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Convergence theorems derived from the theory of Carleman integral operators

by

György I. Targonski

This paper is concerned with the convergence almost everywhere of series of the type

$$\sum_{n=0}^{\infty} c_n \, |\varphi_n(x)|^2 \tag{1}$$

where $\{\varphi_n\}$ is a complete orthonormal system in a Hilbert function space. We derive our results from the theory of Carleman integral operators. Now we list some definitions and results of that theory.

DEFINITION 1. An operator K on the function space $L^2(a, b)$ of complex functions of one real variable is a Carleman operator, if a function $k(x, y)$ exists with the property

$$\int_a^b |k(x, y)|^2 \, dy < \infty \text{ for } a.a.x \quad \text{i.e.} \tag{2}$$

$$k(x, y) \in L^2(a, b; y) \text{ for } a.a.x. \tag{3}$$

the latter notation indicating that $k(x, y)$, considered as a function of y is, for almost all fixed x, an element of $L^2(a, b)$ and

$$(Kf)(x) = \int_a^b k(x, y) f(y) dy \text{ for all } f \in L^2(a, b) \tag{4}$$

DEFINITION 2. An operator K on the function space $L^2(a, b)$ is of Carleman type, if there exists a unitary operator U such that U^*KU is a Carleman operator, i.e. if K is unitarily equivalent to a Carleman operator.

DEFINITION 3. An operator Ω on the function space $L^2(a, b)$ is a Hilbert-Schmidt operator if a function $\omega(x, y)$ exists with the property

$$\int_a^b \int_a^b |\omega(x, y)|^2 \, dx dy < \infty \tag{5}$$

such that

$$(6) \qquad (\Omega f)(x) = \int_a^b \omega(x, y) f(y) dy \text{ for all } f \in L^2(a, b).$$

A Hilbert-Schmidt operator is a fortiori a Carleman operator: it is also bounded. We quote a number of theorems about Carleman operators and Hilbert-Schmidt operators. ([1], [2], [3].)

I. The Hilbert Schmidt class of operators is a two-sided ideal in the algebra of bounded linear operators; thus $A\Omega B$ is a Hilbert-Schmidt operator if Ω is one and if A and B are bounded linear operators. In particular:

II. $U\Omega U^*$ is a Hilbert-Schmidt operator if Ω is one and U is unitary; the Hilbert-Schmidt class is invariant under the transformation of unitary equivalence; the Carleman property, on the contrary, is not invariant under this transformation. This makes Definition 2 meaningful.

III. For a Hilbert-Schmidt operator and any complete orthonormal system $\{\varphi_n\}$

$$(7) \qquad \sum_{n=0}^{\infty} ||\Omega\varphi_n||^2 < \infty$$

and the sum is independent of the choice of $\{\varphi_n\}$.

IV. A normal operator is of Carleman type if and only if zero is Weyl limit point of its spectrum, i.e. if zero is eigenvalue with infinite multiplicity, or limit point of eigenvalues, or part of the continuous spectrum.

V. If K is a — not necessarily bounded — Carleman operator and B a bounded operator, then KB is Carleman operator. A corresponding statement about BK would be false.

VI. The Carleman class is closed under addition.

VII. No unitary operator is a Carleman operator

A general remark: the theory of integral operators is interesting partly because it links the properties of the operator to the properties of the kernel, a function of two variables. Any property of the operator must be somehow reflected in the property of the kernel and vice versa. In what follows, we attempt to explore some implications of this connection.
Our *notations* shall be these:

$L^2(a, b)$ is the Hilbert space of complex-valued functions of one real variable, defined on the interval $[a, b]$ with $-\infty \leqq a \leqq x \leqq b \leqq \infty$; $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ complete, orthonormal systems in $L^2(a, b)$: U the unitary operator defined by

$$U\varphi_n = \psi_n \tag{8}$$

$\{c_n\}$ a sequence of non-negative real numbers with

$$\underline{\lim}\ c_n = 0 \text{ and} \tag{9}$$

$\{b_n\}$ a sequence of non-negative real numbers with

$$\underline{\lim}\, b_n = 0,\ b_n \leqq M < \infty \tag{10}$$

K the — not necessarily bounded — self-adjoint operator defined by

$$K\varphi_n = \sqrt{c_n} \cdot \varphi_n \tag{11}$$

μ_n a bounded sequence of complex numbers. Ω shall denote a Hilbert-Schmidt operator, H a Carleman operator, B a bounded linear operator, A an arbitrary linear operator.

Statements regarding convergence, and adherence to L^2 containing the variable x shall admit exceptions on a set of measure zero.

Lemma 1. *For every sequence* $\{c_n\}$ *see* (9) *there exists a system* $\{\varphi_n\}$ *such that*

$$\sum_n c_n\, |\varphi_n(x)|^2 < \infty \text{ in } [a, b]. \tag{12}$$

Proof. According to IV, $\{\sqrt{c_n}\}$ can be regarded as the spectrum of a family of unitarily equivalent operators of Carleman type. By definition, there exists at least one member of this family, K, which is not only of Carleman integral type, but actually a Carleman operator. Denoting the corresponding kernel (see (4)) by $k(x, y)$ and the corresponding system of eigenfunctions by $\{\varphi_n\}$, we have

$$(K\varphi_n)(x) = \int_a^b k(x, y)\varphi_n(y)dy = \sqrt{c_n}\varphi_n(x). \tag{13}$$

But $\{\sqrt{c_n}\varphi_n(x)\}$ can be considered, according to (13), as the sequence of Fourier coefficients of the $L^2(a, b; y)$ function $\overline{k(x, y)}$ (see (3)) with respect to $\{\varphi_n(y)\}$. Thus it is quadratically convergent and (12) follows.

Once a system $\{\varphi_n\}$ has been found such that (12) holds for a given sequence $\{c_n\}$ and a given interval $[a, b]$, it is possible to construct complete orthonormal systems for the same $\{c_n\}$ but a different interval $[a', b']$. This is achieved by unitary transformations described by J. v. Neumann (see [1], for a detailed discussion Appendix 2 of [2]). If $[a, b]$ and $[a', b']$ are of the same type (both finite, or both semifinite, or both infinite) a linear transformation of the variable achieves the goal; between types the "von Neumann" transforms ([2]) have to be applied.

LEMMA 2. *if* (12) *holds, then* K *is a Carleman operator.*

PROOF: if $\{\sqrt{c_n}\varphi_n(x)\}$ is square convergent, then it is the sequence of Fourier coefficients of some $L^2(a, b; y)$ function $\overline{k(x, y)}$, i.e. (13) holds, defining K, which is by (12) a Carleman operator.

The following theorem is of very general form; corollaries relevant to special cases shall be derived.

THEOREM. *If* $\sum_n c_n |\varphi_n(x)|^2 < \infty$ *then* $\sum c_n |\psi_n(x)|^2 < \infty$ *if and only if* UK *is a Carleman operator.*

PROOF: the condition is sufficient. For let be UK a Carleman-operator; then by V., UKU^* is also a Carleman-operator. Therefore there exists a kernel $h(x, y) \in L^2(a, b; y)$ (see Def. 1) such that

$$(UKU^*f)(x) = (Hf)(x) = \int_a^b h(x, y)f(y)dy \tag{14}$$

By definition of K (see (11)) and U (see (8))

$$UKU^* \psi_n = \sqrt{c_n}\psi_n \tag{15}$$

(14) and (15) imply

$$\int_a^b h(x, y)\psi_n(y)dy = \sqrt{c_n}\psi_n(x) \tag{16}$$

$\{\sqrt{c_n}\psi_n(x)\}$ is the set of Fourier coefficients of the function $\overline{h(x, y)} \in L^2(a, b; y)$ with respect to $\{\psi_n(y)\}$; thus it is square convergent:

$$\sum_n c_n |\psi_n(x)|^2 < \infty.$$

The condition is also necessary. For let be $\{\sqrt{c_n}\psi_n(x)\}$ square convergent, then it is the sequence of Fourier coefficients with respect to $\{\psi_n(y)\}$ of some function $\overline{h(x, y)} \in L^2(a, b; y)$; i.e.

$$\int_a^b h(x, y)\psi_n(y)dy = \sqrt{c_n}\,\psi_n(x)$$

and this defines the Carleman operator

$$(Hf)(x) = \int_a^b k(x, y)f(y)dy \text{ for all } f \in L^2(a, b)$$

But from (8) and (11), $H = UKU^*$, and using $V.$, $UKU^*U = UK$ is a Carleman operator. This completes the proof of the Theorem.

To derive criteria for the applicability of this rather generally formulated Theorem, we prove

LEMMA 3. (*"Decomposition Principle"*) *UK is a Carleman operator if and only if a (not necessarily bounded) linear operator A exists such that $(U-A)K$ and AK are both Carleman operators.*

PROOF: If UK is a Carleman operator, the lemma follows with A being the zero operator. If A exists, according to VI.

$$(U-A)K+AK = UK \tag{17}$$

is a Carleman operator.

The decomposition (17) is useful for the following reason. In view of (8), criteria referring to the term $(U-A)K$ involve $\{\varphi_n\}$ and $\{\psi_n\}$, while AK may be independent of $\{\psi_n\}$. We shall see some applications of this decomposition method.

COROLLARY 1. (see (10)) $\sum_n b_n\,|\psi_n(x)|^2 < \infty$ *if* $\sum_n b_n\,|\varphi_n(x)|^2 < \infty$ *and if in* (17) *A is bounded and commutes with K and $U-A$ is a Carleman-operator.*

PROOF: In the decomposition (17) $U-A$ is a Carleman-operator by assumption and $(U-A)K$ by V since now K is bounded. $AK = KA$ is a Carleman operator according to V.

Before specializing further Corollary 1 we state a negative result; it is of no advantage to make the seemingly obvious choice $A = K^m$, except for $m = 0$. Under the conditions of Corollary 1, $U-K^m$ should be a Carleman operator K_1, i.e.

$$U = K^m+K_1 \tag{18}$$

Since now K is bounded, it follows from V that for $m \geqq 1$ $K^m = K \cdot K^{m-1}$ is a Carleman operator and from VI (see (18)) that U is a Carleman operator. But this contradicts VII. A decomposition (17) with $A = K^m$ is of course not shown to be useless by this argument if K is unbounded.

Thus $A = I$ is a possible choice and we formulate the decomposition at once in a general form and return then to the case $A = I$.

Corollary 2. *If for some bounded complex sequence* $\{\mu_n\}$

(19) $$\sum_n ||\psi_n - \mu_n \varphi_n||^2 < \infty \quad \textit{and} \quad \text{(see (10))}$$

$$\sum_n b_n \mid \varphi_n(x)|^2 < \infty, \ \textit{then}$$

$$\sum_n b_n \, |\psi_n(x)|^2 < \infty.$$

Before proving the Corollary, we remark that $\{\mu_n\}$ cannot be entirely arbitrary; e.g. it cannot contain zero infinitely many times, for then the sequence (19) will contain $||\psi_n|| = 1$ infinitely many times and thus will diverge. More generally, $\{\mu_n\}$ cannot have a limit point at zero; for then the linear operator defined by

(20) $B\varphi_n = \mu_n \varphi_n$ is by IV an operator of Carleman type;

and by Def. 2 there exists a unitary operator V such that VBV^* is a Carleman operator.
From (19) and (8)

(21) $$\sum_n ||U-B)\varphi_n||^2 < \infty$$

therefore $U-B$ is a Hilbert-Schmidt operator by III, and so is by II, $V(U-B)V^* = \Omega$. This means

$VUV^* = VBV^* + \Omega$; but here the right hand side is by VI a Carleman operator, being the sum of two Carleman operators while the left hand side is unitary; this contradicts VII.

To prove Corollary 2, we remark that B as defined by (20) is a bounded operator which also by (20) commutes with K, since they have a common complete system of eigenfunctions; K is also bounded since by definition (cf. (11)) $K\varphi_n = \sqrt{b_n}\varphi_n$. Thus Corollary 1 applies.
If $\mu_n = 1$, for all n, we obtain a special case which also corresponds to $m = O$ in $A = K^m$. We formulate this separately as

Corollary 3. *If* $\sum_n ||\psi_n - \varphi_n||^2 < \infty$ *and* $\sum_n b_n \, |\varphi_n(x)|^2 < \infty$, *then* $\sum_n b_n \, |\psi_n(x)|^2 < \infty$.

For Corollary 3, the following elegant elementary proof has been given by K. Tandori [1].

In $|\alpha+\beta|^2 \leqq 2(|\alpha|^2+|\beta|^2)$ choose $\alpha = b_n\varphi_n(x)$ and $\beta = b_n(\psi_n(x)-$

[1] personal communication.

$\varphi_n(x))$. On summation with respect to n one obtains, for every fixed x,

$$\sum_n b_n |\psi_n(x)|^2 \leqq 2 \sum_n b_n |\varphi_n(x)|^2 + b_n |\psi_n(x) - \varphi_n(x)|^2 \leqq$$

$$\leqq 2 \sum_n b_n |\varphi_n(x)|^2 + M \sum_n |\psi_n(x) - \varphi_n(x)|^2 \quad \text{see (10)}$$

From $\sum_n ||\psi_n - \varphi_n||^2 < \infty$ it follows by Levi's Theorem that $\sum_n |\psi_n(x) - \varphi_n(x)|^2$ converges for almost all x, and $\sum_n b_n |\varphi_n(x)|^2$ converges by assumption; this completes the proof.

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