particular, in the order of λ_0 as a zero of the integral function $\varDelta_n^0(\lambda)$ of the scalar λ. In view of Theorem 2.1, this order can be expressed in terms of the vanishing of the Taylor coefficients $\varDelta_n^r(\lambda_0)$. I now introduce some notations to describe the situation (for brevity, the dependence on λ_0 will not be made explicit).

DEFINITION 2.3. *For each integer* $n \geqq 0$, *we define the number* $p(n)$ *to be the smallest integer p for which* $\Delta_n^p(\lambda_0) \neq \Theta$, *with the convention that, when no such integer p exists, then* $p(n) = \infty$.

Thus $p(n)$ is the order referred to above.

In order to study the variation of $p(n)$ with n, we introduce some further notation.

DEFINITION 2.4. *For every integer* $n \geqq 0$, *we define the number* $q(n)$ *to be* $p(n-1)-p(n)$, *with the conventions* [11] *that* $q(0) = \infty$ *and that* $q(n) = -\infty$ *when* $p(n) = \infty$.

It is convenient to represent the situation diagrammatically. The Fredholm formulae (for the parameter λ_0) can be conveniently arranged in a doubly infinite array:

$$
\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\Delta_3^0(\lambda_0) & \Delta_3^1(\lambda_0) & \Delta_3^2(\lambda_0) & \Delta_3^3(\lambda_0) & \cdots \\
\Delta_2^0(\lambda_0) & \Delta_2^1(\lambda_0) & \Delta_2^2(\lambda_0) & \Delta_2^3(\lambda_0) & \cdots \\
\Delta_1^0(\lambda_0) & \Delta_1^1(\lambda_0) & \Delta_1^2(\lambda_0) & \Delta_1^3(\lambda_0) & \cdots \\
\Delta_0^0(\lambda_0) & \Delta_0^1(\lambda_0) & \Delta_0^2(\lambda_0) & \Delta_0^3(\lambda_0) & \cdots \cdot
\end{array}
$$

Here the formulae $\{\Delta_n^r(\lambda_0)\}$ in any row are the Taylor coefficients of the corresponding Fredholm minor $\Delta_n^0(\lambda)$, higher rows corresponding to Fredholm minors of higher order. Then the orders of the zeros can be represented by putting a white spot to represent $\Delta_n^r(\lambda_0)$ for $r < p(n)$ (when $\Delta_n^r(\lambda_0) = \Theta$), and possibly a black spot to represent $\Delta_n^{p(n)}(\lambda_0)$ (which does not vanish) if $p(n)$ is finite.

On the face of it, the diagram so obtained might be infinite in extent. However, it has certain properties which enable us to concentrate our attention on a finite part of it. These properties we now discuss (others will appear when we discuss the Riesz theory). To this end I now prove two lemmas.

LEMMA 2.1. *If* $X \wedge A = \Theta$, *where A is an n-operator and X a 1-operator, then either* $A = \Theta$ *or X is of rank at most n.*

Let us suppose that $X \wedge A = \Theta$, but that $A \neq \Theta$. Then we can choose elements $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ of $\mathfrak{B}$ and continuous linear functionals $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ on $\mathfrak{B}$ so that

$$(f^{(1)} \otimes f^{(2)} \otimes \ldots \otimes f^{(n)}) A (x^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(n)}) \neq 0.$$

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[11] These conventions are consistent, since by hypothesis $p(0)$ is finite (cf. Definition 2.1).

Then, for any $x \in \mathfrak{B}$, we have

$$\Theta = ((1^*) \otimes f^{(1)} \otimes \ldots \otimes f^{(n)})[X \wedge A](x \otimes x^{(1)} \otimes \ldots \otimes x^{(n)})$$
$$= (f^{(1)} \otimes \ldots \otimes f^{(n)})A(x^{(1)} \otimes \ldots \otimes x^{(n)}) \cdot Xx$$
$$+ \sum_{r=1}^{n} (-1)^r (f^{(1)} \otimes \ldots \otimes f^{(n)})$$
$$\cdot A(x \otimes x^{(1)} \otimes \ldots \otimes x^{(r-1)} \otimes x^{(r+1)} \otimes \ldots \otimes x^{(n)}) \cdot Xx^{(r)},$$

and so Xx is a linear combination of $Xx^{(1)}, Xx^{(2)}, \ldots, Xx^{(n)}$. It follows that the range of X is spanned by $Xx^{(1)}, Xx^{(2)}, \ldots, Xx^{(n)}$, and so that X is of rank at most n. In fact (using the notation of [33] p. 110)

$$X = \sum_{r=1}^{n} (Xx^{(r)}) \otimes$$

$$\left[\frac{(f^{(1)} \otimes \ldots \otimes f^{(n)})A(x^{(r)} \otimes \ldots \otimes x^{(r-1)} \otimes (1) \otimes x^{(r+1)} \otimes \ldots \otimes x^{(n)})}{(f^{(1)} \otimes \ldots \otimes f^{(n)})A(x^{(1)} \otimes \ldots \otimes x^{(n)})} \right].$$

NOTE. The condition is not sufficient, as can be seen by taking X to be a 1-operator of rank precisely unity and A to be a 1-operator of rank greater than unity.

LEMMA 2.2. $X_{n+1}^r = \Theta$ *for all* $r \geqq 1$ *if (and only if)* X *is of rank at most* n.

By [34] Theorem 3.1.5 (p. 356) $X_{n+1}^1 = \Theta$ if and only if X is of rank at most n. But, if $X_{n+1}^1 = \Theta$, then

$$X^{\mu_1} \wedge X^{\mu_2} \wedge \ldots \wedge X^{\mu_{n+1}}$$
$$= X^{\mu_1 - 1}\{1\}(X^{\mu_2 - 1}\{2\}(\ldots (X^{\mu_{n+1} - 1}\{n+1\}X_{n+1}^1) \ldots))$$
$$= \Theta$$

for any set of positive integers $\mu_1, \mu_2, \ldots, \mu_{n+1}$, and so (by [34] Theorem 3.1.4, p. 356) $X_{n+1}^r = \Theta$ for any positive integer r.

THEOREM 2.3. *If* $q(n) < 0$, *then* $q(m) = -\infty$ *for every* $m \geqq n$.

In other words, $p(n) \leqq p(n-1)$ unless $p(n) = \infty$, and then $p(m) = \infty$ for every $m \geqq n$.

Let us suppose that $q(n) < 0$, and let n_0 be the *smallest* positive integer with $q(n_0) < 0$ (so that $n \geqq n_0$). Then $p(n_0-1)$ must be finite (else $q(n_0-1) = -\infty < 0$). Since $q(n_0) < 0$, we have $p(n_0) > p(n_0-1)$, and so $\Delta_{n_0}^r(\lambda_0) = \Theta$ when $r \leqq p(n_0-1)$. It follows (by Theorem 2.2 (iii) — or (i) if $p(n_0-1) = 0$) that

$$K \wedge \Delta_{n_0-1}^{p(n_0-1)}(\lambda_0) = \Theta.$$

But, by definition of $p(n_0-1)$,

$$\Delta_{n_0-1}^{p(n_0-1)}(\lambda_0) \neq \Theta.$$

Hence, by Lemma 2.1, K is of rank at most n_0-1. It now follows, by Lemma 2.2, that $K_m^r = \Theta$ whenever $m \geq n_0$, $r \geq 1$. Hence $\Delta_m^0(\lambda)$ is identically zero for all $m \geq n_0$, and so for all $m \geq n$. Thus $p(m) = \infty$ and $q(m) = -\infty$ for all $m \geq n$.

NOTE. The rank of K is, in fact, precisely n_0-1, else we should have $q(n_0-1) = -\infty < 0$ by the above argument.

We now introduce two more numbers.

DEFINITION 2.5. *We define* μ *to be the smallest non-negative integer such that* $\Delta_n^\mu(\lambda_0) \neq \Theta$ *for some non-negative integer* n, *and we define* d *to be the least such integer* n.
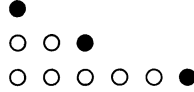
Cf. [34] Theorem 4.2.1, p. 380. Clearly $\mu = p(d)$.

The number d is known in German as the "Defekt" of λ_0. In the past I have translated this "defect" (cf. [34] loc. cit., [35] Theorem 2.4, p. 373), but "deficiency" would be a more idiomatic translation. The number μ is of the nature of a "coefficient of irrelevancy", since the integral functions $\Delta_0^0(\lambda)$, $\Delta_1^0(\lambda)$, $\Delta_2^0(\lambda)$, ... have a common factor $(\lambda-\lambda_0)^\mu$, which could be divided out and contributes nothing to the Fredholm theory (indeed we could do this simultaneously for *all* scalar λ_0 by dividing by a suitable scalar integral function of λ). It will be observed that $\mu = 0$ for the formulae "constructed" in [35] (see [35] Theorem 2.4, p. 373) [12]. It is still an open question whether the same is always true for the formulae constructed for operators in the trace class (cf. [34] Corollary to Theorem 4.2.1, p. 381; see also [16] Chap. I § 5, pp. 164—191).

We can now see how it is that we can concentrate our attention on a finite part of the diagram of spots mentioned above. For, in the first place, either $q(n) = 0$ for all $n > d$, or there is an integer $r_0 \geq d$ such that $q(n) = 0$ for $d < n \leq r_0$ and $q(n) = -\infty$ for $n > r_0$ (this can be proved by induction, using Theorem 2.3 and the definition of d). In the first case K is not of finite rank; in the second case it is of rank r_0. Thus the "shape" of the part of the diagram of spots corresponding to $n > d$ depends only on the rank of K, and is of no great interest in connection with the Riesz theory. Thus we need only concern ourselves with the part of the diagram corresponding to $n \leq d$, for which values of

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[12]) It is also true (in effect) for the classical Fredholm theory.

n (again appealing to Theorem 2.3) $q(n) \geqq 0$ (of course $q(d) > 0$). In the second place, the diagram includes μ columns of white spots, which are (as we have seen) not really significant, and these can be omitted. What is left is that part of the diagram of spots corresponding to $n \leqq d$ and $r \geqq \mu$. I call this the *spot diagram*. This will be a diagram such as

$$\bullet$$
$$\circ \ \circ \ \bullet$$
$$\circ \ \circ \ \circ \ \circ \ \bullet$$

It will have $d+1$ rows; the bottom row (which we call row 0 since it corresponds to $n = 0$) will have $p(0)-\mu$ white spots, the next row (row 1) will have $p(1)-\mu$ white spots, and so on.

3. The Riesz theory

We shall continue to concentrate our attention on a fixed scalar λ_0, which will not be mentioned explicitly in our notations.

If T is a continuous linear operator on $\mathfrak{B}$ into itself, then we call $T^{-1}(\Theta)$ (the set of solutions x of the equation $Tx = \Theta$) the *kernel* of T (following common usage in algebra [13] — I have elsewhere called this the "zerospace" of T), and we call $T\mathfrak{B}$ (the set of values taken by Tx for x in $\mathfrak{B}$) the *range* of T.

DEFINITION 3.1. *For each integer $n \geqq 0$, we denote by $\mathfrak{M}_n$ the kernel, and by $\mathfrak{R}_n$ the range, of the operator $(I - \lambda_0 K)^n$.*

We have seen elsewhere ([36] Lemma 2.1, p. 320) that $\mathfrak{M}_n$ is finite-dimensional for every non-negative integer n (this will also follow from Lemma 3.2). The principal aim of this paper is to show how the structure of these spaces is related to the Fredholm formulae, and in particular how the number of dimensions of $\mathfrak{M}_n$ can be read off from the spot diagram.

DEFINITION 3.2. *For each integer $n \geqq 0$, we denote by m_n the number of dimensions of $\mathfrak{M}_n$.*

It has been known for some time that [14] $m_1 = d$, that is the number of white spots in the first column of the spot diagram, and that [15] $m_\nu = p(0)-\mu$, that is the number of white spots in

[13]) Cf. [2] p. 151, [13] p. 39, [21] p. 470. This use of the word "kernel" should not be confused with that in connection with integral equations, cf. [48] p. 2, [56] p. 177.

[14]) Cf. [34] Theorem 4.2.2, p. 381.

[15]) This lay behind the argument in [35].

the bottom row of the spot diagram (where ν is the *index* in the Riesz theory — cf. [31] p. 81, [32] p. 179 — such that $m_n = m_\nu$ when $n \geqq \nu$ and $m_n < m_{n+1}$ when $n < \nu$). It was these facts that led me to suspect a connection between the dimension numbers $\{m_n\}$ and the spot diagram.

The main tools we shall use in our investigations are given in Lemmas 3.1 and 3.2.

Let n be any integer with $1 \leqq n \leqq d$, and let $p = p(n)$, $q = q(n)$ and $q' = q(n+1)$. By definition of $p(n)$, $\varDelta_n^p(\lambda_0) \neq \Theta$. Let us then choose elements $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ of $\mathfrak{B}$ and continuous linear functionals $f_1^{(n)}, f_2^{(n)}, \ldots, f_n^{(n)}$ on $\mathfrak{B}$ so that [16]

$$(f_1^{(n)} \otimes f_2^{(n)} \otimes \ldots \otimes f_n^{(n)})\varDelta_n^p(\lambda_0)(x_1^{(n)} \otimes x_2^{(n)} \otimes \ldots \otimes x_n^{(n)}) = 1.$$

Then we put

$$\eta_\theta^{(n)} = (f_1^{(n)} \otimes \ldots \otimes f_{\theta-1}^{(n)} \otimes (1^*) \otimes f_{\theta+1}^{(n)} \otimes \ldots \otimes f_n^{(n)})$$
$$\times \varDelta_n^p(\lambda_0)(x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$$

and

$$\xi_\theta^{(n)} = \lambda_0^{q-1}(f_1^{(n)} \otimes \ldots \otimes f_{\theta-1}^{(n)} \otimes (1^*) \otimes f_{\theta+1}^{(n)} \otimes \ldots \otimes f_n^{(n)})$$
$$\times \varDelta_n^{p+q-1}(\lambda_0)(x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$$

for $\theta = 1, 2, \ldots, n$ (cf. [34] p. 381, [35] p. 374).

LEMMA 3.1 *With the above notation,*

$$(I - \lambda_0 K)^{q-r}\xi_1^{(n)}, \ (I - \lambda_0 K)^{q-r}\xi_2^{(n)}, \ldots, \ (I - \lambda_0 K)^{q-r}\xi_n^{(n)}$$

are elements of $\mathfrak{M}_r$ *linearly independent modulo* [17] $\mathfrak{M}_{r-1}$ *whenever* $1 \leqq r \leqq q$.

We know that

$$\varDelta_{n-1}^r(\lambda_0) = \Theta$$

whenever $r < p+q = p(n-1)$. It follows, by repeated application of Theorem 2.2 (iii), that

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[16] It would be sufficient to arrange for this expression to be non-zero, but making it unity simplifies our calculations. Note, however, that, when this argument is adapted for the classical Fredholm theory, we must be content with arranging for the corresponding "kernel"

$$d_n^p \begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} \lambda_0 \bigg)$$

to be non-zero (the duplicated notes for my talk at Amsterdam [cf. 37] require amendment accordingly).

[17] That is to say $\sum_{\theta=1}^n \alpha_\theta (I - \lambda_0 K)^{q-r}\xi_\theta^{(n)} \in \mathfrak{M}_{r-1}$ only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

$$(I-\lambda_0 K)^{q-1}\{1\}\Delta_n^{p+q-1}(\lambda_0)$$
$$= (I-\lambda_0 K)^{q-2}K\{1\}\Delta_n^{p+q-2}(\lambda_0)$$
$$= \ldots$$
$$= K^{q-1}\{1\}\Delta_n^p(\lambda_0),$$

and so that

$$(I-\lambda_0 K)^{r-1} \cdot (I-\lambda_0 K)^{q-r}\xi_\theta^{(n)}$$
$$= (-1)^{\theta-1}\lambda_0^{q-1}$$
$$\times ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_{\theta-1}^{(n)} \otimes f_{\theta+1}^{(n)} \otimes \ldots \otimes f_n^{(n)})$$
$$\times [(I-\lambda_0 K)^{q-1}\{1\}\Delta_n^{p+q-1}(\lambda_0)](x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$$
$$= (-1)^{\theta-1}\lambda_0^{q-1}$$
$$\times ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_{\theta-1}^{(n)} \otimes f_{\theta+1}^{(n)} \otimes \ldots \otimes f_n^{(n)})$$
$$\times [K^{q-1}\{1\}\Delta_n^p(\lambda_0)](x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$$
$$= (\lambda_0 K)^{q-1}\eta_\theta^{(n)}.$$

But

$$(I-\lambda_0 K)\eta_\theta^{(n)}$$
$$= (-1)^{\theta-1}$$
$$\times ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_{\theta-1}^{(n)} \otimes f_{\theta+1}^{(n)} \otimes \ldots \otimes f_n^{(n)})$$
$$\times [(I-\lambda_0 K)\{1\}\Delta_n^p(\lambda_0)](x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$$
$$= \Theta$$

(that is $\eta_\theta^{(n)} \in \mathfrak{M}_1$), and so $\lambda_0 K\eta_\theta^{(n)} = \eta_\theta^{(n)}$. Hence

$$(I-\lambda_0 K)^{r-1} \cdot (I-\lambda_0 K)^{q-r}\xi_\theta^{(n)} = \eta_\theta^{(n)},$$

and

$$(I-\lambda_0 K)^r \cdot (I-\lambda_0 K)^{q-r}\xi_\theta^{(n)} = \Theta,$$

that is

$$(I-\lambda_0 K)^{q-r}\xi_\theta^{(n)} \in \mathfrak{M}_r.$$

Now

$$f_\varphi^{(n)}[(I-\lambda_0 K)^{r-1} \cdot (I-\lambda_0 K)^{q-r}\xi_\theta^{(n)}] = f_\varphi^{(n)}(\eta_\theta^{(n)}) = \delta_{\theta\varphi}$$

for $\varphi = 1, 2, \ldots, n$. It follows [18] that

[18]) If

$$\sum_{\theta=1}^n \alpha_\theta(I-\lambda_0 K)^{q-r}\xi_\theta^{(n)} \in \mathfrak{M}_{r-1},$$

then

$$0 = f_\varphi^{(n)}[(I-\lambda_0 K)^{r-1} \cdot \sum_{\theta=1}^n \alpha_\theta(I-\lambda_0 K)^{q-r}\xi_\theta^{(n)}] = \alpha_\varphi$$

for $\varphi = 1, 2, \ldots, n$.

$$(I-\lambda_0 K)^{q-r}\xi_1^{(n)}, \ (I-\lambda_0 K)^{q-r}\xi_2^{(n)}, \ldots, (I-\lambda_0 K)^{q-r}\xi_n^{(n)}$$

are linearly independent modulo $\mathfrak{M}_{r-1}$. The lemma is thus proved.
The corresponding result for $n = 0$ is trivial.

COROLLARY 1. *If* $1 \leqq r \leqq q(n)$, *then* $m_r - m_{r-1} \geqq n$.

COROLLARY 2. $\Delta_n^r(\lambda_0) \neq \Theta$ *when* $p(n) \leqq r < p(n-1)$.
Thus the spot diagram may be augmented by adding a black spot above any white spot which has not already a spot above it. These black spots correspond to formulae which cannot vanish.

LEMMA 3.2. *With the above notation,* $\mathfrak{M}_1 \cap \mathfrak{R}_{r-1}$ *is contained in the subspace of* $\mathfrak{B}$ *spanned by* $\eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_n^{(n)}$ *whenever* $r > q'$.
Let $(I-\lambda_0 K)^{r-1}x$ be any element of $\mathfrak{M}_1 \cap \mathfrak{R}_{r-1}$. Then $(I-\lambda_0 K)^r x = \Theta$. But, by repeated application of Theorem 2.2 (iv), we have

$$
\begin{aligned}
\Delta_{n+1}^p(\lambda_0)&\{1\}(I-\lambda_0 K)^r \\
&= [K \vee \Delta_n^p(\lambda_0)]\{1\}(I-\lambda_0 K)^{r-1} + \Delta_{n+1}^{p-1}(\lambda_0)\{1\}K(I-\lambda_0 K)^{r-1} \\
&= [K \vee \Delta_n^p(\lambda_0)]\{1\}(I-\lambda_0 K)^{r-1} + \Delta_{n+1}^{p-2}(\lambda_0)\{1\}K^2(I-\lambda_0 K)^{r-2} \\
&= \ldots \\
&= [K \vee \Delta_n^p(\lambda_0)]\{1\}(I-\lambda_0 K)^{r-1} + \Delta_{n+1}^{p-q'}(\lambda_0)\{1\}K^{q'}(I-\lambda_0 K)^{r-q'} \\
&= [K \vee \Delta_n^p(\lambda_0)]\{1\}(I-\lambda_0 K)^{r-1}
\end{aligned}
$$

(unless $q' = 0$ or $-\infty$, when we draw the same conclusion more directly, since then $\Delta_{n+1}^{p-1}(\lambda_0) = \Theta$). Hence

$$
\begin{aligned}
\Theta &= ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_n^{(n)})\Delta_{n+1}^p(\lambda_0)((I-\lambda_0 K)^r x \otimes x_1^{(n)} \otimes \ldots \otimes x_n^{(n)}) \\
&= ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_n^{(n)})[K \vee \Delta_n^p(\lambda_0)] \\
&\quad \times ((I-\lambda_0 K)^{r-1}x \otimes x_1^{(n)} \otimes \ldots \otimes x_n^{(n)}) \\
&= K(I-\lambda_0 K)^{r-1}x \\
&\quad + \sum_{\theta=1}^{n} (-1)^\theta f_\theta^{(n)}(K(I-\lambda_0 K)^{r-1}x) \\
&\quad \times ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_{\theta-1}^{(n)} \otimes f_{\theta+1}^{(n)} \otimes \ldots \otimes f_n^{(n)}) \\
&\quad \times \Delta_n^p(\lambda_0)(x_1^{(n)} \otimes \ldots \otimes x_n^{(n)}) \\
&= K(I-\lambda_0 K)^{r-1}x - \sum_{\theta=1}^{n} f_\theta^{(n)}(K(I-\lambda_0 K)^{r-1}x)\,\eta_\theta^{(n)},
\end{aligned}
$$

and so (since $(I-\lambda_0 K)^{r-1}x = \lambda_0 K(I-\lambda_0 K)^{r-1}x$)

$$(I-\lambda_0 K)^{r-1}x = \sum_{\theta=1}^{n} f_\theta^{(n)}((I-\lambda_0 K)^{r-1}x)\,\eta_\theta^{(n)}.$$

This proves the lemma.

COROLLARY. *If* $r > q(n+1)$, *then* $m_r - m_{r-1} \leqq n$.

Suppose that n is positive. We have just seen that $\mathfrak{M}_1 \cap \mathfrak{R}_{r-1}$ is contained in a space of dimension n. It follows that it is spanned by a set of n of its elements (not necessarily linearly independent), say

$$(I - \lambda_0 K)^{r-1} \zeta_1, \ (I - \lambda_0 K)^{r-1} \zeta_2, \ldots, \ (I - \lambda_0 K)^{r-1} \zeta_n$$

(e.g. we could augment a base of the space by adding as many zeros as are needed to make up n elements; if $r \leqq q = q(n)$, we can clearly take $\zeta_\theta = (I - \lambda_0 K)^{q-r} \xi_\theta^{(n)}$).

Now let x be any element of $\mathfrak{M}_r$. Then $(I - \lambda_0 K)^{r-1} x$ belongs to $\mathfrak{M}_1 \cap \mathfrak{R}_{r-1}$, and so we can write

$$(I - \lambda_0 K)^{r-1} x = \sum_{\theta=1}^{n} \alpha_\theta (I - \lambda_0 K)^{r-1} \zeta_\theta.$$

Thus

$$(I - \lambda_0 K)^{r-1} \left(x - \sum_{\theta=1}^{n} \alpha_\theta \zeta_\theta \right) = \Theta,$$

that is

$$x = \sum_{\theta=1}^{n} \alpha_\theta \zeta_\theta + y$$

where y is a member of $\mathfrak{M}_{r-1}$. It follows that $\mathfrak{M}_r$ is spanned by $\zeta_1, \zeta_2, \ldots, \zeta_n$ modulo $\mathfrak{M}_{r-1}$, and so that $m_r \leqq m_{r-1} + n$. This proves the corollary when $n > 0$.

The proof when $n = 0$ is similar (cf. [36] Lemma 2.2, p. 320).

THEOREM 3.1. *If* $0 \leqq m \leqq n \leqq d$, *then*

$$q(n) \leqq q(m).$$

In other words, the number of "new" white spots in any row of the spot diagram is at least as great as the corresponding number in any higher row.

It will be sufficient to prove that

$$q(n+1) \leqq q(n)$$

when $0 \leqq n < d$. Suppose, on the contrary, that

$$r = q(n+1) > q(n).$$

By Theorem 2.3, $q(n) \geqq 0$, and so $r \geqq 1$. Hence, by Corollary 1 to Lemma 3.1 and the Corollary to Lemma 3.2,

$$n+1 \leqq m_r - m_{r-1} \leqq n-1.$$

This contradiction proves the theorem.

The values of n of most interest are those for which $q(n+1) < q(n) \neq 0$, since then $m_r - m_{r-1} = n$ whenever $q(n+1) < r \leqq q(n)$. In fact we have

THEOREM 3.2. *If $q(n+1) < r \leqq q(n) = q$, then*

$$(I - \lambda_0 K)^{q-r} \xi_1^{(n)}, \ (I - \lambda_0 K)^{q-r} \xi_2^{(n)}, \ldots, (I - \lambda_0 K)^{q-r} \xi_n^{(n)}$$

form a base for $\mathfrak{M}_r$ modulo $\mathfrak{M}_{r-1}$.

DEFINITION 3.3. *We denote the values of n for which $q(n+1) < q(n) \neq 0$, arranged in descending order of magnitude, by $d_1(= d), d_2, \ldots, d_N, d_{N+1} (= 0)$, and put $q_0 = 0$ and $q_\rho = q(d_\rho)$ for $\rho = 1, 2, \ldots, N+1$.*

The most important Fredholm formulae (in view of Theorem 3.2) are $\Delta_{d_\rho}^{p(d_\rho)}(\lambda_0)$. The corresponding spots on the spot diagram can be thought of as the points where a string stretched round the black spots (from "$n = \infty$" to "$r = \infty$") would bend.

In view of Theorem 3.1 and the definition of $d_{\rho+1}$, we have

$$q_\rho = q(d_\rho) = q(d_\rho - 1) = \ldots = q(d_{\rho+1} + 1)$$

for $\rho = 1, 2, \ldots, N$. Hence we have

$$m_r - m_{r-1} = d_\rho$$

whenever $q_{\rho-1} < r \leqq q_\rho$. But

$$0 = q_0 < q_1 < q_2 < \ldots < q_N < q_{N+1} = \infty.$$

Hence, given any positive integer r, there is a unique integer ρ between 1 and $N+1$ such that

$$q_{\rho-1} < r \leqq q_\rho.$$

Thus we can calculate $m_r - m_{r-1}$, and so m_r (since clearly $m_0 = 0$), for any positive integer r. In fact

$$m_1 = d_1,$$
$$m_2 = 2d_1,$$
$$\ldots$$
$$m_{q_1} = q_1 d_1,$$
$$m_{q_1+1} = q_1 d_1 + d_2,$$
$$m_{q_1+2} = q_1 d_1 + 2d_2,$$
$$\ldots$$
$$m_{q_2} = q_1 d_1 + (q_2 - q_1)d_2,$$
$$m_{q_2+1} = q_1 d_1 + (q_2 - q_1)d_2 + d_3,$$

and so on. In general we have

$$m_r = q_1 d_1 + (q_2 - q_1)d_2 + \ldots + (r - q_{\rho-1})d_\rho,$$

where ρ is chosen so that $q_{\rho-1} < r \leqq q_\rho$ — in fact this equation also holds if $r = q_{\rho-1}$.

We can find another formula for m_r as follows. Assuming that $q_{\rho-1} \leqq r \leqq q_\rho$, we have

$$\begin{aligned}
m_r &= q_1 d_1 + (q_2 - q_1)d_2 + \ldots + (r - q_{\rho-1})d_\rho \\
&= q_1(d_1 - d_2) + q_2(d_2 - d_3) + \ldots + q_{\rho-1}(d_{\rho-1} - d_\rho) + rd_\rho \\
&= q(d_1) + q(d_1 - 1) + \ldots + q(d_2 + 1) \\
&\quad + q(d_2) + q(d_2 - 1) + \ldots + q(d_3 + 1) \\
&\quad + \ldots \\
&\quad + q(d_{\rho-1}) + q(d_{\rho-1} - 1) + \ldots + q(d_\rho + 1) + rd_\rho \\
&= p(d_\rho) - p(d_1) + rd_\rho \\
&= p(d_\rho) - \mu + rd_\rho.
\end{aligned}$$

THEOREM 3.3. *The number of dimensions m_n of the kernel $\mathfrak{M}_n$ of $(I - \lambda_0 K)^n$ is given by the equations*

$$\begin{aligned}
m_n &= q_1 d_1 + (q_2 - q_1)d_2 + \ldots + (n - q_{\rho-1})d_\rho \\
&= p(d_\rho) - \mu + nd_\rho,
\end{aligned}$$

where ρ is such that $q_{\rho-1} \leqq n \leqq q_\rho$.

In particular, if $n \geqq q_N$, we have (since $d_{N+1} = 0$)

$$m_n = p(d_{N+1}) - \mu = p(0) - \mu.$$

Thus $q_N \, (= q(1))$ is the index ν of the Riesz theory referred to early in this section [19].

We can express Theorem 3.3 in terms of conjugate partitions (cf. [20] p. 271, [26] p. 94).

THEOREM 3.4. *The sums*

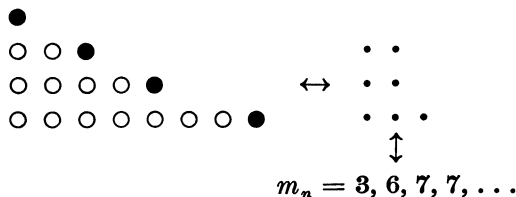$$\{p(d-1) - p(d)\} + \{p(d-2) - p(d-1)\} + \ldots + \{p(0) - p(1)\}$$

and

$$m_1 + (m_2 - m_1) + \ldots + (m_\nu - m_{\nu-1})$$

are conjugate partitions of $m_\nu = p(0) - \mu$.

Cf. [37].

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[19]) Cf. [31] p. 81. We also have, in terms of notations used elsewhere, $d_1 = d(\lambda_0)$ ([34] p. 380), $p(d_1) = \mu(\lambda_0)$ ([34] p. 380), $p(0) = p(\lambda_0)$, ([36] p. 320), $m_{q_N} = m(\lambda_0)$ ([35] p. 369).

This relation can be represented diagrammatically by means of the spot diagram and the usual representation of conjugate partitions. For instance, we have the following diagram [20].

$$m_n = 3, 6, 7, 7, \ldots$$

Here the number of dots in any row is the number of "new" white spots in the corresponding row of the spot diagram; m_n is then the total number of dots in the first n columns.

More detailed use of Theorem 3.2 yields the following base for $\mathfrak{M}_n$:

$$(I-\lambda_0 K)^{q_1-1}\xi_1^{(d_1)}, (I-\lambda_0 K)^{q_1-1}\xi_2^{(d_1)}, \ldots, (I-\lambda_0 K)^{q_1-1}\xi_{d_1}^{(d_1)},$$

$$(I-\lambda_0 K)^{q_1-2}\xi_1^{(d_1)}, \ldots, (I-\lambda_0 K)^{q_1-2}\xi_{d_1}^{(d_1)}, \ldots,$$

$$(I-\lambda_0 K)\xi_1^{(d_1)}, \ldots, (I-\lambda_0 K)\,\xi_{d_1}^{(d_1)}, \xi_1^{(d_1)}, \ldots, \xi_{d_1}^{(d_1)},$$

$$(I-\lambda_0 K)^{q_2-q_1-1}\xi_1^{(d_2)}, \ldots, (I-\lambda_0 K)^{q_2-q_1-1}\xi_{d_2}^{(d_2)}, \ldots,$$

$$\xi_1^{(d_2)}, \ldots, \xi_{d_2}^{(d_2)}, \ldots, \xi_1^{(d_{\rho-1})}, \ldots, \xi_{d_{\rho-1}}^{(d_{\rho-1})},$$

$$(I-\lambda_0 K)^{q_\rho-q_{\rho-1}-1}\xi_1^{(d_\rho)}, \ldots, (I-\lambda_0 K)^{q_\rho-q_{\rho-1}-1}\xi_{d_\rho}^{(d_\rho)}, \ldots,$$

$$(I-\lambda_0 K)^{q_\rho-n}\xi_1^{(d_\rho)}, \ldots, (I-\lambda_0 K)^{q_\rho-n}\xi_{d_\rho}^{(d_\rho)},$$

where ρ is chosen so that $q_{\rho-1} < n \leqq q_\rho$.

NOTE. This does not quite correspond to the arrangement given by Zaanen in [54] (p. 280), [55] (p. 84) and [56] (p. 342). For instance, some of $\xi_1^{(d_1)}, \ldots, \xi_{d_{\rho-1}}^{(d_{\rho-1})}$ could be replaced by elements of the form $(I-\lambda_0 K)^r \xi_\theta^{(d_\rho)}$. The arrangement I have given is that which follows most naturally from the work of this paper.

REMARK. Any spot diagram that accords with the rules which

[20]) This diagram occurs for the operator in 7-dimensional space given by the 'Union Jack" matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\Delta_0(\lambda)=(1-\lambda)^7$ and $\lambda_0 = 1$; see also the remark below.

we have established can arise (for a given non-zero scalar λ_0) provided our space is of a sufficiently large number of dimensions (e.g. if it is infinite-dimensional). For instance, let x_1, $x_2, \ldots, x_{m_\nu}$ be m_ν linearly independent elements, where m_ν is the number of white spots in the bottom row of the required spot diagram, and let $f_1, f_2, \ldots, f_{m_\nu}$ be continuous linear functionals such that $f_\theta(x_\varphi) = \delta_{\theta\varphi}$. Let us arrange the indices $1, 2, \ldots, m_\nu$ in an array corresponding to the diagram of dots associated with the required spot diagram as described above (thus there are d rows, the first (bottom) row has $p(0) - p(1) = q(1)$ entries, the second row has $q(2)$ entries, and so on). Then, for each $\theta = 1, 2, \ldots, m_\nu$, we put $g_\theta = f_\varphi$ if φ is immediately to the right of θ in this array, and put $g_\theta = \Theta$ if θ is on the extreme right of its row. Then it can be shown that the operator

$$\sum_{\theta=1}^{m_\nu} \lambda_0^{-1}\, x_\theta. \otimes (f_\theta - g_\theta)$$

has [21] the required spot diagram for the parameter λ_0 (e.g. with $\Delta_0(\lambda) = (1 - \lambda/\lambda_0)^{m_\nu}$).

So far we have concentrated our attention on the kernels $\mathfrak{M}_n$ (leading to the null space $\mathfrak{M}_\nu$). I should like to consider now the ranges $\mathfrak{R}_n$ (leading to the nucleus space $\mathfrak{R}_\nu$).

The range $\mathfrak{R}_n$ of $(I - \lambda_0 K)^n$ is, of course, the subspace of $\mathfrak{B}$ orthogonal to the kernel $\mathfrak{M}'_n$ of the adjoint $(I - \lambda_0 K^*)^n$ of $(I - \lambda_0 K)^n$ ([36] p. 321, proof of Lemma 2.4; see also [22] Theorem 2.13.6, p. 28, [1] Théorème 9, p. 150). If we put

$$\varphi_\theta^{(n)} = \lambda_0^{q-1}(f_1^{(n)} \otimes \ldots \otimes f_n^{(n)})$$
$$\Delta_n^{p+q-1}(\lambda_0)(x_1^{(n)} \otimes \ldots \otimes x_{\theta-1}^{(n)} \otimes (1) \otimes x_{\theta+1}^{(n)} \otimes \ldots \otimes x_n^{(n)}),$$

where $p = p(n)$ and $q = q(n)$, for each n, then $\mathfrak{R}_n$ will consist of all elements of $\mathfrak{B}$ orthogonal to all of the continuous linear functionals

$$(I - \lambda_0 K^*)^{q_1 - 1} \varphi_1^{(d_1)}, \ldots, (I - \lambda_0 K^*)^{q_1 - 1} \varphi_{d_1}^{(d_1)}, \ldots,$$
$$(I - \lambda_0 K^*)^{q_\rho - n} \varphi_1^{(d_\rho)}, \ldots, (I - \lambda_0 K^*)^{q_\rho - n} \varphi_{d_\rho}^{(d_\rho)},$$

where ρ is chosen so that $q_{\rho-1} < n \leq q_\rho$. Alternatively, we may

---

[21]) This is connected with the Jordan, or "classical", canonical form of a matrix (cf. [2] pp. 333—334, [18] pp. 167—169, [19] pp. 122—132, [27] p. 69, [56] p. 202; [26] pp. 109—113, [28] p. 312; see also [13] p. 563), as well as with Zaanen's tables referred to above. I am indebted to Professor G. E. H. Reuter for suggesting this approach.

adapt the arguments of [34] pp. 382—383 to obtain information about $\mathfrak{R}_n$.

LEMMA 3.3. *If* $1 \leqq r \leqq q(n)$, *then*

$$(I-\lambda_0 K)^{r-1} K x_1^{(n)}, \ (I-\lambda_0 K)^{r-1} K x_2^{(n)}, \ \ldots, \ (I-\lambda_0 K)^{r-1} K x_n^{(n)}$$

are elements of $\mathfrak{R}_{r-1}$ *linearly independent modulo* $\mathfrak{R}_r$.

We have

$$[(I-\lambda_0 K^*)^{q-1} \varphi_\theta^{(n)}](x_\varphi^{(n)}) = \delta_{\theta\varphi}$$

and

$$(I-\lambda_0 K^*)^q \varphi_\theta^{(n)} = \Theta$$

(cf. proof of Lemma 3.1). Hence

$$\begin{aligned}
[\lambda_0 (I-\lambda_0 K^*)^{q-r} \varphi_\theta^{(n)}] &\big((I-\lambda_0 K)^{r-1} K x_\varphi^{(n)}\big) \\
&= [\lambda_0 K^*(I-\lambda_0 K^*)^{q-1} \varphi_\theta^{(n)}](x_\varphi^{(n)}) \\
&= [(I-\lambda_0 K^*)^{q-1} \varphi_\theta^{(n)}](x_\varphi^{(n)}) \\
&= \delta_{\theta\varphi}.
\end{aligned}$$

It follows that, if $\sum_{\theta=1}^{n} \alpha_\theta (I-\lambda_0 K)^{r-1} K x_\theta^{(n)}$ is an element of $\mathfrak{R}_r$, $(I-\lambda_0 K)^r x$ say, then

$$\begin{aligned}
\alpha_\theta &= \sum_{\varphi=1}^{n} \alpha_\varphi \delta_{\theta\varphi} \\
&= [\lambda_0 (I-\lambda_0 K^*)^{q-r} \varphi_\theta^{(n)}]\big((I-\lambda_0 K)^r x\big) \\
&= [\lambda_0 (I-\lambda_0 K^*)^q \varphi_\theta^{(n)}](x) \\
&= 0
\end{aligned}$$

for $\theta = 1, 2, \ldots, n$. Since the elements all manifestly belong to $\mathfrak{R}_{r-1}$, this completes the proof of the lemma.

LEMMA 3.4. *If* $r > q(n+1)$, *then* $\mathfrak{R}_{r-1}$ *is contained in the subspace of* $\mathfrak{B}$ *spanned by*

$$(I-\lambda_0 K)^{r-1} K x_1^{(n)}, \ (I-\lambda_0 K)^{r-1} K x_2^{(n)}, \ \ldots, \ (I-\lambda_0 K)^{r-1} K x_n^{(n)}$$

modulo $\mathfrak{R}_q$.

Let $y = (I-\lambda_0 K)^{r-1} x$ be any element of $\mathfrak{R}_{r-1}$. We put

$$x' = x + \lambda_0 ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_n^{(n)}) \Delta_{n+1}^p(\lambda_0)(x \otimes x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$$

(where $p = p(n)$). Then

$(I-\lambda_0 K)^r x'$

$\quad = (I-\lambda_0 K)^r x$

$\qquad + \lambda_0 (I-\lambda_0 K)^r ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_n^{(n)})$

$\qquad\qquad\qquad \Delta_{n+1}^p(\lambda_0)(x \otimes x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$

$\quad = (I-\lambda_0 K)^r x$

$\qquad + \lambda_0 (I-\lambda_0 K)^{r-1} ((1^*) \otimes f_1^{(n)} \otimes \ldots \otimes f_n^{(n)})$

$\qquad\qquad\qquad [K \wedge \Delta_n^p(\lambda_0)](x \otimes x_1^{(n)} \otimes \ldots \otimes x_n^{(n)})$

$\quad = (I-\lambda_0 K)^r x$

$\qquad + \lambda_0 (I-\lambda_0 K)^{r-1} \{Kx - \sum_{\theta=1}^{n} [(I-\lambda_0 K^*)^{q-1} \varphi_\theta^{(n)}](x) \cdot Kx_\theta^{(n)}\}$

$\quad = (I-\lambda_0 K)^{r-1} x - \lambda_0 \sum_{\theta=1}^{n} [(I-\lambda_0 K^*)^{q-1} \varphi_\theta^{(n)})](x) \cdot (I-\lambda_0 K)^{r-1} Kx_\theta^{(n)}$

(cf. proof of Lemma 3.2), and so

$y = (I-\lambda_0 K)^{r-1} x$

$\quad = (I-\lambda_0 K)^r x' + \lambda_0 \sum_{\theta=1}^{n} [(I-\lambda_0 K^*)^{q-1} \varphi_\theta^{(n)}](x) \cdot (I-\lambda_0 K)^{r-1} Kx_\theta^{(n)}.$

This proves the lemma.

Combining these results, we get

THEOREM 3.5. *If* $q(n+1) < r \leqq q(n)$, *then*

$$(I-\lambda_0 K)^{r-1} Kx_1^{(n)}, \ (I-\lambda_0 K)^{r-1} Kx_2^{(n)}, \ \ldots, \ (I-\lambda_0 K)^{r-1} Kx_n^{(n)}$$

form a base for $\Re_{r-1}$ *modulo* $\Re_r$.

This result is less informative than Theorem 3.2, in that it does not enable us to find a base for $\Re_n$, but only a base for $\Re_n$ modulo $\Re_\nu$. It is perhaps of more interest for the light it throws on the choice of $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$.

4. The Riesz decomposition

Finally we consider the Riesz decomposition $K = K_1 + K_2$ of K, where $K_1 x = \Theta$ for all null elements of $I-\lambda_0 K$ and $K_2 x = \Theta$ for all nucleus elements ([31] Satz 10, p. 88). Then $H(\lambda) = H_1(\lambda) + H_2(\lambda)$ for any regular value λ, $H(\lambda) = K(I-\lambda K)^{-1}$, $H_1(\lambda) = K_1(I-\lambda K_1)^{-1}$ and $H_2(\lambda) = K_2(I-\lambda K_2)^{-1}$ being the resolvents of K, K_1 and K_2 respectively (cf. [35] § 2, p. 370). Using the notation of [35] § 2, we have

$$H_2(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)},$$

where $D_0(\lambda) = (1 - \lambda/\lambda_0)^{m_\nu}$ ([35] § 2, p. 371), and $D_1(\lambda)$ is a polynomial of degree at most $m_\nu - 1$ in λ ([34] Theorem 3.3.4 Corollary, p. 364). Hence, for all regular values of λ,

$$H_2(\lambda) = \frac{B_1}{\lambda - \lambda_0} + \frac{B_2}{(\lambda - \lambda_0)^2} + \cdots + \frac{B_{m_\nu}}{(\lambda - \lambda_0)^{m_\nu}},$$

where $B_1, B_2, \ldots, B_{m_\nu}$ are continuous linear operators (independent of λ) on $\mathfrak{B}$ into itself. On the other hand, $H_1(\lambda_0)$ exists ([31] Satz 11, p. 89), and so $H_1(\lambda)$ is holomorphic in λ at λ_0. It follows that $H_2(\lambda)$ is the principal part ([50] p. 92) of $H(\lambda)$ at λ_0. Since

$$H(\lambda) = \frac{\Delta_1^0(\lambda)}{\Delta_0^0(\lambda)}$$

whenever $\Delta_0^0(\lambda) \neq 0$, we can express $H_2(\lambda)$ in terms of the Fredholm formulae $\Delta_n^r(\lambda_0)$. In fact it is clear that each of $B_1, B_2, \ldots, B_{m_\nu}$ is a linear combination of $\Delta_1^0(\lambda_0), \Delta_1^1(\lambda_0), \Delta_1^2(\lambda_0), \ldots, \Delta_1^{p(0)-1}(\lambda_0)$. Hence

$$K_2 = H_2(0) = \frac{B_1}{-\lambda_0} + \frac{B_2}{(-\lambda_0)^2} + \cdots + \frac{B_{m_\nu}}{(-\lambda_0)^{m_\nu}}$$

is also a linear combination of $\Delta_1^0(\lambda_0), \Delta_1^1(\lambda_0), \ldots, \Delta_1^{p(0)-1}(\lambda_0)$. In fact detailed calculation shows that

$$K_2 = \sum_{r=1}^{p} \frac{\Delta_1^{p-r}(\lambda_0)}{[\lambda_0 \Delta_0^p(\lambda_0)]^r} \times$$

$$\begin{vmatrix}
(-\lambda_0)^{r-1} \ \Delta_0^p(\lambda_0) & 0 & 0 & \ldots & 0 & 0 \\
(-\lambda_0)^{r-2} \ \Delta_0^{p+1}(\lambda_0) & \Delta_0^p(\lambda_0) & 0 & \ldots & 0 & 0 \\
(-\lambda_0)^{r-3} \ \Delta_0^{p+2}(\lambda_0) & \Delta_0^{p+1}(\lambda_0) & \Delta_0^p(\lambda_0) & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \ldots & \ldots \\
(-\lambda_0)^2 & \Delta_0^{p+r-3}(\lambda_0) & \Delta_0^{p+r-4}(\lambda_0) & \Delta_0^{p+r-5}(\lambda_0) \ldots \Delta_0^p(\lambda_0) & 0 \\
-\lambda_0 & \Delta_0^{p+r-2}(\lambda_0) & \Delta_0^{p+r-3}(\lambda_0) & \Delta_0^{p+r-4}(\lambda_0) \ldots \Delta_0^{p+1}(\lambda_0) & \Delta_0^p(\lambda_0) \\
1 & \Delta_0^{p+r-1}(\lambda_0) & \Delta_0^{p+r-2}(\lambda_0) & \Delta_0^{p+r-3}(\lambda_0) \ldots \Delta_0^{p+2}(\lambda_0) & \Delta_0^{p+1}(\lambda_0)
\end{vmatrix}$$

(where $p = p(0)$), but we shall not here need the details of this expression.

It will be convenient to denote by Φ_n the polynomial given by

$$\Phi_n(\mu) = \sum_{r=1}^{n} \binom{n}{r} (-\mu)^{r-1} = \{1 - (1-\mu)^n\}/\mu,$$

so that

$$(1 - \mu)^n = 1 - \mu\Phi_n(\mu).$$

LEMMA 4.1. *For any scalar λ, and any continuous linear operator K, the null space [22] of $I-\lambda K$ is contained in the range of K.*

If x belongs to the null space of $I-\lambda K$, then for some integer n we have $(I-\lambda K)^n x = \Theta$, and so

$$x = K\{\lambda \Phi_n(\lambda K)x\}.$$

LEMMA 4.2. *For any scalar λ, and any continuous linear operator K, the nucleus space [22] of $I-\lambda K$ contains the kernel of K.*

If $Kx = \Theta$, then for every non-negative integer n we must have

$$(I-\lambda K)^n x = x.$$

THEOREM 4.1. *The null space and nucleus space of $I-\lambda_0 K$ coincide with the range and kernel (respectively) of K_2.*

By [31] Satz 10 (p. 88), the range of K_2 is contained in the null space of $I-\lambda_0 K$, and the kernel of K_2 contains the nucleus space of $I-\lambda_0 K$. But, by [31] Satz 11 (p. 89), the null space and nucleus space of $I-\lambda_0 K$ coincide with those of $I-\lambda_0 K_2$. The theorem now follows from Lemmas 4.1 and 4.2.

THEOREM 4.2. *The null space and nucleus space of $I-\lambda_0 K$ coincide with the range and kernel (respectively) of $\Delta_1^{p(0)-1}(\lambda_0)$.*

For $r \leqq p-1 = p(0)-1$ we have (by successive application of Theorem 2.2 (iv))

$$\begin{aligned}
\Delta_1^r(\lambda_0)(I-\lambda_0 K)^p &= \Delta_1^{r-1}(\lambda_0)K(I-\lambda_0 K)^{p-1} \\
&= \ldots \\
&= \Delta_1^0(\lambda_0)K^r(I-\lambda_0 K)^{p-r} \\
&= \Theta,
\end{aligned}$$

and so, if $r < p-1$,

$$\begin{aligned}
\Delta_1^r(\lambda_0) &= \Delta_1^r(\lambda_0)\lambda_0 K\Phi_p(\lambda_0 K) \\
&= \Delta_1^{r+1}(\lambda_0)\lambda_0(I-\lambda_0 K)\Phi_p(\lambda_0 K).
\end{aligned}$$

Moreover all these operators commute with each other. Thus the range of $\Delta_1^r(\lambda_0)$ is contained in that of $\Delta_1^{r+1}(\lambda_0)$, and so (by induction) in that of $\Delta_1^{p-1}(\lambda_0)$. Hence the range of K_2 (which, by Theorem 4.1, is the null space of $I-\lambda_0 K$) is contained in the range of $\Delta_1^{p-1}(\lambda_0)$. But, since $(I-\lambda_0 K)^p\Delta_1^{p-1}(\lambda_0) = \Theta$, the range of $\Delta_1^{p-1}(\lambda_0)$ is contained in the null space of $I-\lambda_0 K$, and so the two must coincide.

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[22]) We interpret the *null space* of a general continuous linear operator T on $\mathfrak{B}$ into itself as the union of the kernels of T^n for $n = 1, 2, 3, \ldots$, and the *nucleus space* of T as the intersection of the ranges of T^n for $n = 1, 2, 3, \ldots$.

Similarly, if $r < p-1$,

$$\Delta_1^r(\lambda_0) = \lambda_0(I-\lambda_0 K)\Phi_p(\lambda_0 K)\Delta_1^{r+1}(\lambda_0),$$

and so the kernel of $\Delta_1^r(\lambda_0)$ contains that of $\Delta_1^{r+1}(\lambda_0)$, and so (by induction) that of $\Delta_1^{p-1}(\lambda_0)$. Hence the kernel of K_2 (which, by Theorem 4.1, is the nucleus space of $I-\lambda_0 K$) contains the kernel of $\Delta_1^{p-1}(\lambda_0)$. But, since $\Delta_1^{p-1}(\lambda_0)(I-\lambda_0 K)^p = \Theta$, the kernel of $\Delta_1^{p-1}(\lambda_0)$ contains the nucleus space of $I-\lambda_0 K$, and so the two must coincide.

We can deduce [23] from this that the range and kernel of $\Delta_1^{p-1-r}(\lambda_0)$ $(0 \leqq r \leqq \nu)$ are $\mathfrak{M}_{\nu-r} \cap \mathfrak{R}_r$ and the subspace of $\mathfrak{B}$ spanned by $\mathfrak{R}_{\nu-r} \cup \mathfrak{M}_r$ respectively.

NOTE. In [36] (p. 319), I defined a continuous linear operator K on a complex Banach space $\mathfrak{B}$ to be a *Riesz* operator iff it had the following three properties (cf. also [35], p. 376):

(i)   For every scalar λ, the set of solutions x of the equation $(I-\lambda K)^n x = \Theta$ forms a finite-dimensional subspace of $\mathfrak{B}$, which is independent of n provided n is sufficiently large.

(ii)   For every scalar λ, the range of $(I-\lambda K)^n$ is a closed subspace of $\mathfrak{B}$, which is independent of n provided n is sufficiently large.

(iii) The characteristic values of K have no finite limit point.

In fact condition (iii) is redundant, being a consequence of (i) and (ii). For the Riesz decomposition $K = K_1 + K_2$ follows from (i) and (ii) (cf. [31] Satz 10, p. 88), as also does the fact that $I-\lambda_0 K_1$ has an inverse ([31] Satz 11, p. 89). We conclude that $I-\lambda K_1$ has an inverse when λ is near λ_0 (cf. for instance [35] p. 370). But $I-\lambda K_2$ has an inverse when $\lambda \neq \lambda_0$ (cf. [31] Satz 13, p. 91) — in fact we can easily verify that

$$(I-\lambda K_2)^{-1} = I + \frac{\lambda_0 \lambda}{\lambda_0 - \lambda} K_2 \Phi_\nu\left(\frac{\lambda_0(I-\lambda K)}{\lambda_0 - \lambda}\right).$$

---

[23]) This can be proved by induction, using the facts that the range of $\Delta_1^{p-1-(r+1)}(\lambda_0)$ is the direct image under $I-\lambda_0 K$ of the range of $\Delta_1^{p-1-r}(\lambda_0)$ and that the kernel of $\Delta_1^{p-1-(r+1)}(\lambda_0)$ is the inverse image under $I-\lambda_0 K$ of the kernel of $\Delta_1^{p-1-r}(\lambda_0)$. These follow from the equations

$$\Delta_1^{p-1-(r+1)}(\lambda_0) = (I-\lambda_0 K)\Delta_1^{p-1-r}(\lambda_0)\lambda_0 \Phi_p(\lambda_0 K)$$

and

$$(I-\lambda_0 K)\Delta_1^{p-1-r}(\lambda_0) = \Delta_1^{p-1-(r+1)}(\lambda_0)K$$

(remembering that all the operators concerned commute with each other).

Hence $I-\lambda K = (I-\lambda K_1)(I-\lambda K_2)$ has an inverse when $\lambda \neq \lambda_0$ is near λ_0, and so λ_0 is not a limit point of characteristic values of K. Since λ_0 is an arbitrary scalar, condition (iii) follows (see also [11] p. 199, [12] p. 645).

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