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by

Vinod B. Goyal

Let $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_n^2$ be the n-dimensional Laplace operator and let D_r and S_r denote the open sphere $x_1^2+x_2^2+\ldots+x_n^2 < r^2$ $(r > 0)$ and its boundary $x_1^2+x_2^2+\ldots+x_n^2 = r^2$, respectively. The aim of this paper is to find explicit bounds for the max. $\omega(Q)$ where $\omega(Q)$ satisfies the differential equation

$$(1) \qquad \Delta\omega = P(r)f(\omega)$$

or, more generally, the differential inequality

$$(2) \qquad \Delta\omega \geqq P(r)f(\omega)$$

where $Q \in D_R$ with $R > r$ and $P(r)$ is positive, monotonically increasing and twice continuously differentiable. In theorem 1 an upper bound is obtained where $P(r)$ is either $C_1 e^{c_2 r^2}$ or Ar^n where C_1, C_2 and A are arbitrary constants, C_1 and A being positive and C_2 non-negative. However, in 2-dimensional case, an upper bound is given if $P(r)$ is αr^β where α and β are arbitrary positive constants. In theorem 2 we find a lower bound in case $P(r)$ is αr^β and Δ is a 2-dimensional Laplace operator. Also, the behaviour of these solutions at an isolated singularity is investigated.

THEOREM 1. *Let $f(\omega)$ be positive, non-decresaing, differentiable function in* $(-\infty, \infty)$, *for which*

$$\int_\omega^\infty \frac{dt}{f(t)} \qquad (\omega > -\infty)$$

exists and

$$(3) \qquad f'(\omega)\int_\omega^\infty \frac{dt}{f(t)} \leqq 1.$$

If

$$(G) \qquad u(r) = \sup_{Q\in S_r} \omega(Q)$$

where $\omega(Q)$ *ranges over all functions of class* C^2 *in* D_r *which satisfy* (2), *then*

$$\frac{C_1 e^{c_2 r^2}(R^2-r^2)^2}{4nR^2} \leqq \int_{u(r)}^{\infty} \frac{dt}{f(t)} \tag{4}$$

provided $P(r) = C_1 e^{c_2 r^2}$ $(C_1 > 0$ *and* $C_2 \geqq 0)$ *and*

$$\frac{Ar^n(R^2-r^2)^2}{4nR^2} \leqq \int_{u(r)}^{\infty} \frac{dt}{f(t)} \tag{5}$$

in case $P(r) = Ar^n$ $(A > 0)$. *Also if* Δ *is a 2-dimensional Laplace operator and* $P(r) = \alpha r^\beta$, *then*

$$\frac{\alpha r^\beta(R^2-r^2)^2}{8R^2} \leqq \int_{u(r)}^{\infty} \frac{dt}{f(t)} \tag{6}$$

α *and* β *being arbitrary positive constants. The inequalities* (4), (5) *and* (6) *are sharp.*

Proof: Consider the function $g = g(r)$ defined by

$$\frac{C(R^2-r^2)^2}{R^2} P(r) = \int_{g}^{\infty} \frac{dt}{f(t)} \tag{7}$$

where C is a positive constant to be chosen later. Denoting by x one of the variables x_k and differentiating with respect to x, we have

$$-\frac{4CxP(r)(R^2-r^2)}{R^2} + \frac{2Cx\dot{P}(r)(R^2-r^2)^2}{R^2} = -\frac{g_x}{f(g)} \tag{8}$$

where the dot denotes differentiation with respect to r^2. Differentiating again with respect to x

$$-\frac{4CP(r)(R^2-r^2)}{R^2} + \frac{8Cx^2P(r)}{R^2} - \frac{16Cx^2\dot{P}(r)(R^2-r^2)}{R^2}$$
$$+\frac{2C\dot{P}(r)(R^2-r^2)^2}{R^2} + \frac{4Cx^2\ddot{P}(r)(R^2-r^2)^2}{R^2} = -\frac{g_{xx}}{f(g)} + \frac{g_x^2}{f^2(g)} f'(g).$$

With the help of (7) and (8), we obtain,

$$\frac{g_{xx}}{P(r)f(g)} = -\frac{8Cx^2}{R^2} + \frac{4C(R^2-r^2)}{R^2} + \frac{16Cx^2\dot{P}(r)(R^2-r^2)}{R^2P(r)} + \frac{4Cx^2}{R^2}$$
$$\cdot\frac{CP(r)(R^2-r^2)^2 f'(g)}{R^2}\cdot\left[2-\frac{\dot{P}(r)(R^2-r^2)}{P(r)}\right]^2$$
$$-\frac{2}{P^2(r)}\left(2x^2\ddot{P}(r)+\dot{P}(r)\right)\int_{g}^{\infty}\frac{dt}{f(t)}.$$

Summing over all x_k, we have

$$\frac{\Delta g}{P(r)f(g)} = -\frac{8Cr^2}{R^2} + \frac{4nC(R^2-r^2)}{R^2} + \frac{16Cr^2\dot{P}(r)(R^2-r^2)}{R^2P(r)} + \frac{4Cr^2}{R^2} \cdot \frac{CP(r)(R^2-r^2)^2f'(g)}{R^2} \cdot \left[2 - \frac{\dot{P}(r)(R^2-r^2)}{P(r)}\right]^2 - \frac{2}{P^2(r)}\left(2r^2\ddot{P}(r)+\dot{P}(r)n\right)\int_g^\infty \frac{dt}{f(t)}.$$

Using (3) it reduces to

$$\frac{\Delta g}{P(r)f(g)} \leqq 4C\left[n - \frac{r^2}{R^2}(n-2)\right] - \frac{2C(R^2-r^2)^2}{R^2}\left\{\frac{2r^2\ddot{P}(r)+n\dot{P}(r)}{P(r)} - \frac{2r^2\dot{P}^2(r)}{P^2(r)}\right\}. \tag{9}$$

Now we consider three cases:

Case I: Choose $P(r)$ such that $\ddot{P}(r)-\dot{P}^2(r)/P(r) = 0$ or $P = C_1e^{c_2r^2}$ where $C_1 > 0$ and $C_2 \geqq 0$ are arbitrary constants. Then (9) reduces to

$$\frac{\Delta g}{C_1e^{c_2r^2}f(g)} \leqq 4C\left\{n - \frac{r^2}{R^2}(n-2)\right\}.$$

If, $n \geqq 2$ and $C = 1/4n$, it follows that

$$\Delta g \leqq C_1e^{c_2r^2}f(g).$$

Since $g(0) = 0$ and $g(r)$ increases to ∞ as $r \to R$ the proof of (4) will follow from the following lemma:

LEMMA: *Let $f(t)$ be monotonically increasing continuous function defined for all t. Suppose the functions g and ω are subject to the inequalities*

$$\Delta g \leqq P(r)f(g)$$

and

$$\Delta\omega \geqq P(r)f(\omega)$$

respectively, for $0 \leqq r_0 < r < R$. If $g \to \infty$ for $r \to R$, then

$$\omega \leqq g$$

for $r_0 < r < R$.

A proof of this lemma (for $r_0 = 0$) can be found in [3]. The changes required to provide it for $r_0 > 0$ are obvious.

Case II. Assume $P(r)$ to satisfy

$$\frac{\dot{P}(r)}{P(r)}\left(\frac{2r^2\dot{P}(r)}{P(r)}-n\right) = 0.$$

(i) If $\dot{P}(r)/P(r) = 0$ then $P = k$ where k is an arbitrary positive constant. This case is implied by Case I if we choose $C_1 = k$ and $C_2 = 0$.

(ii) If $2r^2\dot{P}(r)/P(r) - n = 0$ or, $P = Ar^n$ (A being an arbitrary positive constant) the inequality (9) becomes

$$\frac{\Delta g}{Ar^n f(g)} \leqq 4C\left\{n - \frac{r^2}{R^2}(n-2)\right\}.$$

Again, if $n \geqq 2$ and $C = 1/4n$, we have

$$\Delta g \leqq Ar^n f(g).$$

Now the proof of (5) will follow from the above lemma.

Case III: Choose P such that $2r^2P(r)\ddot{P}(r) + nP(r)\dot{P}(r) - 2r^2\dot{P}^2(r) = 0$ or, $P = \alpha r^\beta$ where α and β are arbitrary positive constants and $n = 2$. Then (9) gives

$$\frac{\Delta g}{\alpha r^\beta f(g)} \leqq 8C.$$

If $C = \frac{1}{8}$, it follows that

$$\Delta g \leqq \alpha r^\beta f(g).$$

Again, with the help of the Lemma, we get (6).

This completes the proof of theorem 1. Now, we derive the following corollaries.

COROLLARY 1. *In case of a function ω satisfying*

$$\Delta\omega = C_1 e^{\omega + c_2 r^2}$$

which is regular in D_R

$$\omega \leqq 2\log\frac{2\sqrt{n}R}{\sqrt{C_1}(R^2-r^2)} - C_2 r^2 \qquad (n \geqq 2)$$

where $C_1 > 0$ and $C_2 \geqq 0$ are arbitrary constants.

Indeed, setting $f(t) = e^t$ in (4), we get, where $\omega = u$

$$\frac{C_1 e^{c_2 r^2}(R^2-r^2)^2}{4nR^2} \leqq e^{-\omega}.$$

Taking logarithm on both sides

$$\omega \leqq 2 \log \frac{2\sqrt{n}R}{\sqrt{C_1}(R^2-r^2)} - C_2 r^2.$$

REMARK: Nehari [2] proved that in case of a solution u of

$$\Delta u = e^u$$

which is regular in D_R

$$\phi(r) \leqq 2 \log \frac{2\sqrt{n}R}{(R^2-r^2)}$$

where $\phi(r) = \sup_{Q \in S_r} u(Q)$.

If we take $C_1 = 1$ and $C_2 = 0$ the above result becomes a particular case of this corollary.

COROLLARY 2: *If ω satisfies the equation*

$$\Delta\omega = Ar^n e^\omega$$

where A is an arbitrary positive constant, it is subject to the inequality

$$\omega \leqq 2 \log \frac{2\sqrt{n}R}{\sqrt{A}r^{n/2}(R^2-r^2)}. \tag{10}$$

At an isolated singularity of ω, the behaviour of ω is such that

$$\overline{\lim_{r\to 0}} \left(\frac{\omega}{\log \frac{1}{r}}\right) \leqq n$$

set $f(t) = e^t$ in (5). With $u = \omega$, we obtain

$$e^{-\omega} \geqq \frac{Ar^n(R^2-r^2)^2}{4nR^2}.$$

Taking logarithm

$$\omega \leqq 2 \log \frac{2\sqrt{n}R}{\sqrt{A}(R^2-r^2)r^{n/2}}.$$

Now (10) could be written

$$\omega \leqq \log \frac{4nR^2}{A(R^2-r^2)^2} + n \log \frac{1}{r}.$$

Dividing by $\log 1/r$ and letting $r \to 0$

$$\overline{\lim_{r\to 0}} \left(\frac{\omega}{\log \frac{1}{r}}\right) \leqq n.$$

REMARK: This is a generalisation of and an improvement upon a result of the author [1], namely, if $\omega = \omega\ (x_1, x_2, \ldots, x_n)$ is a solution of

$$\Delta\omega \geqq r^2 e^{\omega}$$

which is regular for $0 < r < R$, then

$$\omega \leqq \log \frac{4(n+4)R^4}{r^4(R^2-r^2)^2}$$

and at an isolated singularity of ω,

$$\overline{\lim_{r\to 0}} \left(\frac{\omega}{\log \frac{1}{r}}\right) \leqq 4.$$

COROLLARY 3. *Every solution* ω *of*

$$\Delta\omega = \alpha r^{\beta} e^{\omega}$$

where α *and* β *are arbitrary positive constants and* Δ *is* 2*-dimensional Laplace operator, satisfies*

$$\omega \leqq 2 \log \frac{2\sqrt{2}R}{\sqrt{\alpha r^{n}}\ (R^2-r^2)}.$$

At an isolated singularity of ω

$$\overline{\lim_{r\to 0}} \left(\frac{\omega}{\log \frac{1}{r}}\right) \leqq \beta.$$

Setting $f(t) = e^t$ in (6), it could be proved exactly as Corollary 2.

In the next theorem we find a lower bound for the maximum of the solutions of the differential inequality

$$\Delta\omega \geqq \alpha r^{\beta} f(\omega) \tag{11}$$

where Δ is a 2-dimensional Laplace operator and α, β are arbitrary positive constants.

THEOREM 2. *Let* $f(\omega)$ *satisfy the conditions of theorem* 1 *with* (3) *replaced by*

$$f'(\omega)\int_{\omega}^{\infty} \frac{dt}{f(t)} = 1. \tag{3'}$$

If

$$v(r) = \sup_{Q\in S_r} \omega(Q) \tag{G'}$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r ($r^2 = x_1^2+x_2^2$) and which satisfy the inequality (11), *then*

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\alpha r^\beta (R^2-r^2)}{4}.$$

PROOF: Consider the function $h = h_\rho(r)$ defined by

$$\frac{\rho^2-r^2}{4} = \frac{1}{P(r)} \int_h^\infty \frac{dt}{f(t)} \qquad (\rho > R > r) \tag{12}$$

where $P(r)$ is positive, monotonically increasing and twice continuously differentiable. Clearly, $h_\rho(r)$ belongs to the class C^2 in D_R and satisfies the differential inequality (11) if $P(r) = \alpha r^\beta$. Differentiating (12) twice with respect to $x = x_k$ ($k = 1, 2$), we obtain

$$-\frac{x}{2} = -\frac{hx}{f(h)P(r)} - \frac{2x\dot{P}(r)}{P^2(r)} \int_h^\infty \frac{dt}{f(t)} \tag{13}$$

$$-\frac{1}{2} = -\frac{h_{xx}}{f(h)P(r)} + \frac{2xh_x\dot{P}(r)}{f(h)P^2(r)} + \frac{h_x^2}{f^2(h)P(r)} f'h - \frac{2\dot{P}(r)}{P^2(r)} \int_h^\infty \frac{dt}{f(t)}$$

$$-\frac{4x^2}{P^2(r)} \ddot{P}(r) \int_h^\infty \frac{dt}{f(t)} + \frac{8x^2\dot{P}(r)}{P^3(r)} \int_h^\infty \frac{dt}{f(t)} + \frac{2x\dot{P}(r)}{P^2(r)} \cdot \frac{h_x}{f(h)}.$$

Using (13) and rearranging, we get,

$$-\frac{1}{2} = -\frac{h_{xx}}{P(r)f(h)} + \frac{2x^2\dot{P}(r)}{P(r)}$$

$$+ x^2P(r)f'(h)\left[\frac{\rho^2-r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2}\right]^2 - \frac{2x^2\ddot{P}(r)+\dot{P}(r)}{P(r)} \cdot \frac{\rho^2-r^2}{2}.$$

Summing over both x_k, we have

$$\frac{\Delta h}{P(r)f(h)} = 1 + \frac{2r^2\dot{P}(r)}{P(r)}$$

$$+ r^2P(r)f'(h)\left[\frac{\rho^2-r^2}{2} \cdot \frac{\dot{P}(r)}{P(r)} - \frac{1}{2}\right]^2 - \frac{2r^2\ddot{P}(r)+2\dot{P}(r)}{P(r)} \cdot \frac{\rho^2-r^2}{2}.$$

Since $f' > 0$, we obtain with the help of (3')

$$\frac{\Delta h}{P(r)f(h)} \geqq 1 - \frac{2r^2\ddot{P}P+2P\dot{P}-2r^2\dot{P}^2}{P^2(r)} \cdot \frac{\rho^2-r^2}{2}.$$

Now choose $P(r)$ such that $2r^2P(r)\ddot{P}(r)+2P(r)\dot{P}(r)-2r^2\dot{P}^2(r) = 0$ or, $P = \alpha r^\beta$ where α and β are arbitrary positive constants. Hence,

$$\Delta h \geqq \alpha r^\beta f(h). \tag{14}$$

Consequently (G') and (14) imply

$$h(r) \leqq v(r).$$

Since we can take ρ arbitrary, close to R, we have

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\alpha r^\beta (R^2 - r^2)}{4} \tag{15}$$

which proves theorem 2.

COROLLARY 4: *If ω satisfies the equation*

$$\Delta\omega = \alpha r^\beta e^\omega \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \tag{16}$$

and is regular in Dr, then

$$\log \frac{4}{\alpha r^\beta (R^2 - r^2)} \leqq \omega. \tag{17}$$

Moreover, at an isolated singularity of ω, the behaviour of ω is such that

$$\overline{\lim_{r\to 0}} \left(\frac{\omega}{\log \frac{1}{r}}\right) \geqq \beta.$$

Setting $f(t) = e^t$ in (15), this could be proved exactly as Corollary 2.

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