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On the mean values of integral functions represented by Dirichlet series[1]

by

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Consider the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, where

$$\lambda_{n+1} > \lambda_n, \quad \lambda_1 \geqq 0, \quad \lim_{n \to \infty} \lambda_n = \infty, \quad s = \sigma + it$$

and

(1.1)
$$\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = 0$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. If $\sigma_c = \infty$, σ_a is also infinite, since according to a known result ([1], p. 4) a Dirichlet series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence and therefore $f(s)$ represents an integral function.

The mean value of $f(s)$ is defined as

(1.2)
$$I_2(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n}.$$

We extend this to

(1.3)
$$m_{2,k}(\sigma) = \lim_{T \to \infty} \frac{1}{Te^{k\sigma}} \int_0^{\sigma} \int_{-T}^{T} |f(x+it)|^2 e^{kx} dx dt$$

$$= 2 \sum_{n=1}^{\infty} |a_n|^2 \frac{(e^{2\sigma\lambda_n} - e^{-k\sigma})}{k+2\lambda_n},$$

where k is any positive number.
We shall obtain some of the properties of $m_{2,k}(\sigma)$ and $I_2(\sigma)$.

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THEOREM 1. *Let $f(s)$ be an integral function. Then, for $0 < \sigma_1 < \sigma_2$,*

$$I_2(\sigma_1) \leqq \frac{k}{2} \left\{ \frac{e^{k\sigma_2} m_{2,k}(\sigma_2) - e^{k\sigma_1} m_{2,k}(\sigma_1)}{e^{k\sigma_2} - e^{k\sigma_1}} \right\} \leqq I_2(\sigma_2),$$

where k is any positive number.

PROOF: From (1.2) and (1.3), we have

$$(2.1) \qquad \tfrac{1}{2} m_{2,k}(\sigma) = \frac{1}{e^{k\sigma}} \int_0^\sigma I_2(x) e^{kx} dx.$$

From (2.1) follows

$$(2.2) \qquad e^{k\sigma_2} m_{2,k}(\sigma_2) - e^{k\sigma_1} m_{2,k}(\sigma_1) = 2 \int_{\sigma_1}^{\sigma_2} I_2(x) e^{kx} dx$$

and the inequalities follow since $I_2(x)$ is an increasing function of x.

We may note that if $f(s)$ is an integral function, other than a constant, and $\alpha(0 < \alpha < 1)$ is a constant,

$$\lim_{\sigma \to \infty} \left\{ \frac{1}{m_{2,k}(\sigma) - e^{k\sigma(\alpha-1)} m_{2,k}(\alpha\sigma)} \right\} = 0.$$

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THEOREM 2. *If $f(s)$ be an integral function of order $\rho(0 < \rho < \infty)$, type τ and lower type v, then*

$$(3.1) \qquad \lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log m_{2,k}(\sigma)}{e^{\rho\sigma}} = \frac{2\tau}{2v}.$$

PROOF: From (1.3), we have

$$m_{2,k}(\sigma) \leqq \frac{2}{k} \{M(\sigma)\}^2 (1 - e^{-k\sigma}),$$

where

$$M(\sigma) = \operatorname*{l.u.b.}_{-\infty < t < \infty} |f(\sigma + it)|.$$

Taking limits, we get

$$(3.2) \qquad \lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log m_{2,k}(\sigma)}{e^{\rho\sigma}} \leqq 2 \lim_{\sigma \to \infty} \frac{\sup}{\inf} \frac{\log M(\sigma)}{e^{\rho\sigma}}.$$

Also, from (2.1), we have for $h > 0$

$$m_{2,k}(\sigma+h) = \frac{2}{e^{k(\sigma+h)}} \int_0^{\sigma+h} I_2(x)e^{kx}\,dx$$

(3.3)
$$\geqq \frac{2}{e^{k(\sigma+h)}} \int_\sigma^{\sigma+h} I_2(x)e^{kx}\,dx$$

$$\geqq \frac{2}{k} I_2(\sigma)(1-e^{-kh})$$

since $I_2(x)$ is an increasing function of x. Further, from (1.2), we have

(3.4)
$$I_2(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} \geqq \{\mu(\sigma)\}^2$$

where $\mu(\sigma) = |a_{N(\sigma)}|e^{\sigma\lambda_{N(\sigma)}}$ is the maximum term of rank $N(\sigma)$, for Re $s = \sigma$, in the series for $f(s)$. Therefore, from (3.3) and (3.4), we get

$$m_{2,k}(\sigma+h) \geqq \frac{2}{k} \{\mu(\sigma)\}^2(1-e^{-kh}).$$

Taking limits, we get

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log m_{2,k}(\sigma+h)}{e^{\rho(\sigma+h)}} \geqq \frac{2}{e^{\rho h}} \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log \mu(\sigma)}{e^{\rho\sigma}}$$

or

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log m_{2,k}(\sigma)}{e^{\rho\sigma}} \geqq \frac{2}{e^{\rho h}} \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log \mu(\sigma)}{e^{\rho\sigma}}.$$

Since left hand side is independent of h, therefore making $h \to 0$, we get

(3.5)
$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log m_{2,k}(\sigma)}{e^{\rho\sigma}} \geqq 2 \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log \mu(\sigma)}{e^{\rho\sigma}}.$$

The result (3.1) follows easily from (3.2) and (3.5) since for functions of finite order

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log \mu(\sigma)}{e^{\rho\sigma}} = \frac{\tau}{\nu}.$$

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THEOREM 3. *Let $f(s)$ be an integral function. Then*

$$\limsup_{\sigma\to\infty} \frac{m_{2,k}(\sigma)}{\{M(\sigma)\}^2} \leqq \limsup_{\sigma\to\infty} \frac{m_{2,k}(\sigma)}{I_2(\sigma)} \leqq \frac{2}{k},$$

where $M(\sigma) = \underset{-\infty < t < \infty}{\text{l.u.b.}} |f(\sigma + it)|$ and k is any positive number.

PROOF. Since $I_2(x)$ is an increasing function of x, therefore from (2.1), we have

$$m_{2,k}(\sigma) \leqq \frac{2}{e^{k\sigma}} I_2(\sigma) \int_0^\sigma e^{kx} dx$$

$$= \frac{2}{k} I_2(\sigma)(1 - e^{-k\sigma}).$$

Taking limits, we get

(4.1) $\underset{\sigma \to \infty}{\lim \sup} \dfrac{m_{2,k}(\sigma)}{I_2(\sigma)} \leqq \dfrac{2}{k}.$

Also, from (1.2), we have

(4.2) $I_2(\sigma) \leqq \{M(\sigma)\}^2.$

Therefore, from (4.1) and (4.2), follows

$$\underset{\sigma \to \infty}{\lim \sup} \frac{m_{2,k}(\sigma)}{\{M(\sigma)\}^2} \leqq \underset{\sigma \to \infty}{\lim \sup} \frac{m_{2,k}(\sigma)}{I_2(\sigma)} \leqq \frac{2}{k}.$$

REFERENCE

V. BERNSTEIN

[1] Leçons sur les progrès récents de la théorie des séries de Dirichlet, Gauthier Villars, Paris, 1933.