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Complex interpolation

by

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1. Introduction

In this paper we consider certain intermediate spaces "between" two Banach spaces. Our method generalizes the complex interpolation method introduced by Calderón [1] and Lions [2]. Let X_0, X_1 be given Banach spaces and let Ω denote the strip $0 < \xi < 1$ in the $\zeta = \xi + i\eta$ plane. If $\rho(\zeta)$ is continuous in $\bar{\Omega}$, we let $\mathscr{H}(X_0, X_1; \rho)$ denote the set of vector valued functions $f(\zeta)$ which are continuous in $\bar{\Omega}$, analytic in Ω, have values in X_j on $\xi = j$, $j = 0, 1$, and grow no faster than ρ at infinity (for a precise definition see Section 2). Let T be a (two dimensional) distribution with compact support in Ω. The Banach space $X_{T,\rho} = [X_0, X_1]_{T,\rho}$ is defined as the set of elements of the form $T(f)$, $f \in \mathscr{H}(X_0, X_1; \rho)$. The Banach space $X^{T,\rho} = [X_0, X_1]^{T,\rho}$ is defined as the set of elements x which satisfy $xT = fT$ in the sense of distributions for some $f \in \mathscr{H}(X_0, X_1; \rho)$. Basic properties of the spaces $X_{T,\rho}$, $X^{T,\rho}$ are studied (Propositions 2.1-2.6, 2.9, 2.10) and a general interpolation theorem is given (Theorem 2.8). For the case when the support of T is finite, further results are obtained (Theorems 2.11-2.16). The duals of both X_T and X^T are determined (Theorem 2.13) and it is shown that when one of the spaces X_0, X_1 is reflexive, then

$$(X_T)^* = [X_0^*, X_1^*]^T; \ (X^T)^* = [X_0^*, X_1^*]_T.$$

(Here $X_T = X_{T,1}$, $X^T = X^{T,1}$.) However, in general, these spaces are larger. This necessitated the study of larger spaces X'_T, X'^T along the lines of Calderón [3] (cf. Section 2). When $T = \delta(\theta)$, $0 < \theta < 1$, these results are due to Calderón [3]. When T is a one dimensional distribution the definition of X_T is due to Lions [2].

In Section 3 we consider the special case of $T = \delta^{(n)}(\theta)$, $0 < \theta < 1$,

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$n = 0, 1, 2, \ldots$. Some specialized results are proved. Two applications of the general theory are given in the following two sections. In Section 4 we consider the spaces $B^{\lambda,p}$, $H^{\lambda,p}$, where λ is a positive function on E^n and $1 \leqq p \leqq \infty$. A function $u(x)$ on E^n is in $B^{\lambda,p}$ if $\lambda\mathscr{F}u \in L^p(E^n)$, where $\mathscr{F}$ denotes the Fourier transform. Such spaces were studied by Hörmander [4]. The function $u(x)$ is in $H^{\lambda,p}$ if $\mathscr{F}^{-1}\lambda\mathscr{F}u \in L^p(E^n)$. These spaces were studied by several authors. Under mild assumptions on λ_0, λ_1 we show that

$$\{B^{\lambda_0,p}, B^{\lambda_1,p}]_T = B^{\lambda_3,p}, \quad [B^{\lambda_0,p}, B^{\lambda_1,p}]^T = B^{\lambda_2,p},$$
$$[H^{\lambda_0,p}, H^{\lambda_1,p}]_T = H^{\lambda_3,p}, \quad [H^{\lambda_0,p}, H^{\lambda_1,p}]^T = H^{\lambda_2,p},$$

where λ_2, λ_3 are positive functions depending on λ_0, λ_1, T.

The second application concerns functions of a closed operator A in a Banach space X. Under mild assumption on A, operators $\psi(\zeta, A)$ are defined in such a way as to be continuous in $\bar{\Omega}$ and analytic in Ω. From this one obtains an operator $\psi_T(A)$ which maps X into $[X, D(A)]_{T,\rho}$. For $0 < \theta < 1$ the operator $A^{-\theta}$ is such an operator with $T = \delta^{(2)}(\theta)$. Details and other results are given in Section 5. The last section, Section 6, is devoted to the proofs of the results of Section 2.

The idea to introduce a two dimensional distribution T is due to A. Lebow. Further results were obtained with him, and we hope to publish them soon. We are also grateful to J. L. Lions for interesting correspondence and H. Tanabe for stimulating conversations.

2. Complex interpolation

We consider interpolation spaces of the type introduced by Calderón [1] and Lions [2]. We shall call two Banach spaces X_0, X_1 compatible if they can be continuously embedded in a topological vector space V. We let X_0+X_1 denote the set of those elements $x \in V$ which can be written in the form

$$x = x^{(0)}+x^{(1)}, \tag{2.1}$$

where $x^{(j)} \in X_j$, $j = 0, 1$. Set

$$||x||_{X_0+X_1} = \inf \{||x^{(0)}||_{X_0}+||x^{(1)}||_{X_1}\}, \tag{2.2}$$

where the infinium is taken over all pairs $x^{(j)} \in X_j$ satisfying (2.1). One easily checks that (2.2) gives a norm on X_0+X_1 when X_0 and X_1 are compatible. Moreover, when X_0+X_1 is equipped with this norm, it becomes a Banach space (cf. [3]).

Let $\mathscr{A}$ denote the set of complex valued functions of the complex variable $\zeta = \xi + i\eta$ which are continuous on bounded subsets of $\bar{\Omega}$, where Ω is the strip $0 < \xi < 1$ in the $\zeta = \xi + i\eta$ plane. Let $\mathscr{B}$ be the set of those functions in $\mathscr{A}$ which are holomorphic in Ω and nonvanishing in $\bar{\Omega}$. For $\rho \in \mathscr{A}$ the space $\mathscr{H}(X_0, X_1; \rho)$ will consist of those functions $f(\zeta)$ with values in $X_0 + X_1$ such that

a) $f(\zeta)$ is continuous on bounded subsets of $\bar{\Omega}$

b) $f(\zeta)$ is holomorphic in Ω

c) $f(j+i\eta) \in X_j$, $j = 0, 1$, η real, and

$$\|f(j+i\eta)\|_{X_j} \leqq \text{const. } |\rho(j+i\eta)|. \tag{2.3}$$

Under the norm

$$\|f\|_{\mathscr{H}(X_0, X_1; \rho)} = \max_{j=0,1} \sup_{\eta} |\rho(j+i\eta)|^{-1} \|f(j+i\eta)\|_{X_j}, \tag{2.4}$$

$\mathscr{H}(X_0, X_1; \rho)$ becomes a Banach space. We set $\mathscr{H}(X_0, X_1) = \mathscr{H}(X_0, X_1; 1)$.

Let T be a distribution with compact support in Ω (i.e., $T \in \mathscr{E}'(\Omega)$). For $\rho \in \mathscr{A}$ we let $X_{T,\rho} \equiv [X_0, X_1]_{T,\rho}$ denote the set of those $x \in X_0 + X_1$ for which there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ satisfying

$$x = T(f). \tag{2.5}$$

If we introduce the norm

$$\|x\|_{T,\rho} = \|x\|_{X_{T,\rho}} = \inf \|f\|_{\mathscr{H}(X_0, X_1; \rho)},$$

where the infimum is taken over all f which satisfy (2.5), then we have

PROPOSITION 2.1. *$X_{T,\rho}$ is a Banach space.*

Let C denote the (Banach space of) complex numbers. One checks easily

PROPOSITION 2.2. *If $\rho, \sigma \in \mathscr{A}$ and $\omega \in \mathscr{H}(C, C; \sigma)$, then $X_{\omega T,\rho} \subseteqq X_{T,\rho\sigma}$ with*

$$\|x\|_{T,\rho\sigma} \leqq \|\omega\|_{\mathscr{H}(C,C;\sigma)} \|x\|_{\omega T,\rho}, \quad x \in X_{\omega T,\rho}. \tag{2.6}$$

Set $X_T = X_{T,1}$. Then we have

PROPOSITION 2.3. *If $\rho \in \mathscr{B}$, then $X_{T,\rho} \equiv X_{T\rho}$ with the same norm.*

Spaces closely related to the $X_{T,\rho}$ as defined as follows. We let $X^{T,\rho} \equiv [X_0, X_1]^{T,\rho}$ designate the set of those $x \in X_0 + X_1$ for which there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that

$$(2.7) \qquad fT = xT$$

in the sense of distributions. The norm in $T^{T,\rho}$ is given by

$$(2.8) \qquad ||x||^{T,\rho} = ||x||_{X^{T,\rho}} = \inf ||f||_{\mathscr{H}(X_0, X_1; \rho)},$$

where the infimum is taken over all $f \in \mathscr{H}(X_0, X_1; \rho)$ satisfying (2.7).

PROPOSITION 2.4. *$X^{T,\rho}$ is a Banach space.*

The counterpart of Proposition 2.2 is

PROPOSITION 2.5. *If $\rho, \sigma \in \mathscr{A}$ and $\omega \in C^\infty(\Omega)$, then $X^{T,\rho} \subseteqq X^{\omega T,\rho}$ and*

$$||x||^{\omega T,\rho} \leqq ||x||^{T,\rho}, \qquad x \in X^{T,\rho}.$$

If $\omega \neq 0$ on the support of T, the spaces are identical.

We set $X^T = X^{T,1}$. The analogue of Corollary 2.3 is not true in general.

PROPOSITION 2.6. *If $\rho, \sigma \in \mathscr{B}$, $\rho T = \sigma T$, then $X_{T,\rho} \equiv X_{T,\sigma}$, $X^{T,\rho} \equiv X^{T,\sigma}$ with identical norms.*

In studying duality for the spaces $X_{T,\rho}$ and $X^{T,\rho}$ one is led to consider spaces which are slightly larger. We employ ideas of Calderón [3]. Let $\mathscr{H}'(X_0, X_1; \rho]$ denote the space of (X_0+X_1)-valued functions on $\bar{\Omega}$ which are continuous on bounded subsets of $\bar{\Omega}$, holomorphic on Ω and such that $f(j+it_1)-f(j+it_2)$ is in X_j for all real t_1, t_2, $j = 0, 1$, and

$$(2.9) \qquad ||f(j+it_2)-f(j+it_1)|| \leqq M \int_{t_1}^{t_2} |\rho(j+it)|dt, \; t_1 < t_2, \; j = 0, 1.$$

The smallest constant M which works in (2.9) is the semi-norm of f in $\mathscr{H}'(X_0, X_1; \rho)$. If one considers $\mathscr{H}'(X_0, X_1; \rho)$ modulo the constant functions, this becomes a norm and the resulting space is a Banach space. We say that $x \in X'_{T,\rho}$ if $x = T(f')$ for some $f \in \mathscr{H}'(X_0, X_1; \rho)$ and its norm is the infimum of the semi-norms of all such f. Similarly, $x \in X'^{T,\rho}$ if $f'T = xT$ for some such f and its norm is defined correspondingly. If $f \in \mathscr{H}(X_0, X_1; \rho)$, then one checks easily that $\int_0^\zeta f(\zeta)d\zeta$ is in $\mathscr{H}'(X_0, X_1; \rho)$ with semi-norm not greater than the norm of f. Hence

$$(2.10) \qquad X_{T,\rho} \subseteqq X'_{T,\rho} \qquad X^{T,\rho} \subseteqq X'^{T,\rho}$$

with continuous injections. The primed spaces are similar to the unprimed ones. In fact we have

PROPOSITION 2.7. *Propositions 2.1–2.6 hold true if each space is replaced by its primed counterpart.*

For all of the spaces constructed we can state a general interpolation theorem.

THEOREM 2.8. *Let Y_0, Y_1 be another pair of compatible Banach spaces and define $Y_{T,\rho}$, etc., in the same way. If L is a linear mapping of X_0+X_1 into Y_0+Y_1 which is bounded from X_j to Y_j, $j = 0, 1$, then it is a bounded mapping from $X_{T,\rho}$ to $Y_{T,\rho}$ and from $X^{T,\rho}$ to $Y^{T,\rho}$. The same holds true for the primed spaces.*

Let $Z_1, \ldots, Z_N$ be Banach spaces continuously imbedded in a topological vector space V. We let $\sum Y_n$ denote the Banach space consisting of those elements of V of the form $y = \sum y_n$, $y_n \in Y_n$, with norm given by

$$||y|| = \inf \sum ||y_n||_{Y_n}.$$

The space $\cap Y_n$ is the set of those y common to all the Y_n with norm

$$||y|| = \max ||y||_{Y_n}.$$

PROPOSITION 2.9. *Assume that there are distributions $T_1, \ldots, T_N$ and functions $\omega_1, \ldots, \omega_N$; $\tau_1, \ldots, \tau_N$ in $\mathscr{H}(\boldsymbol{C}, \boldsymbol{C})$ such that*

$$T = \sum \omega_n T_n, \quad T_n = \tau_n T. \tag{2.11}$$

Then $X_{T,\rho} \equiv \sum X_{T_n,\rho}$, $X^{T,\rho} \equiv \cap X^{T_n,\rho}$, with equivalent norms. The same relationship holds for the primed spaces.

PROPOSITION 2.10. *If $\rho, \sigma \in \mathscr{A}$ satisfy*

$$\left|\frac{\rho(j+it)}{\sigma(j+it)}\right| \leqq M, \qquad j = 0, 1, t \text{ real},$$

then $X_{T,\rho} \subseteqq X_{T,\sigma}$, $X^{T,\rho} \subseteqq X^{T,\sigma}$ and

$$||x||_{T,\sigma} \leqq M\, ||x||_{T,\rho},\ ||x||^{T,\sigma} \leqq M ||x||^{T,\rho}.$$

The same holds for the primed spaces.

We now assume that the support of T consists of a finite number of points $z_1, \ldots, z_N$ in Ω. When acting on holomorphic functions T can be written in the form

$$T = \sum_{k=1}^{N} \sum_{l=0}^{m_k} a_{kl}\delta^{(l)}(z_k). \tag{2.12}$$

In this case we have

THEOREM 2.11. *If $\rho \in \mathscr{A}$ and $\omega \in C^\infty(\Omega)$, then $X_{\omega T,\rho} \subseteqq X_{T,\rho}$ with continuous injection. If $\omega \neq 0$ on the support of T, then the* *spaces are equivalent. Similar statements are true for the primed spaces.*

COROLLARY 2.12. *If* $\rho \in \mathscr{B}$, *then* $X_{T,\rho} \equiv X_T$.

THEOREM 2.13. *If* $X_0 \cap X_1$ *is dense in both* X_0 *and* X_1, *then the dual of* X_T *is isomorphic to* $[X_0^*, X_1^*]'^T$; *that of* X^T *to* $[X_0^*, X_1^*]'_T$.

THEOREM 2.14. *If either* X_0 *or* X_1 *is reflexive, then* $X'_T \equiv X_T$ *and* $X'^T \equiv X^T$. *Moreover, both of these spaces are reflexive when* $X_0 \cap X_1$ *is dense in both* X_0 *and* X_1.

If $a_{k,m_k} \neq 0$, we shall say that the distribution (2.12) is of order m_k at z_k. We shall say that T_0 is contained in T and write $T_0 \subseteqq T$ if the support of T_0 is contained in that of T and it is not of greater order than T at any point of its support.

THEOREM 2.15. *If* $T_0 \subseteqq T$, *then* $X_{T_0,\rho} \subseteqq X_{T,\rho}$ *and* $X^{T_0,\rho} \supseteqq X^{T,\rho}$ *with continuous inclusions. The same holds for the primed spaces.*

THEOREM 2.16. *If* $T = \sum T_n$ *and each* $T_n \subseteqq T$, *then*

$$X_{T,\rho} \equiv \sum X_{T_n,\rho}, \quad X^{T,\rho} \equiv \cap X^{T_n,\rho}.$$

A similar statement is true for the primed spaces.

REMARK 2.17. Since $e^{-\zeta^2} T \subseteqq T$, we see by Theorem 2.15 that if $x \in X_{T,\rho}$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that $x = T(e^{\zeta^2}f)$. Hence, when the support of T is finite, we can restrict ourselves to those $f \in \mathscr{H}(X_0, X_1; \rho)$ which satisfy $f/\rho \to 0$ as $|\zeta| \to \infty$ when defining the spaces $X_{T,\rho}$.

REMARK 2.18. For $T = \delta(\theta)$, $0 < \theta < 1$, Theorems 2.13 and 2.14 are due to Calderón [3]. Our proofs are modeled after his. Proofs for all of the results of this section are given in Section 6.

3. The case $T = \delta^{(n)}(\theta)$

Let $0 < \theta < 1$, $n \geqq 0$ and consider the special case when $T = \delta^{(n)}(\theta)$. We set $X_{T,\rho} = X_{\theta,\rho}^{(n)}$, $X_T = X_\theta^{(n)}$, $X_{\theta,\rho}^{(0)} = X_{\theta,\rho}$, $X_\theta^{(0)} = X_\theta$. By Theorem 2.11 we have

PROPOSITION 3.1. *For each* $\rho \in \mathscr{A}$

$$X_{\theta,\rho}^{(n)} \subseteqq X_{\theta,\rho}^{(n+1)}$$

with continuous injection.

For $\rho \equiv 1$, this result is due to Lions [9]. We also have by Corollary 2.12

PROPOSITION 3.2. *If* $\rho, \sigma \in \mathscr{B}$, *then*

$$X^{(n)}_{\theta,\rho} \equiv X^{(n)}_{\theta,\sigma}$$

with equivalent norms.

The case $n = 0$ is of special interest.

PROPOSITION 3.3. *If* $\rho, \sigma \in \mathscr{B}$ *then*

$$|\rho(\theta)| \, ||x||_{\delta(\theta),\rho} = |\sigma(\theta)| \, ||x||_{\delta(\theta),\sigma}, \quad x \in X_\theta. \tag{3.1}$$

PROPOSITION 3.4. *Assume* $\rho, \sigma \in \mathscr{B}$. *If* $f \in \mathscr{H}(X_0, X_1; \rho\sigma)$ *and*

$$||f(j+i\eta)||_{X_j} \leqq M_j |\rho(j+i\eta)\sigma(j+i\eta)|, \qquad j = 0, 1,$$

then $f(\theta) \in X_{\theta,\sigma}$ *and*

$$||f(\theta)||_{\delta(0),\sigma} \leqq M_0^{1-\theta} M_1^\theta |\rho(\theta)|. \tag{3.2}$$

PROPOSITION 3.5. *Suppose* $\rho \in \mathscr{B}$. *If* $f \in \mathscr{H}(X_0, X_1; \rho)$ *and*

$$||f(j+i\eta)||_{X_j} \leqq M_j |\rho(j+i\eta)|, \qquad j = 0, 1,$$

then for each real t, $f(\theta+it) \in X_\theta$ *and*

$$||f(\theta+it)||_{\delta(\theta)} \leqq M_0^{1-\theta} M_1^\theta |\rho(\theta+it)|. \tag{3.3}$$

For any Banach spaces Z_1, Z_2, we shall denote the (Banach) space of continuous linear mappings from Z_1 to Z_2 by $\{Z_1, Z_2\}$. If $Z_1 \equiv Z_2$ we denote it by $\{Z_1\}$. If $H(\zeta) \in \{Z_1, Z_2\}$ for each ζ in a set Ξ, it is said to be continuous (holomorphic) on Ξ if for each $z \in Z_1$ the Z_2-valued function $H(\zeta)z$ is continuous (holomorphic) in Ξ.

THEOREM 3.6. *Let* $H(\zeta)$ *be in* $\{X_0+X_1, Y_0+Y_1\}$ *for* $\zeta \in \bar{\Omega}$, *continuous on* $\bar{\Omega}$ *and holomorphic on* Ω, *where* Y_0, Y_1 *is another compatible pair of Banach spaces. Assume that* $H(j+i\eta) \in \{X_j, Y_j\}$ *with norm* $\leqq M_j \, |\rho(j+i\eta)|$, $j = 0, 1$, *where* $\rho \in \mathscr{B}$. *Then for each real* t, $H(\theta+it) \in \{X_\theta, Y_\theta\}$ *with norm* $\leqq M_0^{1-\theta} M_1^\theta \, |\rho(\theta+it)|$.

COROLLARY 3.7.

$$[\{X_0, Y_0\}, \{X_1, Y_1\}]_{\delta(\theta)} \subseteqq \{X_\theta, Y_\theta\}$$

with continuous injection.

THEOREM 3.8. *Suppose* $\rho \in \mathscr{B}$ *and that* $H(\zeta) \in \{X_0+X_1\}$ *for* $\zeta \in \bar{\Omega}$, *continuous in* $\bar{\Omega}$ *and holomorphic in* Ω. *Assume, in addition, that* $H(i\eta) \in \{X_j\}$ *with norm* $\leqq M_j |\rho(i\eta)|, j = 0, 1$, *while* $H(1+i\eta)$ $\in \{X_0, X_1\}$ *with norm* $\leqq M_2 |\rho(1+i\eta)|$. *Then for* $0 \leqq \theta+\theta_0 \leqq 1$, $H(\theta) \in \{X_{\theta_0}, X_{\theta_0+\theta}\}$ *with norm* $\leqq M_0^{1-\theta-\theta_0} M_1^{\theta_0} M_2^\theta |\rho(\theta)|$.

COROLLARY 3.9. *Under the same hypotheses,*

$H(\theta)H(\theta_0) \in \{X_0, X_{\theta+\theta_0}\}$ *with norm* $\leqq M_0^{2-2\theta_0-\theta} M_1^{\theta_0} M_2^{\theta+\theta_0} |\rho(\theta)\rho(\theta_0)|$.

REMARK 3.10. One sees easily that results similar to those of this section hold for the primed spaces as well.

PROOF OF PROPOSITION 3.3. By Proposition 2.3,

$$||x||_{\delta(\theta),\rho} = ||x||_{\rho\delta(\theta)} = ||x||_{\rho(\theta)\delta(\theta)} = |\rho(\theta)|\ ||x||_{\delta(\theta)}.$$

PROOF OF PROPOSITION 3.5. Set $g(\zeta) = M_0^{\zeta-1} M_1^{-\zeta} f(\zeta+it)/\rho(\zeta+it)$. Then one checks easily that $g \in \mathscr{H}(X_0, X_1)$ and $||g||_{\mathscr{H}(X_0,X_1)} \leqq 1$. Hence $||g(\theta)||_{\delta(\theta)} \leqq 1$. But $g(\theta) = M_0^{\theta-1} M_1^{-\theta} f(\theta+it)/\rho(\theta+it)$.

PROOF OF PROPOSITION 3.4. By Proposition 3.5,

$$||f(\theta)||_{\delta(\theta)} \leqq M_0^{1-\theta} M_1^{\theta} |\rho(\theta)\sigma(\theta)|.$$

But by Proposition 3.3

$$||f(\theta)||_{\delta(\theta)} = |\sigma(\theta)|\ ||f(\theta)||_{\delta(\theta),\sigma}.$$

PROOF OF THEOREM 3.6. If $x \in X_\theta$, then for every $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1)$ such that

$$x = f(\theta), \quad ||f||_{\mathscr{H}(X_0,X_1)} \leqq ||x||_{X_\theta} + \varepsilon.$$

Set $g(\zeta) = H(\zeta)f(\zeta - it)$. Then $g \in \mathscr{H}(X_0, X_1; \rho)$ and

$$||g(j+i\eta)||_{Y_j} \leqq M_j |\rho(j+i\eta)|\ ||f||_{\mathscr{H}(X_0,X_1)}, \qquad j = 0, 1.$$

Hence by Proposition 3.5,

$$||g(\theta+it)||_{Y_\theta} \leqq M_0^{1-\theta} M_1^{\theta} |\rho(\theta+it)|(||x||_{X_\theta} + \varepsilon).$$

But $g(\theta+it) = H(\theta+it)x$. Let $\varepsilon \to 0$.

PROOF OF THEOREM 3.8. By Theorem 3.6 $H(\theta+it) \in \{X_{1-\theta}, X_1\}$ with norm $\leqq M_1^{1-\theta} M_2^{\theta} |\rho(\theta+it)|$. If $x \in X_{\theta_0}$, then for every $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1)$ such that $f(\theta_0) = x$, $||f||_{\mathscr{H}(X_0,X_1)} \leqq ||x||_{\delta(\theta_0)} + \varepsilon$. Set

$$g(\zeta) = H\left(\frac{\theta\zeta}{\theta_0+\theta}\right) f\left(\frac{\theta_0\zeta}{\theta_0+\theta}\right).$$

Then

$$||g(i\eta)||_{X_0} \leqq M_0 \left|\rho\left(\frac{i\eta\theta}{\theta_0+\theta}\right)\right|\ ||f||_{\mathscr{H}(X_0,X_1)}$$

$$||g(1+i\eta)||_{X_1} \leqq M_1^{\theta_0/\theta_0+\theta} M_2^{\theta/\theta_0+\theta} \left|\rho\left(\frac{\theta(1+i\eta)}{\theta_0+\theta}\right)\right|\ ||f||_{\mathscr{H}(X_0,X_1)}.$$

The last inequality follows from the fact that

$$\left\|f\left(\frac{\theta_0(1+i\eta)}{\theta_0+\theta}\right)\right\|_{\mathscr{s}(\theta_0/\theta_0+\theta)} \leqq \|f\|_{\mathscr{H}(X_0,X_1)}.$$

Consequently we have $H(\theta)x = g(\theta_0+\theta) \in X_{\theta_0+\theta}$ and

$$\|H(\theta)x\|_{\mathscr{s}(\theta_0+\theta} \leqq M_0^{1-\theta_0-\theta} M_1^{\theta_0} M_2^{\theta} |\rho(\theta)|(\|x\|_{\mathscr{s}(\theta_0)}+\varepsilon).$$

We now let $\varepsilon \to 0$.

Proof of Corollary 3.9. $H(\theta_0) \in \{X_0, X_{\theta_0}\}$ with norm $\leqq M_0^{1-\theta_0} M_2^{\theta_0} |\rho(\theta_0)|$. We now apply Theorem 3.8.

4. Some examples

In this section we apply some of the results of the preceding sections to two very useful families of function spaces. Further applications are given in the next section.

We consider distributions $u(x)$ on E^n, $x = (x_1, \ldots, x_n)$. Let $\mathscr{F}u(\xi)$ denote the Fourier transform of $u(x)$, $\xi = (\xi_1, \ldots \xi_n)$. For a positive function $\lambda(\xi)$ we let $B^{\lambda,p}$ denote the set of those $u(x)$ such that $\lambda\mathscr{F}u \in L^p(E^n)$, $1 \leqq p \leqq \infty$. The norm of $u(x)$ in $B^{\lambda,p}$ is the L^p norm of $\lambda\mathscr{F}u$. These spaces were studied by Hörmander [4] when the function $\lambda(\xi)$ satisfies certain conditions.

Let $\lambda_0(\xi)$, $\lambda_1(\xi)$ be functions satisfying $\lambda_1(\xi) \geqq \lambda_0(\xi) > 0$. We assume that the distribution T is of the form (2.12). Set $\gamma = \lambda_1/\lambda_0$, $s_k = \operatorname{Re} z_k$,

$$\alpha = \sum \gamma^{s_k}(1+\log\gamma)^{m_k}, \quad \beta = \sum \gamma^{-s_k}(1+\log\gamma)^{m_k},$$

$\lambda_2 = \lambda_0\alpha$, $\lambda_3 = \lambda_0/\beta$. We also write

$$B_T = [B^{\lambda_0,p}, B^{\lambda_1,p}]_T, \quad B^T = [B^{\lambda_0,p}, B^{\lambda_1,p}]^T.$$

Theorem 4.1. *$B_T \equiv B^{\lambda_3,p}$, $B^T \equiv B^{\lambda_2,p}$ with equivalent norms.*

The proofs of Theorem 4.1 can be made to depend upon the following lemmas.

Lemma 4.2. *$u \in B_T$ if and only if there is a $g \in \mathscr{H}(L^p, L^p)$ such that*

$$\mathscr{F}u = \frac{1}{\lambda_0} T(\gamma^{-\zeta} g). \tag{4.1}$$

Proof. If $u \in B_T$, then there is an $f(\zeta) \in \mathscr{H}(B^{\lambda_0,p}, B^{\lambda_1,p})$ such that $u = T(f)$. Set $g = \lambda_0\lambda^{\zeta}\mathscr{F}f$. Then $g \in \mathscr{H}(L^p, L^p)$. Thus $\mathscr{F}u = T(\mathscr{F}f) = 1/\lambda_0\, T(\gamma^{-\zeta}g)$. Conversely, if u satisfies (4.1) for some $g \in \mathscr{H}(L^p, L^p)$, define f by $\mathscr{F} f = 1/\lambda_0 \gamma^{-\zeta} g$. One checks easily that $f \in \mathscr{H}(B^{\lambda_0, p}, B^{\lambda_1, p})$. But $\mathscr{F} u = T(\mathscr{F} f)$, i.e., $u = T(f)$ and hence $u \in B_T$. This completes the proof.

LEMMA 4.3. *Let $z_1, \ldots, z_N$ be fixed points in Ω and let $\{\nu_{kl}\}$, $1 \leqq k \leqq N$, $0 \leqq l \leqq m$, be given complex numbers. Then one can find a function $\omega \in \mathscr{H}(C, C)$ satisfying*

$$\omega^{(l)}(z_k) = \nu_{kl}, \; 1 \leqq k \leqq N, \; 0 \leqq l \leqq m$$
$$||\omega||_{\mathscr{H}(C,C)} \leqq K \sum |\nu_{kl}|,$$

where the constant K depends only on the z_k and m.

PROOF. It suffices to consider the case when all but one of the ν_{kl} vanish and the non-vanishing one is 1. For if we can find $\omega_{kl} \in \mathscr{H}(C, C)$ such that $\omega_{kl}^{(i)}(z_j) = \delta_{jk}\delta_{il}$, we can take $\omega = \sum \omega_{kl}\nu_{kl}$ for our desired function. Thus we may assume $\nu_{kl} = 1$, $\nu_{ji} = 0$ for $j \neq k$ or $i \neq l$. We take ω of the form

$$\omega(\zeta) = Q(\zeta) e^{\zeta^2} \prod_{j \neq k} (\zeta - z_j)^{m+1} \tag{4.2}$$

where Q is a polynomial. It is clear that we can pick Q so that ω satisfies the desired requirements.

COROLLARY 4.4. It $T_0 \subseteqq T$, there is a $\tau \in \mathscr{H}(C, C)$ such that $T_0 = \tau T$.

PROOF. The distribution T_0 must be of the form

$$T_0 = \sum_{k=1}^{N} \sum_{l=0}^{m_k} b_{kl} \delta^{(l)}(z_k). \tag{4.3}$$

Thus we need for each $\phi \in \mathscr{H}(C, C)$

$$\sum_{k=1}^{N} \sum_{l=0}^{m_k} \sum_{j=0}^{l} a_{kl} \binom{l}{j} \tau^{(l-j)}(z_k) \phi^{(j)}(z_k) = \sum_{k=1}^{N} \sum_{j=0}^{m_k} b_{kl} \phi^{(j)}(z_k).$$

This means that we must have

$$\sum_{l=j}^{m_k} a_{kl} \binom{l}{j} \tau^{(l-j)}(z_k) = b_{kl}, \quad 1 \leqq k \leqq N, \quad 0 \leqq j \leqq m_k.$$

We can solve these equations for the $\tau^{(j)}(z_k)$ and then apply Lemma 4.3.

PROOF OF THEOREM 4.1. If $u \in B_T$, there is a $g \in \mathscr{H}(L^p, L^p)$ such that (4.1) holds. Now

$$T(\gamma^{-\zeta} g) = \sum_{j,k,l} a_{kl} \binom{l}{j} \gamma^{-z_k} (\log \gamma)^{l-j} g^{(j)}(z_k).$$

Now $g^{(j)}(z_k) \in L^p$ and $\gamma^{-z_k}(\log\gamma)^m/\beta$ is bounded for $m \leqq m_k$. Hence $\beta^{-1}T(\gamma^{-\zeta}g) = \lambda_3\mathscr{F}u$ is in L^p. Thus $u \in B^{\lambda_3,p}$. Conversely, assume that $u \in B^{\lambda_3,p}$. We determine a function $\omega \in \mathscr{H}(C, C)$ satisfying

$$\sum_{l=j}^{m_k} a_{kl}\binom{l}{j}\omega^{(l-j)}(z_k) = \binom{m_k}{j} \tag{4.4}$$

for each k, j, $0 \leqq j \leqq m_k$. This can be done by solving (4.4) for the values of $\omega^{(j)}(z_k)$ and then employing Lemma 4.3. Then

$$\sum_{l=0}^{m_k}\sum_{j=0}^{l} a_{kl}\binom{l}{j}\omega^{(l-j)}(z_k)(\log\gamma)^j = (1+\log\gamma)^{m_k}.$$

Thus $T(\gamma^{-\zeta}\omega) = \sum\gamma^{-z_k}(1+\log\gamma)^{m_k} = \beta$. Set $g = \omega\lambda_3\mathscr{F}u$. Then $g \in \mathscr{H}(L^p, L^p)$. Moreover $T(\gamma^{-\zeta}g) = \lambda_3\mathscr{F}uT(\gamma^{-\zeta}\omega) = \lambda_3\beta\mathscr{F}u = \lambda_0\mathscr{F}u$. Hence $u \in B_T$.

Next assume that $u \in B^{\lambda_2,p}$. Set

$$T_k = \sum_{l=0}^{m_k} a_{kl}\delta^{(l)}(z_k).$$

Then $T = \sum T_k$. We now pick $\omega_k \in \mathscr{H}(C, C)$ to satisfy

$$\omega_k^{(j)}(z_k) = (-\log\gamma)^j, \qquad 0 \leqq j \leqq m_k \tag{4.5}$$

and

$$||\omega_k||_{\mathscr{H}(C,C)} \leqq \text{const. } (1+\log\gamma)^{m_k}.$$

This can be done by Lemma 4.3. Thus if $l \geqq 1$ we have at $\zeta = z_k$

$$d^l(\omega_k\gamma^{-\zeta})/d\zeta^l = \gamma^{-z_k}\sum_j\binom{l}{j}(\log\gamma)^{l-j}(-\log\gamma)^j = 0.$$

Thus $T_k(\omega_k\gamma^{-\zeta}\phi) = \gamma^{-z_k}T_k\phi$ for any $\phi \in \mathscr{H}(C, C)$ i.e., $\omega_k\gamma^{-\zeta}T_k = \gamma^{-z_k}T_k$. Since each $T_k \subseteqq T$, there is a $\tau_k \in \mathscr{H}(C, C)$ such that $T_k = \tau_k T$ (Corollary 4.4). We now define f by

$$\mathscr{F}f = \sum\omega_k\tau_k\gamma^{z_k-\zeta}\mathscr{F}u.$$

Hence

$$\mathscr{F}f = \sum\tau_k\frac{\gamma^{z_k}\omega_k}{\alpha}\cdot\frac{1}{\lambda_0}\cdot\gamma^{-\zeta}\lambda_2\mathscr{F}u.$$

This shows that $f \in \mathscr{H}(B^{\lambda_0,p}, B^{\lambda_1,p})$. Moreover,

$$\mathscr{F}fT = \sum\omega_k\gamma^{z_k-\zeta}\mathscr{F}uT_k = \mathscr{F}u\sum T_k = \mathscr{F}uT.$$

Hence $u \in B^T$. Finally, assume that $u \in B^T$. Then there is an $f \in \mathscr{H}(B^{\lambda_0,p}, B^{\lambda_1,p})$ such that $fT = uT$. This means that for each $\phi \in \mathscr{H}(C, C)$

$$\sum a_{kl}\binom{l}{j} f^{(l-j)}(z_k)\phi^{(j)}(z_k) = u \sum a_{kj}\phi^{(j)}(z_k).$$

Thus for each j, k

$$\sum_{l=j}^{m_k} a_{kl}\binom{l}{j} f^{(l-j)}(z_k) = a_{kj}u.$$

This implies that for each k

$$f(z_k) = u,\ f'(z_k) = \ldots = f^{(m_k)}(z_k) = 0.$$

Thus

$$\begin{aligned}\lambda_2 \mathscr{F} u &= \lambda_0 \sum \gamma^{z_k}(1+\log\gamma)^{m_k}\mathscr{F} u \\ &= \lambda_0 \sum \gamma^{z_k}(1+\log\gamma)^{m_k}\mathscr{F} f(z_k) \\ &= \sum \frac{d^{m_k}}{d\zeta^{m_k}}\left(\gamma_0\gamma^{\zeta}\mathscr{F} f(\zeta)\right)\big|_{\zeta=z_k} = \sum g^{(m_k)}(z_k),\end{aligned}$$

where $g \in \mathscr{H}(L^p, L^p)$. Thus $u \in B^{\lambda_2, p}$ and the proof is complete.

The second family of function spaces we shall consider is closely related to the first. For positive $\lambda(\xi)$ we let $H^{\lambda, p}$ denote the set of those distributions $u(x)$ such that $\mathscr{F}^{-1}\lambda\mathscr{F} u \in L^p(E^n)$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. When $\lambda(\xi)$ is of the form

(4.6) $$\lambda(\xi) = (1+|\xi|^2)^a,\ a \text{ real},$$

these spaces were studied by Calderón [5], Aronszajn, Mulla, Szeptycki [6], Lions, Magenes [7] and others.

In order to obtain the counterpart of Theorem 4.1 for the spaces $H^{\lambda, p}$ we shall make further restrictions. Specifically we shall assume that $1 < p < \infty$ and that the functions

(4.7) $$\gamma^{it},\ t \text{ real}$$

(4.8) $$\gamma^{z_k}(\log\gamma)^m/\alpha,\ \gamma^{-z_k}(\log\gamma)^m/\beta,\ 1 \leqq k \leqq N,\ m \leqq m_k$$

all belong to the space M_p of multipliers in L^p. Moreover the norms of γ^{it} in M_p are to be uniformly bounded. By employing Michlin's theorem [8], one can show easily that these assumptions are implied by

(4.9) $$|\xi|^{|\mu|}|D^\mu\gamma(\xi)| \leqq \text{const.}\ |\gamma(\xi)|, \qquad |\mu| \leqq n,$$

which in turn holds when λ_0 and λ_1 are of the form (4.6).

Under the above assumptions we can state

THEOREM 4.5. *$H_T \equiv H^{\lambda_3, p}$, $H^T \equiv H^{\lambda_2, p}$ with equivalent norms, where*

$$H_T = [H^{\lambda_0, p}, H^{\lambda_1, p}]_T, \quad H^T = [H^{\lambda_0, p}, H^{\lambda_1, p}]^T.$$

The proof of Theorem 4.5 follows that of Theorem 4.1 very closely. In place of Lemma 4.2 we have

LEMMA 4.6. *$u \in H_T$ if and only if there is a $g \in \mathscr{H}(L^p, L^p)$ such that*

$$. \mathscr{F}u = \frac{1}{\lambda_0} T(\gamma^{-\zeta} \mathscr{F}g).$$

One now follows the proofs of Lemma 4.2 and Theorem 4.1 word for word replacing the fact that the expressions (4.7, 8) are bounded by the fact that they are multipliers in L^p. In obtaining the ω_k to satisfy (4.5) we note, as in the proof Lemma 4.3, that they may be taken as linear combination of $(\log \gamma)^j$. Thus the expressions $\omega_k \gamma^{2k}/\alpha$ are in M_p.

5. Functions of an operator

Let X be a Banach space and A a closed linear operator in X with dense domain $D(A)$. We assume that the resolvent of A contains the negative real axis and that [1]

$$(5.1) \qquad ||(\lambda+A)^{-1}|| \leqq M(1+\lambda)^{-1}, \qquad \lambda > 0.$$

Let $\phi(\zeta, \lambda)$ be a complex valued function defined for $\zeta \in \bar{\Omega}$ and $0 < \lambda < \infty$ and satisfying the following conditions

(a) for each $\zeta \in \bar{\Omega}$, $\phi(\zeta, \lambda)$ is measurable (as a function of λ) and

$$(5.2) \qquad \int_0^\infty |\phi(\zeta, \lambda)|(1+\lambda)^{-1} d\lambda$$

is finite.

(b) considered as a function ζ, $\phi(\zeta, \lambda)$ is continuous in $\bar{\Omega}$ with respect to the norm (5.2).

(c) $\phi(\zeta, \lambda)$ is an analytic function of ζ in Ω with respect to the same norm.

(d) there is a $\rho \in \mathscr{B}$ such that

$$(5.3) \qquad \int_0^\infty |\phi(j+i\eta, \lambda)|(1+\lambda)^{j-1} d\lambda \leqq |\rho(j+i\eta)|,$$
$$\eta \text{ real}, \; j = 0, 1.$$

Under the above assumptions we define the following family of operators

$$\psi(\zeta, A) = \int_0^\infty \phi(\zeta, \lambda)(\lambda+A)^{-1} d\lambda, \qquad \zeta \in \bar{\Omega}.$$

[1] It is possible to weaken this assumption.

By (5.1) and (a), $\psi(\zeta, A)$ is a bounded operator on X for each $\zeta \in \bar{\Omega}$. For

$$||\psi(\zeta, A)|| \leqq M \int_0^\infty |\phi(\zeta, \lambda)|(1+\lambda)^{-1} d\lambda$$

and in particular

$$||\psi(i\eta, A)|| \leqq M|\rho(i\eta)| \tag{5.4}$$

Proposition 5.1. *For real η the operator $\psi(1+i\eta, A)$ maps X into $D(A)$ and*

$$||A\psi(1+i\eta, A)|| \leqq (M+1)|\rho(1+i\eta)|. \tag{5.5}$$

Proof. We first note that the adjoint A^* of A satisfies the same hypotheses as A. In particular, $||(\lambda+A^*)^{-1}|| \leqq M(1+\lambda)^{-1}$ and hence $||A^*(\lambda+A^*)^{-1}|| \leqq M+1$. Thus if $y = \psi(1+i\eta, A)x$ and $z \in D(A^*)$, then

$$\begin{aligned}\langle y, A^*z\rangle &= \int_0^\infty \phi(1+i\eta, \lambda) \langle(\lambda+A)^{-1}x, A^*z\rangle d\lambda \\ &= \int_0^\infty \phi(1+i\eta, \lambda) \langle x, (\lambda+A^*)^{-1}A^*z\rangle d\lambda,\end{aligned}$$

and hence

$$|\langle y, A^*z\rangle| \leqq (M+1)||x|| \, ||z|| \, |\rho(1+i\eta)|.$$

Thus $y \in D(A)$ and

$$||Ay|| \leqq (M+1)||x|| \, |\rho(1+i\eta)|,$$

which gives the result.

Next we consider $D(A)$ as a Banach space contained in X with norm $||x||+||Ax||$. We set

$$\psi^{(m)}(\zeta, A) = \frac{d^m}{d\zeta^m}\psi(\zeta, A) = \int_0^\infty \frac{\partial^m \phi(\zeta, \lambda)}{\partial \zeta^m}(\lambda+A)^{-1} d\lambda, \qquad m = 0, 1, 2, \ldots,$$

$$\psi_T(A) = T[\psi(\zeta, A)] = \int_0^\infty T[\phi(\zeta, \lambda)](\lambda+A)^{-1} d\lambda, \qquad T \in \mathscr{E}'(\Omega).$$

Theorem 5.2. *The operator $\psi_T(A)$ maps X boundedly into*

$$X_{T,\rho} = [X, D(A)]_{T,\rho}$$

with norm $\leqq M+2$. In particular, $\psi^{(m)}(\theta, A)$ maps X into

$$X_{\theta,\rho}^{(m)} = [X, D(A)]_{\delta^{(m)}(\theta),\rho}.$$

Proof. For $x \in X$ we have by (5.4) and (5.5) that $\psi(\zeta, A)x \in (X, D(A); \rho)$ and

$$||\psi(\zeta, A)x||_{\mathscr{H}(X, D(A); \rho)} \leqq (M+2)||x||.$$

Hence $\psi_T(A)x \in X_{T,\rho}$ and has norm $\leqq (M+2)||x||$.

Lemma 5.3. *For real η the operator $\psi(i\eta, A)$ maps $D(A)$ into itself with norm $\leqq M|\rho(i\eta)|$.*

Proof. If $x \in D(A)$, $z \in D(A^*)$ and $y = \psi(i\eta, A)x$, then

$$\begin{aligned}\langle y, A^*z\rangle &= \int_0^\infty \phi(i\eta, \lambda)\langle(\lambda+A)^{-1}x, A^*z\rangle d\lambda \\ &= \int_0^\infty \phi(i\eta, \lambda)\langle(\lambda+A)^{-1}Ax, z\rangle d\lambda.\end{aligned}$$

Hence

$$|\langle y, A^*z\rangle| \leqq M||Ax||\ ||z||\ |\rho(i\eta)|,$$

showing that $y \in D(A)$ and

$$||Ay|| \leqq M||Ax||\ |\rho(i\eta)|,$$

proving the lemma.

Theorem 5.4. *For $0 \leqq \theta+\theta_0 \leqq 1$, $\psi(\theta, A)$ maps X_{θ_0} boundedly into $X_{\theta_0+\theta}$ with norm $\leqq M^{1-\theta}(M+2)^\theta|\rho(\theta)|$.*

Proof. By (5.4) and Lemma 5.3, $\psi(i\eta, A)$ is in $\{X\}$ and $\{D(A)\}$ with norms $\leqq M\ |\rho(i\eta)|$, while $\psi(1+i\eta, A)$ is in $\{X, D(A)\}$ with norm $\leqq (M+2)|\rho(1+i\eta)|$ by Proposition 5.1. We now apply Theorem 3.8.

Let Y be a Banach space and let B be an operator defined in Y with the same properties as A. In particular, we assume

$$(5.6) \qquad ||(\lambda+B)^{-1}|| \leqq M(1+\lambda)^{-1} \qquad \lambda > 0.$$

Let L be a linear operator which maps X into Y in such a way that $D(A)$ maps into $D(B)$. Assume that

$$(5.7) \qquad ||Lx|| \leqq K_0||x|| \qquad x \in X$$

$$(5.8) \qquad ||BLx|| \leqq K_1||Ax|| \qquad x \in D(A).$$

Then we have

Theorem 5.5. *For $0 \leqq \theta \leqq 1$ the operator $\psi(1-\theta, B)L\psi(\theta, A)$ maps X into $D(B)$ and*

$$(5.9) \qquad ||B\psi(1-\theta, B)L\psi(\theta, A)|| \leqq M(M+2)K_0^{1-\theta}K_1^\theta|\rho(\theta)\rho(1-\theta)|.$$

PROOF. By Theorem 5.4, $\psi(\theta, A)$ maps X into X_θ with norm $\leqq M^{1-\theta}(M+2)^\theta|\rho(\theta)|$. Moreover, L maps X_θ into Y_θ with norm $\leqq K_0^{1-\theta}K_1^\theta$. Finally, $\psi(1-\theta, B)$ maps Y_θ into $D(B)$ with norm $\leqq M^\theta(M+2)^{1-\theta}|\rho(1-\theta)|$, again by Theorem 5.4. Combining these we get (5.9).

Next consider the operators A^θ defined by

$$A^{-\theta} = \frac{\sin \pi\theta}{\pi}\int_0^\infty \lambda^{-\theta}(\lambda+A)^{-1}d\lambda, \qquad 0 < \theta < 1,$$

as given by Kato [10]. If we set

$$\phi(\zeta, \lambda) = \lambda^{-\zeta}[(\log\lambda)^2+\pi^2]^{-2}\{[(\log\lambda)^2-\pi^2)\sin\pi\zeta+2\pi\log\lambda\cos\pi\zeta\},$$

then the corresponding operator $\psi(\zeta, A)$ satisfies all of the requirements above with $\rho = 1$. Moreover

$$A^{-\theta} = \psi^{(2)}(\theta, A).$$

Hence we have by Theorem 5.2.

PROPOSITION 5.6. *The domain of* A^θ *is contained in* $X_\theta^{(2)}$.

We also have

PROPOSITION 5.7. *The operator* $A^{-\theta}$ *maps* X_{θ_0} *boundedly into* $X_{\theta_0+\theta-\varepsilon}$ *for every* $\varepsilon > 0$, *and the operator* A^θ *maps* X_{θ_0} *boundedly into* $X_{\theta_0-\theta-\varepsilon}$ *for each* $\varepsilon > 0$. *In particular, the domain of* A^θ *is contained in* $X_{\theta-\varepsilon}$ *for each* $\varepsilon > 0$.

Our proof of Proposition 5.7 rests on

LEMMA 5.8. For $x \in X_{\theta_0}$ we have

$$(5.10) \qquad ||(\lambda+A)^{-1}x||_{s(\theta_0+\theta)} \leqq M^{1-\theta}(M+2)^\theta(1+\lambda)^{\theta-1}||x||_{s(\theta_0)}$$

$$(5.11) \qquad ||A(\lambda+A)^{-1}x||_{s(\theta_0-\theta)} \leqq (M+1)^{1-\theta}M^\theta(1+\lambda)^{-\theta}||x||_{s(\theta_0)}$$

PROOF. We first note that $(\lambda+A)^{-1}$ maps X into X and $D(A)$ into $D(A)$ each with norm $\leqq M(1+\lambda)^{-1}$. Moreover, it maps X into $D(A)$ with norm $\leqq M+2$. We now apply Theorem 3.8 to obtain (5.10). Next we note that the operator $A(\lambda+A)^{-1}$ maps X into X and $D(A)$ into $D(A)$ each with norm $\leqq M+1$ and it takes $D(A)$ into X with norm $\leqq M(1+\lambda)^{-1}$. We apply Theorem 3.8 again, taking note that $X_{\theta_0} = [D(A), X]_{s(1-\theta_0)}$.

PROOF OF PROPOSITION 5.7. We have by (5.10)

$$\begin{aligned}||A^{-\theta}x||_{s(\theta_0+\theta-\varepsilon)} &\leqq \frac{1}{\pi}\int_0^\infty \lambda^{-\theta}||(\lambda+A)^{-1}x||_{s(\theta_0+\theta-\varepsilon)}\,d\lambda \\ &\leqq \frac{1}{\pi}M^{1-\theta+\varepsilon}(M+2)^{\theta-\varepsilon}||x||_{s(\theta_0)}\int_0^\infty \lambda^{-\theta}(1+\lambda)^{\theta-\varepsilon-1}\,d\lambda\end{aligned}$$

and since $A^{\theta} = A^{\theta-1}A$,

$$\begin{aligned}||A^{\theta}x||_{s(\theta_0-\theta-\varepsilon)} &\leqq \frac{1}{\pi}\int_0^{\infty}\lambda^{\theta-1}||A(\lambda+A)^{-1}x||_{s(\theta_0-\theta-\varepsilon)}\,d\lambda \\ &\leqq \frac{1}{\pi}(M+1)^{1-\theta-\varepsilon}M^{\theta+\varepsilon}||x||_{s(\theta_0)}\int_0^{\infty}\lambda^{\theta-1}(1+\lambda)^{-\theta-\varepsilon}\,d\lambda\end{aligned}$$

by (5.11). This completes the proof.

6. Proofs

Proof of Proposition 2.1. One sees immediately that $X_{T,\rho}$ is a factor space of $\mathscr{H}(X_0, X_1; \rho)$, which is a Banach space.

Proof of Proposition 2.2. If $x \in X_{\omega T,\rho}$, then for every $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that

$$T(\omega f) = x, \quad ||f||_{\mathscr{H}(X_0,X_1;\rho)} \leqq ||x||_{\omega T,\rho}+\varepsilon.$$

Set $h = \omega f$. Then $h \in \mathscr{H}(X_0, X_1; \rho\sigma)$, $T(h) = x$, and

$$||h||_{\mathscr{H}(X_0,X_1;\rho\sigma)} \leqq ||\omega||_{\mathscr{H}(C,C;\sigma)}||f||_{\mathscr{H}(X_0,X_1;\rho)}.$$

Thus $x \in X_{T,\rho\sigma}$ and

$$||x||_{T,\rho\sigma} \leqq ||\omega||_{\mathscr{H}(C,C;\sigma)}(||x||_{\omega T,\rho}+\varepsilon)$$

Letting $\varepsilon \to 0$ we obtain (2.6).

Proof of Proposition 2.3. If $\rho \in \mathscr{B}$, then it is in $\mathscr{H}(C, C; \rho)$ with norm 1. Hence by Proposition 2.2 $X_{T\rho} = X_{\rho T,1} \subseteqq X_{T,\rho}$ and $||x||_{T,\rho} \leqq ||x||_{T\rho}$. Moreover $1/\rho$ is in $\mathscr{B}$ and hence in $\mathscr{H}(C, C; 1/\rho)$. Hence $X_{T,\rho} = X_{\rho T/\rho,\rho} \subseteqq X_{T\rho,\rho/\rho} = X_{T\rho}$ and $||x||_{T\rho} \leqq ||x||_{T,\rho}$.

Proof of Proposition 2.4. We must show that $X^{T,\rho}$ is complete. We first note that since T has compact support in Ω, there is a constant K_1 such that

$$(6.1) \qquad ||T(f)||_{X_0+X_1} \leqq K_1||f||_{\mathscr{H}(X_0,X_1;\rho)}, \quad f \in \mathscr{H}(X_0, X_1; \rho).$$

Let $\mathscr{S}$ be the manifold in $\mathscr{H}(X_0, X_1; \rho)$ consisting of those f for which there is an $x \in X_0+X_1$ satisfying (2.7). Since $X^{T,\rho}$ is a factor space of $\mathscr{S}$, the completeness of $X^{T,\rho}$ follows from the closedness of $\mathscr{S}$. Thus we must show that $\mathscr{S}$ is closed in $\mathscr{H}(X_0, X_1; \rho)$. Suppose $f_nT = x_nT$ and $||f_n-f_m||_{\mathscr{H}(X_0,X_1;\rho)} \to 0$. Then by (6.1)

$$\|x_n - x_m\|_{X_0+X_1} |T\phi| = \|T(x_n\phi - x_m\phi)\|_{X_0+X_1}$$
$$= \|T(f_n\phi - f_m\phi)\|_{X_0+X_1} \leqq K_1\|\phi\|_{\mathscr{H}(C,C)}\|f_n - f_m\|_{\mathscr{H}(X_0,X_1;\rho)}$$
$$\to 0$$

for each $\phi \in \mathscr{H}(C, C)$. Thus if $T\phi \neq 0$ we see that there is an $x \in X_0 + X_1$ such that $x_n \to x$ is X_0+X_1. Moreover, there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that $f_n \to f$ is $\mathscr{H}(X_0, X_1; \rho)$. Thus for $\phi \in \mathscr{H}(C, C)$

$$T(f\phi) = \lim T(f_n\phi) = \lim T(x_n\phi) = T(x\phi)$$

and hence $fT = xT$. Thus $\mathscr{S}$ is closed and the proof is complete.

Proof of Proposition 2.5. If $x \in X^{T,\rho}$, then for any $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that

$$(6.2) \qquad fT = xT, \ \|f\|_{\mathscr{H}(X_0,X_1;\rho)} \leqq \|x\|^{T,\rho} + \varepsilon.$$

Then $f\omega T = x\omega T$. Hence $x \in X^{\omega T,\rho}$ and

$$\|x\|^{\omega T,\rho} \leqq \|f\|_{\mathscr{H}(X_0,X_1;\rho)} \leqq \|x\|^{T,\rho} + \varepsilon.$$

Letting $\varepsilon \to 0$ gives the result.

Proof of Proposition 2.6. If $x \in X_{T,\rho}$ then for $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that

$$(6.3) \qquad x = T(f), \ \|f\|_{\mathscr{H}(X_0,X_1;\rho)} \leqq \|x\|_{T,\rho} + \varepsilon.$$

Set $h = f\sigma/\rho$. Then $h \in \mathscr{H}(X_0, X_1; \sigma)$ and

$$\|h\|_{\mathscr{H}(X_0,X_1;\sigma)} \leqq \|f\|_{\mathscr{H}(X_0,X_1;\rho)}.$$

Moreover $T(h) = \sigma T(f/\rho) = \rho T(f/\rho) = x$. Hence $x \in X_{T,\sigma}$ and $\|x\|_{T,\sigma}$ and $\|x\|_{T,\sigma} \leqq \|x\|_{T,\rho} + \varepsilon$. Letting $\varepsilon \to 0$ we get the result in one direction. The other direction follows from symmetry. For $X^{T,\rho}$ we use (6.2). In this case $hT = (f/\rho)\sigma T = (f/\rho)\rho T = xT$ and the rest is the same.

Proof of Proposition 2.7. Consider the counterpart of Proposition 2.2. If $x \in X'_{\omega T,\rho}$ then there is an $f \in \mathscr{H}'(X_0, X_1; \rho)$ such that $x = T(f'\omega)$ and

$$\|f(j+it_2) - f(j+it_1)\|_{X_j} \leqq (\|x\|'_{\omega T,\rho} + \varepsilon) \int_{t_1}^{t_2} |\rho(j+it)|dt,$$
$$t_1 < t_2, \quad j = 0, 1.$$

Set $h(\zeta) = \int_a^\zeta f'\omega d\zeta$, where a is some point of Ω. Then $h(\zeta)$ is obviously an (X_0+X_1)-valued function holomorphic in Ω. We can also write $h(\zeta)$ as a Riemann-Stieltjes integral

$$h(\zeta) = \int_a^\zeta \omega(\zeta)df(\zeta).$$

Since $f(\zeta)$ is Lipschitz continuous on bounded subsets of $\bar{\Omega}$, h can be extended to be continuous up to the boundary. But

$$h(j+it_2)-h(j+it_1) = \int_{t_1}^{t_2} \omega(j+it)df(j+it)$$

and hence $h(j+it_2)-h(j+it_1) \in X_j$ with

$$||h(j+it_2)-h(j+it_1)||_{X_j} \leqq (||x||'_{\omega T,\rho}+\varepsilon) \int_{t_1}^{t_2} |\omega(j+it)\rho(j+it)|dt.$$

Making use of the fact that

$$|\omega(j+it)| \leqq ||\omega||_{\mathscr{H}(C,C;\sigma)} |\sigma(j+it)|, \qquad j = 0, 1,$$

we have

$$||h||_{\mathscr{H}'(X_0,X_1;\rho\sigma)} \leqq ||\omega||_{\mathscr{H}(C,C;\sigma)}(||x||'_{\omega T,\rho}+\varepsilon).$$

Now $x = T(h')$. Hence $x \in X'_{T,\rho\sigma}$ and

$$||x||'_{T,\rho\sigma} \leqq ||\omega||_{\mathscr{H}(C,C;\sigma)}||x||'_{\omega T,\rho}.$$

Next consider Proposition 2.6. Set $\omega = \sigma/\rho$. Then $\omega \in \mathscr{H}(C, C; \sigma/\rho)$. Thus by the above,

$$X'_{\omega T,\rho} \subseteqq X'_{T,\sigma} \quad \text{and} \quad ||x||'_{T,\sigma} \leqq ||x||'_{\omega T,\rho}.$$

We now note that $\omega T = \sigma T/\rho = T$ and the rest follows from symmetry. The remaining proofs are similar to those for the unprimed cases.

Proof of Theorem 2.8. If $x \in X_{T,\rho}$, then for every $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that (6.3) holds. Set $g = Lf$. Since

$$||g(j+it)||_{Y_j} \leqq M_j||f(j+it)||_{X_j}, \qquad j = 0, 1,$$

we see that $g \in \mathscr{H}(Y_0, Y_1; \rho)$ and that

$$||g||_{\mathscr{H}(Y_0,Y_1;\rho)} \leqq \max(M_0, M_1)||f||_{\mathscr{H}(X_0,X_1;\rho)}.$$

Since $Lx = LT(f) = T(g)$, we have $Lx \in Y_{T,\rho}$ and

$$||Lx||_{Y_{T,\rho}} \leqq \max(M_0, M_1)(||x||_{X_{T,\rho}}+\varepsilon).$$

Letting $\varepsilon \to 0$ we have the first statement. The second follows in a similar way.

Proof of Proposition 2.9. If $x \in X_{T,\rho}$, then for every $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that (6.3) holds. Thus $x = T(f) = \sum \omega_n T_n(f) = \sum x_n$, where $x_n = T_n(\omega_n f)$. Now $\omega_n f \in \mathscr{H}(X_0, X_1; \rho)$

$$||\omega_n f||_{\mathscr{H}(X_0, X_1; \rho)} \leqq ||\omega_n||_{\mathscr{H}(C, C)} ||f||_{\mathscr{H}(X_0, X_1; \rho)}.$$

Thus

$$||x_n||_{T_n, \rho} \leqq ||\omega_n||_{\mathscr{H}(C, C)} (||x||_{T, \rho} + \varepsilon).$$

Letting $\varepsilon \to 0$, we have

$$\sum ||x_n||_{T_n, \rho} \leqq \sum ||\omega_n||_{\mathscr{H}(C, C)} ||x||_{T, \rho},$$

showing that $X_{T, \rho} \subseteqq \sum X_{T_n, \rho}$. Conversely, if

$$x = \sum x_n, \quad x_n = T_n(f_n), \quad f_n \in \mathscr{H}(X_0, X_1; \rho),$$

then $x = \sum \tau_n T(f_n) = T(f)$, where $f = \sum \tau_n f_n$. Since

$$||f||_{\mathscr{H}(X_0, X_1; \rho)} \leqq \sum ||\tau_n||_{\mathscr{H}(C, C)} ||f_n||_{\mathscr{H}(X_0, X_1; \rho)},$$

we have $x \in X_{T, \rho}$ and

$$||x||_{T, \rho} \leqq (\max ||\tau_k||_{\mathscr{H}(C, C)}) \sum ||x_n||_{T_n, \rho}.$$

Next, if $x \in X^{T, \rho}$, there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that (6.2) holds. Then $xT_n = x\tau_n T = f\tau_n T = fT_n$. Hence $x \in X^{T_n, \rho}$ and

$$||x||_{X^{T_n, \rho}} \leqq ||x||_{X^{T, \rho}} + \varepsilon.$$

Letting $\varepsilon \to 0$ we have $X^{T, \rho} \subseteqq \cap X^{T_n, \rho}$. Conversely, if $x \in X^{T_n, \rho}$ for each n, there are $f_n \in \mathscr{H}(X_0, X_1; \rho)$ such that $xT_n = f_n T_n$ and

$$||f_n||_{\mathscr{H}(X_0, X_1; \rho)} \leqq ||x||_{X^{T_n, \rho}} + \varepsilon.$$

Then $xT = x \sum \omega_n T_n = \sum \omega_n f_n T_n = \sum \omega_n f_n \tau_n T$. Setting $f = \sum \omega_n \tau_n f_n$, we have $f \in \mathscr{H}(X_0, X_1; \rho)$ and

$$||f||_{\mathscr{H}(X_0, X_1; \rho)} \leqq \sum ||\omega_n \tau_n||_{\mathscr{H}(C, C)} ||x||_{X^{T_n, \rho}} + \varepsilon)$$

showing that $\cap X^{T_n, \rho} \subseteqq X^{T, \rho}$ and the proof is complete.

Proof of Proposition 2.10. If $x \in X_{T, \rho}$, then for $\varepsilon > 0$ there is an $f \in \mathscr{H}(X_0, X_1; \rho)$ such that (6.3) holds. Now

$$||f(j+it)||_{X_j} |\sigma(j+it)|^{-1} \leqq M \, ||f(j+it)||_{X_j} |\rho(j+it)|^{-1}.$$

Hence $f \in \mathscr{H}(X_0, X_1; \sigma)$ and

$$||f||_{\mathscr{H}(X_0, X_1; \sigma)} \leqq M ||f||_{\mathscr{H}(X_0, X_1; \rho)}.$$

Thus $x \in X_{T, \sigma}$ and $||x||_{T, \sigma} \leqq M||x||_{T, \rho}$. The proof for $X^{T, \rho}$ is similar.

Proof of Theorem 2.11. If we can find a $\phi \in \mathscr{H}(C, C)$ such that $\omega T = \phi T$, then the first statement will follow from Proposition 2.2. But obviously $\omega T \subseteqq T$. Thus the existence of the required

ϕ follows from Corollary 4.4. If $\omega \neq 0$ on the support of T, we can set $T_0 = \omega T$, $T = \omega^{-1}T_0$ and employ the same reasoning in the other direction.

Corollary 2.12 follows immediately from Theorem 2.11 and Proposition 2.3.

The proofs of Theorems 2.13 and 2.14 are postponed until the end of this section.

Proof of Theorem 2.15. By Corollary 4.4 there is a $\tau \in \mathscr{H}(C, C)$ such that $T_0 = \tau T$. We now apply Theorem 2.11.

Proof of Theorem 2.16. By Corollary 4.4 there is a $\tau_n \in \mathscr{H}(C, C)$ such that $T_n = \tau_n T$. Thus the hypotheses of Proposition 2.9 are satisfied.

Let $\mu_0(\zeta, t)$, $\mu_1(\zeta, t)$ be the Poisson kernels for Ω (cf. [3, § 9.4]). Thus every function $h(\zeta)$ harmonic in Ω, continuous in $\overline{\Omega}$ and $0(e^{a|\zeta|})$, $a < \pi$, can be represented by

$$h(\zeta) = \sum_{j=0}^{1} \int_{-\infty}^{\infty} \mu_j(\zeta, t) h(j+it)\, dt. \tag{6.4}$$

In proving Theorems 2.13 and 2.14 we shall make use of the following.

Lemma 6.1. *If the support of T is finite, then there are distributions $T_1, \ldots, T_M$ contained in T and a constant K_3 such that for every $\omega \in C^\infty(\Omega)$ one can find a $\tau \in \mathscr{H}(C, C)$ satisfying*

$$\tau T = \omega T, \quad ||\tau||_{\mathscr{H}(C,C)} \leqq K_3 \sum |T_n(\omega)|. \tag{6.5}$$

Proof. If T is of the form (2.12) we can take the T_n to be the distributions $\delta^{(l)}(z_k)$, $1 \leqq k \leqq N$ $0 \leqq l \leqq m_k$. Then $\tau T = \omega T$ if $T_n(\tau - \omega) = 0$ for each n. We then apply Lemma 4.3.

Theorem 6.2. *For each compact set $\Xi \subset \Omega$ having the support of T in its interior there is a constant K with the following property. For each $f \in \mathscr{H}(X_0, X_1)$ and each $\varepsilon > 0$ there is an $h \in \mathscr{H}(X_0, X_1)$ satisfying $hT = fT$ and*

$$||h||_{\mathscr{H}(X_0,X_1)} \leqq K \max_{\zeta \in \Xi} e^{\phi(\zeta)+\varepsilon}, \tag{6.6}$$

where

$$\phi(\zeta) = \sum_{j=0}^{1} \int_{-\infty}^{\infty} \mu_j(\zeta, t) \log ||f(j+it)||_{X_j}\, dt. \tag{6.7}$$

Proof. Let Ξ and ε be given. Let $\psi_j(t)$ be bounded functions in $C^\infty(-\infty, \infty)$ satisfying $\psi_j(t) \geqq \log ||f(j+it)||_{X_j}$, $j = 0, 1$. Set

(6.8) $$\psi(\zeta) = \sum_{j=0}^{1} \int_{-\infty}^{\infty} \mu_j(\zeta, t)\psi_j(t)\,dt.$$

We can choose the functions ψ_j so that $\psi \leqq \phi+\varepsilon$ in Ξ. Let $\Psi \in \mathscr{H}(C, C)$ be such that $\psi(\zeta) = \operatorname{Re} \Psi(\zeta)$. By Lemma 6.1 there is a $\tau \in \mathscr{H}(C, C)$ such that

(6.9) $$e^{\Psi}T = \tau T, \ \ ||\tau||_{\mathscr{H}(C,C)} \leqq K_3 \sum |T_n(e^{\Psi})|.$$

Let Ξ_1 be a closed set in the interior of Ξ but containing the support of T in its interior. Since Ψ is holomorphic

(6.10) $$\begin{aligned} |T_n(e^{\Psi})| &\leqq K_4 \max_{\Xi_1} \sum_{j=0}^{m} |d^{(j)} e^{\Psi}/d\zeta^j| \\ &\leqq K_5 \max_{\Xi} e^{\psi}, \end{aligned}$$

where m is the maximum of the numbers m_k. Set $h = \tau e^{-\Psi} f$. Then $fT = fe^{-\Psi}e^{\Psi}T = hT$ and by (6.10)

$$||h||_{\mathscr{H}(X_0,X_1)} \leqq ||\tau||_{\mathscr{H}(C,C)} \leqq K \max_{\Xi} e^{\phi+\varepsilon}.$$

This completes the proof.

Corollary 6.3. *The inequality*

(6.11) $$||T(f)||_T \leqq K \max_{\zeta \in \Xi} e^{\phi(\zeta)}$$

holds for all $f \in \mathscr{H}(X_0, X_1)$. Moreover if $xT = fT$, then $||x||^T$ is bounded by the same expression.

Proof. By Theorem 6.2 for any $\varepsilon > 0$ there is an $h \in \mathscr{H}(X_0, X_1)$ such that $hT = fT$ and (6.6) holds. Thus $T(f) = T(h)$ and

$$||T(f)||_T \leqq ||h||_{\mathscr{H}(X_0,X_1)}.$$

Hence

$$||T(f)||_T \leqq K \max_{\Xi} e^{\phi+\varepsilon}.$$

Now let $\varepsilon \to 0$. Similarly, if $xT = fT = hT$, $||x||^T$ is bounded by the same expression and the same reasoning applies.

Theorem 6.4. *There is a constant K_6 depending only on T with the following property. For each $f \in \mathscr{H}(X_0, X_1)$ and $\varepsilon > 0$ there is an $h \in \mathscr{H}(X_0, X_1)$ such that $hT = fT$ and*

(6.12) $$||h||_{\mathscr{H}(X_0,X_1)} \leqq K_6 e^{\varepsilon} \sum_{j=0}^{1} \int_{-\infty}^{\infty} |T[\mu_j(\cdot, t)]|\ ||f(j+it)||_{X_j}\,dt.$$

Proof. We follow the proof of Theorem 6.2. The only difference is that we replace (6.10) by the following reasoning. Since $T_n \subseteqq T$, there is a $\tau_n \in \mathscr{H}(C, C)$ such that $T_n = \tau_n T$. Since Ψ is holomorphic we have by (6.4)

$$\tau_n(\zeta)e^{\Psi(\zeta)} = \sum_{j=0}^{1}\int_{-\infty}^{\infty} \mu_j(\zeta, t)e^{\Psi(j+it)}\tau_n(j+it)\,dt.$$

Hence

$$|T_n(e^{\Psi})| \leqq ||\tau_n||_{\mathscr{H}(C,C)}\sum_{j=0}^{1}\int_{-\infty}^{\infty} |T[\mu_j(\cdot, t)]|e^{\psi_j(t)}\,dt.$$

We now merely make use of the fact that $\phi(j+it) = \log ||f(j+it)||_{X_j}$ and choose the ψ_j so that $\psi(j+it) \leqq \phi(j+it)+\varepsilon$ for real t.

Corollary 6.5. *The inequality*

$$(6.13)\qquad ||T(f)||_T \leqq K_6 \sum_{j=0}^{1}\int_{-\infty}^{\infty} |T[\mu_j(\cdot, t)]|\ ||f(j+it)||_{X_j}dt$$

holds for all $f \in \mathscr{H}(X_0, X_1)$. *If* $xT = fT$, *then* $||x||^T$ *is bounded by the same expression.*

Theorem 6.6. *Let* $\{f_n\}$ *be a sequence of elements in* $\mathscr{H}(X_0, X_1)$ *such that*

$$||f_n||_{\mathscr{H}(X_0, X_1)} \leqq K_7$$

and assume that $f_n(it)$ *converges in* X_0 *for each* t *in a set* E *of positive measure. Then*

(a) $T(f_n)$ *converges in* X_T.

(b) *There is a subsequence (also denoted by* $\{f_n\}$*) and a sequence* $\{h_n\}$ *of elements in* $\mathscr{H}(X_0, X_1)$ *such that* $f_nT = h_nT$ *and* h_n *converges in* $\mathscr{H}(X_0, X_1)$.

(c) *If* $x_nT = f_nT$, *then* x_n *converges in* X^T.

Proof. (a)(c) We may assume that $K_7 \geqq 1$ and that E is bounded. Set

$$\phi_{mn}(\zeta) = \sum_{j=0}^{1}\int_{-\infty}^{\infty} \mu_j(\zeta, t)\log ||f_m(j+it)-f_n(j+it)||_{X_j}dt.$$

Then

$$\phi_{mn}(\zeta) \leqq \int_E \mu_0(\zeta, t)\log ||f_m(it)-f_n(it)||_{X_0}dt+\log 2K_7.$$

Let Ξ be a compact set in Ω containing the support of T in its interior. Since $\mu_0(\zeta, t)$ is positive and bounded away from zero for $\zeta \in \Xi$ and $t \in E$, we have $\phi_{nm} \to -\infty$ as $m, n \to \infty$ uniformly on Ξ. By Corollary 6.3

$$||T(f_m-f_n)||_T \leqq K \max_{\Xi} e^{\phi_{mn}} \to 0$$

as $m, n \to \infty$. The same reasoning applies to (c).

(b) By deleting members of the sequence, if necessary, we may always guarantee that

$$Ke^{\phi_{n,n-1}} < 1/n^2$$

for $\zeta \in \Xi$. Thus by Theorem 6.2 we can find a $g_n \in \mathscr{H}(X_0, X_1)$ such that $g_n T = (f_n - f_{n-1})T$ and

$$||g_n||_{\mathscr{H}(X_0, X_1)} < 1/n^2.$$

(Here we have taken $f_0 = 0$.) Set $h_n = \sum_{k=1}^n g_k$. Then clearly $\{h_n\}$ is a convergent sequence in $\mathscr{H}(X_0, X_1)$. Moreover $h_n T = \sum_{k=1}^n (f_k - f_{k-1})T = f_n T$. This completes the proof.

We shall employ some additional ideas of Calderón [3]. For a Banach space X, let $\Gamma(X)$ denote the space of X-valued continuous functions $h(t)$ on the real line such that

$$||h||_{\Gamma(X)} = \int_{-\infty}^{\infty} ||h(t)||_X \, dt$$

is finite. The space of Lipschitz continuous X-valued functions $g(t)$ is denoted by $\Lambda(X)$ with semi-norm

$$||g||_{\Lambda(X)} = \sup |t_2 - t_1|^{-1} ||g(t_2) - g(t_1)||_X.$$

When $h \in \Gamma(X)$ has compact support and $g \in \Lambda(X^*)$, then the Riemann-Stieltjes integral

$$\int_{-\infty}^{\infty} \langle j(t), dg(t) \rangle = \lim \sum \langle h(t_j'), g(t_{j+1}) - g(t_j) \rangle, \qquad t_j \leqq t_j' \leqq t_{j+1}, \tag{6.14}$$

is easily seen to exist and satisfy

$$\left| \int_{-\infty}^{\infty} \langle h(t), dg(t) \rangle \right| \leqq ||h||_{\Gamma(X)} ||g||_{\Lambda(X^*)}. \tag{6.15}$$

By continuity, this extends to all $h \in \Gamma(X)$. Thus the integral (6.14) represents a bounded linear functional on $\Gamma(X)$. The converse was also proved by Calderón [3].

LEMMA 6.7. *Every bounded linear functional F on $\Gamma(X)$ can be expressed by an integral of the form* (6.14) *with $g \in \Lambda(X^*)$ and* $||g||_{\Lambda(X^*)} = ||F||$.

We shall also use the following from Calderón [3].

LEMMA 6.8. *Assume that $X_0 \cap X_1$ is dense in both X_0 and X_1. Let $g_j(t) \in \Lambda(X_j^*)$, $j = 0, 1$, be such that for each $x \in X_0 \cap X_1$ the functions $d/dt \langle x, g_j(t) \rangle$ are the boundary values of a holomorphic function in Ω. Then there is a $g \in \mathscr{H}'(X_0^*, X_1^*)$ such that*

$$g(j+it) = ig_j(t) + constant, \qquad j = 0, 1. \tag{6.16}$$

LEMMA 6.9. *If* X_0 *is reflexive,* $g \in \mathscr{H}'(X_0, X_1)$ *and*

$$h_n(\zeta) = \frac{n}{i}\left[g\left(\zeta + \frac{i}{n}\right) - g(\zeta)\right], \qquad n = 1, 2, \ldots,$$

then $h_n(it)$ *converges in* X_0 *for almost all* t.

We shall also use

LEMMA 6.10. *If* $g(t)$ *is a bounded measurable function in* $(0, 2\pi)$, *for any* $f(z)$ *analytic in* $|z| \leqq 1$ *let* $H_f(z)$ *be the harmonic function in* $|z| < 1$ *satisfying*

$$H_f(e^{i\theta}) = f(e^{i\theta})g(e^{i\theta}) \qquad \text{a.e.}$$

Suppose that for some integer $m \geqq 0$,

$$H_f^{(m)}(0) = 0$$

for every f *satisfying* $f^{(m)}(0) = \ldots f(0) = 0$. *Then* $g(e^{i\theta})$ *is the boundary value of a bounded function analytic in* $|z| < 1$.

PROOF. We have

$$H_f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\phi - \theta) f(e^{i\theta})g(e^{i\theta})d\theta,$$

where $z = re^{i\theta}$ and

$$P_r(\phi - \theta) = e^{i\theta}(e^{i\theta} - z)^{-1} + \bar{z}e^{i\theta}(1 - \bar{z}e^{i\theta})^{-1}.$$

Hence

$$H_f^{(m)}(0) = \frac{m!}{2\pi}\int_0^{2\pi} e^{-im\theta} f(e^{i\theta})g(e^{i\theta})d\theta.$$

By hypotheses, this vanishes if we take $f(z) = z^{m+k}$, $k \geqq 1$. Hence

$$\int_0^{2\pi} e^{ik\theta} g(e^{i\theta})d\theta = 0, \qquad k = 1, 2, \ldots.$$

This implies the conclusion.

The following is due to Aronszajn-Gagliardo [11, Th. 8. III].

LEMMA 6.11. *Let* $Z_1, \ldots, Z_n$ *be Banach spaces continuously embedded in a topological sector space such that* $\cap Z_j$ *is dense in each* Z_k. *Then*

$$(\cap Z_j)^* \equiv \sum Z_j^*, \quad (\sum Z_j)^* \equiv \cap Z_j^*.$$

We are now ready for the

PROOF OF THEOREM 2.13. We first note that it suffices to prove the theorem for the case when the support of T consists of a single point. For by Propositions 2.9 and 2.16,

$$(X_T)^* = (\sum X_{T_n})^* = \cap (X_{T_n})^* = \cap [X_0^*, X_1^*]'^{T_n} = [X_0, X_1]'^T$$
$$(X^T)^* = (\cap X^{T_n})^* = \sum [X_0^*, X_1^*]'_{T_n} = [X_0, X_1]'_T,$$

where the T_n are weaker than T and have supports consisting of single points. Suppose that the support of T is at z_0 and it is of order m. Let T_0 be any other distribution equivalent to T. Then $X_T = X_{T_0}$. For every $f \in \mathscr{H}(X_0, X_1)$ the function

$$f(j+it)T_0[\mu_j(\cdot, t)]$$

is in $\Gamma(X_j)$, $j = 0, 1$. For $L \in (X_T)^*$ we define

$$\mathscr{L}\{f(j+it)T_0[\mu_j(\cdot, t)\} = L[T_0(f)].$$

Now consider $\mathscr{L}$ as a linear functional on a submanifold of $\Gamma(X_0) \oplus \Gamma(X_1)$. By Corollary 6.5 it is a bounded functional. Thus we may consider it extended to the whole of $\Gamma(X_0) \oplus \Gamma(X_1)$ with the same norm. Thus by Lemma 6.7 there are $g_j(t) \in \Lambda(X_j^*)$ such that

$$L[T_0(f)] = \sum_{j=0}^{1} \int_{-\infty}^{\infty} T_0[\mu_j(\cdot, t)]\langle f(j+it), dg_j(t)\rangle.$$

Let h be in $\mathscr{H}(\mathrm{C}, \mathrm{C})$ and $x \in X_0 \cap X_1$. Then

$$L[T_0(hx)] = \sum_{j=0}^{1} \int_{-\infty}^{\infty} T_0[\mu_j(\cdot, t)]h(j+it) \frac{d}{dt} \langle x, g_j(t)\rangle dt.$$

Now the functions $d\langle x, g_j(t)\rangle/dt$ are bounded and measurable, and the above expression vanishes whenever $T_0(h) = 0$. We map Ω conformally onto the unit disc $|w| < 1$ in such a way that z_0 is mapped into the origin, e.g., we let

$$\zeta = W(w) = \frac{1}{\pi i} \log \left(\frac{we^{-i\pi z_0} - e^{i\pi z_0}}{w-1}\right).$$

This maps $|w| \leqq 1$ continuously onto $\bar{\Omega}$. Set

$$\tilde{g}(e^{i\theta}) = \frac{d}{dt} \langle x, g_j(ij - iW(e^{i\theta}))\rangle \quad \text{when} \quad \mathscr{Re}\, W(e^{i\theta}) = j.$$

Then $\tilde{g}(e^{i\theta})$ is a bounded measurable function. Now

$$\sum_{j=0}^{1} \int_{-\infty}^{\infty} \mu_j(\zeta, t)h(j+it) \frac{d}{dt} \langle x; g_j(t)\rangle dt$$

is the harmonic function in Ω with boundary values $h(j+it)d\langle x, g_j(t)\rangle/dt$. Under the mapping it is taken into the harmonic function in $|w| < 1$ with boundary values $h(W(e^{i\theta}))\tilde{g}(e^{i\theta})$. Call this function $H_h(w)$. The discussion so far holds for any distribution T_0 equivalent to T. We now pick a particular one — namely the one which maps into $\delta^{(m)}(0)$ under the conformal transformation. We consider the $g_j(t)$ associated with this particular distribution. In this case we have $H_f^{(m)}(0) = 0$ whenever $f^{(m)}(0) = \ldots = f(0) = 0$. This allows us to apply Lemma 6.10 to conclude that the functions $d\langle x, g_j(t)\rangle/dt$ are the boundary values of a holomorphic function in Ω. This in turn shows by Lemma 6.8 that there is a $g \in \mathscr{H}'(X_0^*, X_1^*)$ such that $g(j+it) = ig_j(t)+\text{constant}$. Hence

$$\begin{aligned} L[T_0(f)] &= -i\sum_{j=0}^{1}\int_{-\infty}^{\infty} T_0[\mu_j(\cdot, t)]\langle f(j+it), dg(j+it)\rangle \\ &= T_0(\langle f(\zeta), g'(\zeta)\rangle). \end{aligned}$$

Take $f = hx$, where $h \in \mathscr{H}(C, C)$ and $x \in X_0 \cap X_1$. Then

$$L[T_0(hx)] = T_0(\langle hx, g'\rangle) = \langle x, T_0(hg')\rangle$$

and hence

$$L(x) = \langle x, T_0(hg')/T_0(h)\rangle$$

provided $T_0(h) \neq 0$. This shows that there is an element $x^* \in X_0^*+X_1^*$ which equals $T_0(hg')/T_0(h)$ for all $h \in \mathscr{H}(C, C)$. This means that $g'T_0 = x^*T_0$, i.e., that $x^* \in [X_0^*, X_1^*]'^{T_0}$. Thus

$$L[T_0(f)] = \langle T_0(f), x^*\rangle.$$

Working back, one checks readily that every $x^* \in [X_0^*, X_1^*]'^{T_0}$ does indeed give rise to a bounded linear functional on X_{T_0}. This shows that $(X_{T_0})^* = [X_0^*, X_1^*]'^{T_0}$. But by Proposition 2.7, $[X_0^*, X_1^*]'^{T_0} = [X_0^*, X_1^*]'^{T}$. Hence the first statement is established. To prove the second, let L be a bounded linear functional on X^T. If $xT_0 = fT_0$, where $f \in \mathscr{H}(X_0, X_1)$ we have by Corollary 6.5

$$|L(x)| \leqq K_8 \sum_{j=0}^{1}\int_{-\infty}^{\infty} |T_0[\mu_j(\cdot, t)]| \, \|f(j+it)\|_{X_j} dt.$$

Reasoning as before we see that there are $g_j(t) \in \Lambda(X_j^*)$, $j = 0, 1$, such that

$$L(x) = \sum_{j=0}^{1}\int_{-\infty}^{\infty} T_0[\mu_j(\cdot, t)]\langle f(j+it), dg_j(t)\rangle.$$

We shall prove, as before, that there is a $g \in \mathscr{H}'(X_0, X_1)$ such that (6.16) holds. Assuming this we have

$$L(x) = T_0(\langle f(\zeta), g'(\zeta)\rangle) = T_0(\langle x, g'(\zeta)\rangle) = \langle x, T_0(g')\rangle.$$

Since $g \in \mathscr{H}'(X_0, X_1)$, $T_0(g') \in [X_0^*, X_1^*]'_{T_0}$, which is the desired conclusion. It therefore remains only to show that such a g exists. By Lemma 6.8 it suffices to show that for each $x \in X_0 \cap X_1$ the function $d\langle x, g_j(t)\rangle/dt$ are the boundary values of a holomorphic function in Ω. If $x \in X_0 \cap X_1$ we may take $f = x$ and hence by (6.20)

$$L(x) = \sum_{j=0}^{1} \int_{-\infty}^{\infty} T_0[\mu_j(\cdot, t)] \frac{d}{dt} \langle x, g_j(t)\rangle \, dt.$$

Moreover if $h \in \mathscr{H}(C, C)$ and $h_0 T = 0$, then $xT_0 \equiv (h+1)xT_0$ and hence

$$L(x) = \sum_{j=0,1} \int_{-\infty}^{\infty} T_0[\mu_j(\cdot, t)]\{h(j+it)+1\} \frac{d}{dt} \langle x, g_j(t)\rangle \, dt.$$

Subtracting we have

$$\sum_{j=0,1} \int_{-\infty}^{\infty} T_0[\mu(\cdot, t)]h(j+it) \frac{d}{dt} \langle x, g_j(t)\rangle \, dt = 0$$

whenever $hT_0 = 0$. The result now follows as before.

PROOF OF THEOREM 2.14. Assume X_0 is reflexive. If $x \in X'_T$, there is a $g \in \mathscr{H}'(X_0, X_1)$ such that $x = T(g')$. Set

$$h_n(\zeta) = \frac{n}{i}\left[g\left(\zeta + \frac{i}{n}\right) - g(\zeta)\right], \qquad n = 1, 2, \ldots.$$

Then $h_n(\zeta) \in \mathscr{H}(X_0, X_1)$ and

$$||h_n||_{\mathscr{H}(X_0, X_1)} \leqq ||g||_{\mathscr{H}'(X_0, X_1)}.$$

By Lemma 6.9, $h_n(it)$ converges in X_0 for almost all t. Hence by Theorem 6.6(a) $T(h_n)$ converges in X_T to an element of X_T. But $T(h_n)$ converges in X_0+X_1 to $T(g') = x$. Hence $x \in X_T$. If $x \in X'^T$, then $xT = g'T$ for some $g \in \mathscr{H}'(X_0, X_1)$. Moreover $h_n T(\phi) \to g'T(\phi)$ for each $\phi \in \mathscr{H}(C, C)$. This follows from the fact that h_n and all its derivatives converge to g' uniformly and each compact set in Ω. Now by Theorem 6.6(b) there is a subsequence $\{\tilde{h}_n\}$ of $\{h_n\}$ and a sequence $\{f_n\}$ of elements of $\mathscr{H}(X_0, X_1)$ such that $f_n T = \tilde{h}_n T$ and f_n converges in $\mathscr{H}(X_0, X_1)$ to some $f \in \mathscr{H}(X_0, X_1)$. Hence $f_n T(\phi) \to fT(\phi)$ for each $\phi \in \mathscr{H}(C, C)$. But $f_n T(\phi) = \tilde{h}_n T(\phi) = xT(\phi)$. Hence $xT = fT$ and $x \in X^T$. This completes the proof of the first statement. The second follows from the first and Theorem 2.13.

7. Remarks

Theorem 7.1. *$X_0 \cap X_1$ is dense in X_T.*

Proof: If $x \in X_T$, there is an $f \in \mathscr{H}(X_0, X_1)$ such that $T(f) = x$. We may assume that f vanishes at infinity. Calderón has shown [3.9.2] that sums of the form $\sum \phi_j x_j$ with $\phi_j \in \mathscr{H}(C, C)$ and $x_j \in X_0 \cap X_1$ are dense in $\mathscr{H}(X_0, X_1)$. Hence there is a sequence f_n of such sums which approaches f in $\mathscr{H}(X_0, X_1)$. Thus $T(f_n)$ approaches $T(f) = x$ in X_T.

Theorem 7.2. *If $X_0 \cap X_1$ is dense in both X_0 and X_1, then it is dense in X^T.*

Proof: Suppose $l \in (X^T)^*$ and $l(x) = 0$ for all $x \in X_0 \cap X_1$. Then l can be realized by an element $y \in [X_0^*, X_1^*]_T'$ (Theorem 2.13). This latter space is continuously imbedded in $X_0^* + X_1^*$, which by Lemma 6.11, is isomorphic to $(X_0 \cap X_1)^*$. Since l vanishes on $X_0 \cap X_1$, y must be zero.

Lemma 7.3. For $0 < \theta_1 \leqq \theta_2 < 1$, $0 < \theta < 1$, $\theta_3 = (1-\theta)\theta_1 + \theta\theta_2$,

$$X_{\theta_3} \subseteqq [X_{\theta_1}, X_{\theta_2}]_{s(\theta)} \tag{7.1}$$

$$X'_{\theta_3} \subseteqq [X'_{\theta_1}, X'_{\theta_2}]'_{s(\theta)} \tag{7.2}$$

with continuous injections.

Proof: If $x \in X_{\theta_3}$, there is an $f \in \mathscr{H}(X_0, X_1)$ such that $x = f(\theta_3)$. Set $g(\zeta) = f((1-\zeta)\theta_1 + \zeta\theta_2)$. Then $g \in \mathscr{H}(X_{\theta_1}, X_{\theta_2})$, $g(\theta) = x$ and

$$||g||_{\mathscr{H}(X_{\theta_1}, X_{\theta_2})} \leqq ||f||_{\mathscr{H}(X_0, X_1)}.$$

This proves (7.1). If $y \in X'_{\theta_3}$, there is an $h \in \mathscr{H}'(X_0, X_1)$ such that $y = h'(\theta_3)$. Now for $0 < \alpha < 1$, $h'(\alpha + it)$ is in X'_α and

$$||h'(\alpha+it)||'_{s(\alpha)} \leqq ||h||_{\mathscr{H}'(X_0, X_1)}.$$

Set

$$v(\zeta) = (\theta_2 - \theta_1)^{-1} h((1-\zeta)\theta_1 + \zeta\theta_2).$$

Then

$$\begin{aligned} v(j+it_2) - v(j+it_1) &= (\theta_2-\theta_2)^{-1}\{h(\theta_1 + j(\theta_2-\theta_1) + it_2(\theta_2-\theta_1) \\ &\qquad - h(\theta_1 + j(\theta_2-\theta_1) + it_1(\theta_2-\theta_1))\} \\ &= i\int_{t_1}^{t_2} h'(\theta_1 + j(\theta_2-\theta_1) + it)dt. \end{aligned}$$

Thus $v \in \mathscr{H}'(X_{\theta_1}, X_{\theta_2})$ and $||v||_{\mathscr{H}'(X_{\theta_1}, X_{\theta_2})} \leqq ||h||_{\mathscr{H}'(X_0, X_1)}$. Since $x = v'(\theta)$, this gives (7.2).

Theorem 7.4. *If $X_0 \cap X_1$ is dense in both X_0 and X_1, then*

$$X_{\theta_3} = [X_{\theta_1}, X_{\theta_2}]_{\delta(\theta)}. \tag{7.3}$$

Proof: By (7.1) and (7.2) and the fact that $(X_\alpha)^* = [X_0^*, X_1^*]'_{\delta(\alpha)}$ (Theorem 2.13), we see that X_{θ_3} is a closed subspace of $A = [X_{\theta_1}, X_{\theta_2}]_{\delta(\theta)}$. Now A^* is continuously imbedded in $X_0^* + X_1^* = (X_0 \cap X_1)^*$. If a functional in A^* vanishes on X_{θ_3}, it vanishes on $X_0 \cap X_1$ and hence must be identically zero. Hence X_{θ_3} is dense in A and must therefore be the entire space.

Theorem 7.4 was proved by Calderón [3] under the additional assumption that $X_0 \cap X_1$ is dense in $X_{\theta_1} \cap X_{\theta_2}$. However, this assumption is superfluous since $X_{\theta_1} \cap X_{\theta_2} = X^T$ with $T = \delta(\theta_1) + \delta(\theta_2)$ and we may apply Theorem 7.2. Theorem 7.4 was also proved in [12] under the additional assumption that X_0 and X_1 are reflexive.

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