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# Extremal noints in summability theory

by

J. W. Baker and G. M. Petersen

## 1.

The main purpose of this paper is to study the norms of regular summability methods introduced by Brudno in [4]. In this paper, the term *matrix* will be reserved for regular summability matrices. We shall be studying the summability of bounded sequences throughout. The usual norm for the space of bounded sequences is $||\{s_n\}|| = \sup |s_n|$, and the *unit sphere* is the set of bounded sequences $s = \{s_n\}$ with $||s|| \leqq 1$. If $A$ is a matrix, then $\mathscr{A}$ denotes the set of bounded sequences which are summed by $A$, $\mathscr{A}$ is called the *summability field* of $A$. If $\{s_n\}$ is summed by $A$, $A$-lim $s_n$ denotes the value to which it is summed. Two matrices, $A$ and $B$, are said to be *b-consistent* if whenever a sequence is summed by both matrices, it is summed to the same value by $A$ as by $B$; $A$ is said to be *b-stronger* than $B$ if we have $\mathscr{A} \supset \mathscr{B}$. The following result relates these last two definitions, see [3] and [7].

THEOREM 1. *If $A$ and $B$ are matrices with $A$ b-stronger than $B$, then $A$ and $B$ are b-consistent.*

From this theorem it is clear that if $\mathscr{A} = \mathscr{B}$ then $A$ and $B$ sum exactly the same bounded sequences to the same values. In this case we say that $A$ and $B$ are *b-equivalent*. By the *summability method*, $\mathfrak{U}$, we mean the set of all matrices which have $\mathscr{A}$ as their summability field.

## 2.

Let $A$ be a matrix, then we have,

$$h(A) = \sup_m \sum_{n=1}^{\infty} |a_{m,n}| < \infty;$$

$h(A)$ is called the *matrix norm* of $A$. We may define,

(1) $$N_A = \sup |A\text{-lim } s_n| \leqq h(A),$$

where the supremum is taken over all the bounded sequences $s$, which are in the unit sphere and summed by $A$. If $B$ is $b$-equivalent to $A$, then $N_B = N_A$, so that $N_A$ is a function of the summability field $\mathscr{A}$ rather than of the matrices. For this reason, for each matrix we define the *field norm*, $N(\mathscr{A})$ by

$$N(\mathscr{A}) = N_A.$$

Since every matrix must sum the unit sequence to unity, the following theorem is an immediate consequence of Theorem 1.

THEOREM 2. *If $A$ and $B$ are matrices with $A$ b-stronger than $B$, we have,*

$$N(\mathscr{A}) \geqq N(\mathscr{B}) \geqq 1.$$

We can also consider

$$||\mathfrak{A}|| = \inf h(A),$$

where the infimum is taken over the matrices, $A$, which have $\mathscr{A}$ as their summability field. The following fundamental theorem is due to Brudno, [4].

THEOREM 3. *For every matrix $A$,*

$$||\mathfrak{A}|| = N(\mathscr{A}).$$

If there is an $A'$ in $\mathfrak{A}$, such that

$$h(A') = ||\mathfrak{A}|| = N(\mathscr{A}),$$

we shall say that the field norm of $\mathscr{A}$ is *attained* by the matrix $A'$; otherwise, we shall say that the norm is *not attained*.

We shall subsequently require the following constructions and notation. Suppose that $A^1, A^2, \ldots, A^n, \ldots$ is a countable family of matrices. Firstly, we may obtain a sequence, $\{\mu(k)\}$, of natural numbers, $\mu(k) \nearrow \infty$, such that,

$$\sum_{p=\mu(k)}^{\infty} |a^n_{m,p}| < \frac{1}{k}, \qquad (1 \leqq m, n \leqq k), \tag{2}$$

then it follows that,

$$\lim_{m\to\infty} \sum_{p=\mu(m)}^{\infty} |a^n_{m,p}| = 0, \qquad (n = 1, 2, \ldots). \tag{3}$$

Since,

$$\lim_{m\to\infty} |a^n_{m,p}| = 0, \qquad (n, p = 1, 2, \ldots),$$

it is clear that there is a sequence, $\{m(r)\}$, of natural numbers with $m(r) > m(r-1)$, $(r = 2, 3, \ldots)$, such that

$$\sum_{p=1}^{r} |a^n_{m,p}| < \frac{1}{r}, \qquad (n \leqq r, r = 1, 2, \ldots),$$

if $m \geqq m(r)$. Then a sequence $\{\lambda(m)\}$ may be selected so that $\lambda(m) = 1$, $(m < m(2))$, and $\lambda(m) = r$, $(m(r) \leqq m < m(r+1))$. Then,

$$\lim_{m\to\infty} \sum_{p=1}^{\lambda(m)} |a^n_{m,p}| = 0, \qquad (n = 1, 2, \ldots), \tag{4}$$

and $\{\lambda(m)\}$, $\lambda(m) \nearrow \infty$ satisfies,

$$\lambda(m) - \lambda(m-1) \leqq 1, \qquad (m = 2, 3, \ldots). \tag{5}$$

If necessary, we may also increase the terms of $\{\mu(m)\}$ so that $\lambda(m) < \mu(m)$, $(m = 1, 2, \ldots)$, without changing the other properties of $\{\mu(m)\}$. This construction of $\{\lambda(m)\}$ and $\{\mu(m)\}$ is of course possible in particular for a finite set of matrices, $A^1, A^2, \ldots, A^N$.

Before proceeding further, we require two auxiliary theorems. We first make some definitions.

DEFINITION 1. Let $E$ be a set of bounded sequences, the set is said to be *uniformly summable* to $s$ by the matrix $A$, if there exists a squence, $\{\varepsilon_n\}$, $\varepsilon_n \searrow 0$, such that,

$$\left|\sum_{k=1}^{\infty} a_{m,k}s_k - s\right| < \varepsilon_m, \qquad (m = 1, 2, \ldots),$$

for every sequence, $\{s_n\}$, in $E$. The set is said to be *uniformly bounded* if there exists an $H$ such that $\|\{s_n\}\| \leqq H$, for every $\{s_n\}$ in $E$.

DEFINITION 2. A sequence $\{s_n\}$ is *eventually bounded* by the sequence $\{\xi_n\}$, if there exists an $N$ such that $|s_n| \leqq |\xi_n|$, $(n \geqq N)$.

If $\{\zeta^k_n\}$, $\zeta^k_n \searrow 0$, $(k = 1, 2, \ldots)$, is a countable set of sequences, let $m(k)$, $(k = 1, 2, \ldots)$ be the index such that,

$$|\zeta^r_n| < \frac{1}{k}, \qquad (n \geqq m(k), 1 \leqq r \leqq k).$$

Let $\{\xi_n\}$ be the sequence defined by

$$\xi_n = 1, \quad (n < m(1)), \quad \xi_n = \frac{1}{k}, \qquad (m(k) \leqq n < m(k+1)),$$

then $\{\zeta^k_n\}$ is eventually bounded by $\{\xi_n\}$, for each $k$, $(k = 1, 2, \ldots)$, and $\xi_n \searrow 0$.

THEOREM 4. *Let $A^1$, $A^2, \ldots, A^n, \ldots$ be a countable set of matrices, and let $\{s_k^n\}$, $(n = 1, 2, \ldots)$ be a countable set of uniformly bounded sequences, such that $\{s_k^n\}$, $(n \geqq p)$ is summed to zero by $A^p$, $(p = 1, 2, \ldots)$. Then, there exists a set of bounded sequences, $\{t_k^n\}$, $(n = 1, 2, \ldots)$ which are uniformly summed to zero by $A^p$, $(p \leqq n, p = 1, 2, \ldots)$, and such that,*

$$s_k^n = t_k^n, \qquad (k \geqq \bar{N}(n), n = 1, 2, \ldots),$$
$$|t_k^n| \leqq |s_k^n|, \qquad (n, k = 1, 2, \ldots).$$

PROOF. Let $\zeta_m^{n,p} = \sup_{r \geq m} |A_r^p(s^n)|$, $(m, n = 1, 2, \ldots; n \geqq p)$, where,

$$A_r^p(s^n) = \sum_{k=1}^{\infty} a_{r,k}^p s_k^n.$$

Since $\zeta_m^{n,p} \searrow 0$ $(n = 1, 2, \ldots; p \leqq n)$, there exists a sequence $\{\xi_m\}$, $\xi_m \searrow 0$, such that $\{A_m^p(s^n)\}$ is eventually bounded by $\{\xi_m\}$ for each $n$ and each $p$, $(n = 1, 2, \ldots; p \leqq n)$. Therefore, there exists an $N(n)$ such that,

$$|A_m^p(s^n)| < \xi_m, \qquad (m > N(n), p \leqq n).$$

Let $|s_k^n| \leqq H$, $(k, n = 1, 2, \ldots)$, $h(A^p) = M_p$, $(p = 1, 2, \ldots)$, and let $\{\lambda(m)\}$ and $\{\mu(m)\}$ be chosen as above. Then we have,

$$\sum_{k=1}^{\lambda(m)} |a_{m,k}^p| < \varepsilon_{m,p}, \qquad \sum_{k=\mu(m)}^{\infty} |a_{m,k}^p| < \varepsilon_{m,p},$$

where $\varepsilon_{m,p} \searrow 0$, $(p = 1, 2, \ldots)$. Let $\{\nu(r)\}$ be the sequence of indices such that $\lambda(\nu(r)) = \mu(\nu(r-1))$, $(r = 2, 3, \ldots)$, $\nu(1) = 1$. To construct $\{t_k^n\}$, select $q$ so that,

$$\lambda\left(\nu\left[\frac{q(q-1)}{2}\right]\right) > \mu(N(n)),$$

and set,

$$t_k^n = 0, \qquad \left(k \leqq \lambda\left(\nu\left[\frac{q(q-1)}{2}\right]\right)\right),$$

$$t_k^n = \frac{r}{q} s_k^n, \quad \left(\lambda\left(\nu\left[\frac{q(q-1)}{2} + r - 1\right]\right) < k \leqq \lambda\left(\nu\left[\frac{q(q-1)}{2} + r\right]\right)\right),$$

$$t_k^n = s_k^n, \quad \left(k > \lambda\left(\nu\left[\frac{q(q+1)}{2}\right]\right)\right); \qquad (1 \leqq r \leqq q),$$

and note that $|t_k^n| \leqq |s_k^n|$, $(k, n = 1, 2, \ldots)$.

i) If $m \leqq N(n)$, $p \leqq n$, we have,

$$|\sum_{k=1}^{\infty} a_{m,k}^{p} t_k^n| \leqq H \sum_{k=\mu(m)+1}^{\infty} |a_{m,k}^{p}| \leqq H\varepsilon_{m,p}.$$

ii) If $m > \nu[q(q+1)/2]$, and $p \leqq n$, we have

$$\begin{aligned}|\sum_{k=1}^{\infty} a_{m,k}^{p} t_k^n| &\leqq |\sum_{k=1}^{\lambda(m)} a_{m,k}^{p} t_k^n| + |\sum_{k=\lambda(m)+1}^{\infty} a_{m,k}^{p} s_k^n| \\ &\leqq 2H \sum_{k=1}^{\lambda(m)} |a_{m,k}^{p}| + |\sum_{k=1}^{\infty} a_{m,k}^{p} s_k^n| \\ &\leqq 2H\varepsilon_{m,p} + \xi_m.\end{aligned}$$

iii) If $\nu\{q(q-1)/2+r-1] < m \leqq \nu[q(q-1)/2+r]$, $(1 \leqq r \leqq q)$, $p \leqq n$, and if $\lambda'(r)$ denotes $\lambda(\nu\{q(q-1)/2+r])$, we have,

$$\begin{aligned}|\sum_{k=1}^{\infty} a_{m,k}^{p} t_k^n| &\leqq |\sum_{k=1}^{\lambda(m)} a_{m,k}^{p} t_k^n| + |\sum_{k=\lambda(m)+1}^{\mu(m)} \frac{r}{q} a_{m,k}^{p} s_k^n| \\ &\quad + |\sum_{k=\lambda'(r)+1}^{\mu(m)} \frac{1}{q} a_{m,k}^{p} s_k^n| + |\sum_{k=\mu(m)+1}^{\infty} a_{m,k}^{p} t_k^n| \\ &\leqq H\left(1+\frac{r}{q}\right) \sum_{k=1}^{\lambda(m)} |a_{m,k}^{p}| + \frac{r}{q} |\sum_{k=1}^{\infty} a_{m,k}^{p} s_k^n| \\ &\quad + \frac{H}{q} \sum_{k=\lambda'(r)+1}^{\mu(m)} |a_{m,k}^{p}| + H\left(1+\frac{r}{q}\right) \sum_{k=\mu(m)+1}^{\infty} |a_{m,k}^{p}| \\ &\leqq 4H\varepsilon_{m,p} + \frac{r}{q}\xi_m + \frac{HM_p}{q} \\ &\leqq 4H\varepsilon_{m,p} + \xi_m + \frac{HM_p}{q}.\end{aligned}$$

iv) If $N(n) < m \leqq \nu\{q(q-1)/2]$, and if $\lambda'(0)$ denotes $\lambda(\nu[q(q-1)/2])$, we have,

$$|\sum_{k=1}^{\infty} a_{m,k}^{p} t_k^n| \leqq \frac{1}{q} |\sum_{k=\lambda'(0)+1}^{\mu(m)} a_{m,k}^{p} s_k^n| + |\sum_{k=\mu(m)+1}^{\infty} a_{m,k}^{p} t_k^n|,$$

(the first sum of the right hand side being taken as zero if $\lambda'(0) \geqq \mu(m)$). This implies that,

$$|\sum_{k=1}^{\infty} a_{m,k}^{p} t_k^n| \leqq \frac{HM_p}{q} + H\varepsilon_{m,p}.$$

It follows that for every $m$ and $n$, $(m, n = 1, 2, \ldots)$, and for $p \leqq n$,

$$|\sum_{k=1}^{\infty} a^p_{m,k} t^n_k| \leqq 4H\varepsilon_{m,p} + \xi_m + \frac{HM_p}{q},$$

where $q$ is such that $\nu[q(q-1)/2] < m \leqq \nu[q(q+1)/2]$. For fixed $p$ and $n$, $4H\varepsilon_{m,p}+\xi_m+HM_p/q$ converges to zero as $m \nearrow \infty$, and the proof is completed by observing that if $n \geqq p$, these terms are independent of $n$.

Definition 3. Consider the partial sums $\{u_r\}$ of the series

$$1-1+\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}+ \dots +\tfrac{1}{n}+\tfrac{1}{n}+ \dots$$
$$+\tfrac{1}{n}-\tfrac{1}{n}-\tfrac{1}{n}- \dots -\tfrac{1}{n}+ \dots,$$

where $1/n$ appears in the series $2n$ times, the first $n$ times with the plus sign, the second $n$ times with the minus sign. We shall call the sequence $\{u_r\}$ the *c-sequence.*

It is evident that $0 \leqq u_r \leqq 1$, $(r = 1, 2, \dots)$, and that $u_{q(q+1)} = 0$, $u_{q^2} = 1$, $(q = 1, 2, \dots)$; if $\eta_r = |u_r - u_{r+1}|$, then $\lim_{r\to\infty} \eta_r = 0$.

Suppose that we have a countable set of matrices, $A^1, A^2, \dots, A^n, \dots$, and that $\{\lambda(m)\}$, $\{\mu(m)\}$ and $\{m(r)\}$ have been chosen as above. Let $v = \{v_r\}$ be a sequence of real numbers with $0 \leqq v_r \leqq 1$, $(r = 1, 2, \dots)$. Suppose that the set of zeroes of this sequence is infinite so that $v_{z(k)} = 0$, $(k = 1, 2, \dots)$, say, $z(k) \nearrow \infty$. For each natural number $r$, $r \geqq m[z(1)]$, there exists $q = q(r)$, such that,

$$\lambda(m[z(q)]) \leqq \lambda(r) \leqq \lambda(r+1) < \lambda(m[z(q+1)]).$$

For each $r$, $r \geqq m[z(1)]$, and each $q(r)$, let $I(v, r, q)$ denote the interval,

$$[\lambda(r), \lambda(r+1)] \subset (\lambda(m[z(q)]), \lambda(m[z(q+1)])].$$

Every natural number $k$, $k \geqq \lambda(m[z(1)])$, belongs to a unique interval $I(v, r, q)$.

In particular, if $u = \{u_r\}$ is the $c$-sequence, then $I(u, r, q)$ is the interval,

$$[\lambda(r), \lambda(r+1)] \subset (\lambda(m[q(q-1)]), \lambda(m[q(q+1)]);$$

and if $u' = \{1-u_r\}$, then $I(u', r, q)$ is the interval,

$$[\lambda(r), \lambda(r+1)] \subset (\lambda(m[q^2]), \lambda(m[(q+1)^2]).$$

We now state the following theorem, which is a generalization of Theorem 4 of [2].

Theorem 5. *Let $A = (a_{m,k})$ be a matrix. Choose $\{\lambda(m)\}$, $\{\mu(m)\}$ and $\{m(r)\}$ as in the beginning of this section. Let $v = \{v_r\}$ be a* *sequence as above, with* $0 \leqq v_r \leqq 1$, $(r = 1, 2, \ldots)$ *and* $\lim_{r\to\infty} \gamma_r = 0$, *where* $\gamma_r = |v_r - v_{r+1}|$, $(r = 2, 3, \ldots)$. *Then, if* $\{s_k^q\}$, $(q = 1, 2, \ldots)$ *is a countable set of uniformly bounded sequences, uniformly summed to zero by* $A$, *the bounded sequence* $\{s_k\}$ *is* $A$ *summable to zero, where,*

$$s_k = v_r s_k^q \quad \textit{for} \quad k \in I(v, r, q),$$
$$s_k = s_k^1 \quad \textit{for} \quad k < \lambda(m[z(1)]).$$

Proof. Let $\{s_k\}$ be the sequence described in the theorem, $h(A) = M$, and $|s_k^q| \leqq H$, $(k, q = 1, 2, \ldots)$. Let $\{\zeta_m\}$ be a sequence, $\zeta_m \searrow 0$, such that,

$$|\sum_{k=1}^{\infty} a_{m,k} s_k^q| < \zeta_m, \qquad (m, q = 1, 2, \ldots).$$

For $m$ with $\lambda(m)$ in $I(v, r, q)$,

$$|\sum_{k=1}^{\infty} a_{m,k} s_k| \leqq |\sum_{k=1}^{\lambda(m)} a_{m,k} s_k| + v_r |\sum_{k=\lambda(m)+1}^{\mu(m)} a_{m,k} s_k^q|$$
$$+ |v_r - v_{r+1}| |\sum_{k=\lambda(m(r+1))}^{\mu(m)} a_{m,k} s_k^q| + |\sum_{k=\mu(m)+1}^{\infty} a_{m,k} s_k|$$

(the third sum to be taken as zero if $\lambda(m(r+1)) \geqq \mu(m)$).

It follows that,

$$|\sum_{k=1}^{\infty} a_{m,k} s_k| \leqq 2H \sum_{k=1}^{\lambda(m)} |a_{m,k}| + 2H \sum_{k=\mu(m)}^{\infty} |a_{m,k}| + v_r |\sum_{k=1}^{\infty} a_{m,k} s_k^q| + \gamma_r MH,$$

and the sequence $\{s_k\}$ is $A$ summable to zero.

This theorem is true in particular if $v = \{u_r\}$ or $v = \{1 - u_r\}$. We shall describe the operations performed in Theorem 4 as *interweaving the sequences* $\{s_k^q\}$.

Now suppose that $A$ is a matrix, and that $s^n = \{s_k^n\}$, $(n = 1, 2, \ldots)$, are sequences in the unit sphere with $A$-lim $s^n = \alpha^n$. Suppose that $\lim_{n\to\infty} \alpha^n = \alpha$, and that the sequences $\{s_k^n - \alpha^n\}$, $(n = 1, 2, \ldots)$ are summed uniformly to zero by $A$. Let $\{t_k^2\}$ be defined by

$$t_k^2 = u_r(s_k^{2q-1} - \alpha^{2q-1}) \quad \text{for } k \text{ in } I(u, r, q)$$

and

$$t_k^2 = 0, \; k < \lambda(m(2));$$

and $\{t_k^1\}$ by

$$t_k^1 = (1 - u_r)(s_k^{2q} - \alpha^{2q}) \text{ for } k \text{ in } I(u', r, q),$$

and

$$t_k^1 = 0, \; k < \lambda(m(1)),$$

where $u$ is the $c$-sequence $\{u_r\}$, and $u'$ the sequence, $\{1 - u_r\}$.

From Theorem 4 it follows that $\{t_k^1\}$ and $\{t_k^2\}$ are both summed to zero by $A$. Let $t_k = t_k^1 + t_k^2$, $(k = 1, 2, \ldots)$, then $\{t_k\}$ is also summed to zero by $A$, and $\{t_k + \alpha\}$ is summed to $\alpha$ by $A$. Now if $\lambda(m(q^2)) < \lambda(m(r)) \leqq k \leqq \lambda(m(r+1)) \leqq \lambda(m[q(q+1)])$, we have $k$ in $I(u, r, q) \cap I(u', r, q)$, and

$$t_k + \alpha = u_r(s_k^{2q-1} - \alpha^{2q-1}) + (1-u_r)(s_k^{2q} - \alpha^{2q}) + \alpha,$$

and if $\lambda(m[q(q+1)]) < \lambda(m(r)) \leqq k \leqq \lambda(m(r+1)) \leqq \lambda(m[(q+1)^2])$, we have $k$ in $I(u, r, q+1) \cap I(u', r, q)$, and,

$$t_k + \alpha = u_r(s_k^{2q+1} - \alpha^{2q+1}) + (1-u_r)(s_k^{2q} - \alpha^{2q}) + \alpha.$$

Now the sequence $\{v_k\}$ defined by

$$v_k = \alpha - (1-u_r)\alpha^{2q} - u_r\alpha^{2q-1} \text{ for } k \text{ in } I(u, r, q) \cap I(u', r, q),$$

and

$$v_k = \alpha - (1-u_r)\alpha^{2q} - u_r\alpha^{2q+1} \text{ for } k \text{ in } I(u, r, q+1) \cap I(u', r, q)$$

converges to zero. Thus the sequence $\{s_k\}$, defined by,

$$s_k = t_k + \alpha - v_k, \qquad (k = 1, 2, \ldots),$$

is $A$ summable to $\alpha$, further, $\{s_k\}$ is in the unit sphere.

Hence, we have the following:

THEOREM 6. *Let $A$ be a matrix, and suppose that $\{s_k^n\}$, $(n = 1, 2, \cdots)$ are sequences in the unit sphere with $A$-lim $s^n = \alpha^n$. Suppose that $\lim_{n\to\infty} \alpha^n = \alpha$, and that the sequences $\{s_k^n - \alpha^n\}$, $(n = 1, 2, \ldots)$ are summed uniformly to zero by $A$. Then there is a sequence in the unit sphere, $\{s_k\}$, such that $A$-lim $s_k = \alpha$.*

From this theorem we can immediately deduce the following, which is originally due to Brudno, see [4].

THEOREM 7. *Let $A$ be a matrix. Then $A$ sums a sequence in the unit sphere, $s$, with*

$$A\text{-lim } s = N(\mathscr{A}).$$

PROOF. By definition of $N$, we have sequences $s^n = \{s_k^n\}$, $(n = 1, 2, \ldots)$, in the unit sphere with $A$-lim $s^n = \alpha^n$, and $\lim_{n\to\infty} \alpha^n = N(\mathscr{A})$. After theorem 4, we may assume that these sequences are uniformly summable by $A$, without altering other properties. Theorem 6 then implies that we can construct a sequence $s$ in the unit sphere with

$$A\text{-lim } s = N(\mathscr{A}).$$

In [3], see also [7], the following has been shown.

THEOREM 8. *Let $A^1, A^2, \ldots, A^n, \ldots$ be a countable set of regular matrices, with $\mathscr{A}^n \supset \mathscr{A}^{n+1}$, $(n = 1, 2, \ldots)$, then the intersection of their bounded convergence fields is the bounded convergence field of some matrix $A$.*

Theorem 6 enables us to obtain more information in this direction.

THEOREM 9. *Let $A^1, A^2, \ldots, A^n, \ldots$, be a countable set of regular matrices with $\mathscr{A}^n \supset \mathscr{A}^{n+1}$, $(n = 1, 2, \ldots)$, then if $\mathscr{A} = \bigcap_{n=1}^{\infty} \mathscr{A}^n$,*

$$N(\mathscr{A}) = \lim_{n\to\infty} N(\mathscr{A}^n).$$

PROOF. From Theorem 7, we can construct sequences $s^n = \{s_k^n\}$ in the unit sphere with

$$A^r\text{-lim } s^n = N(\mathscr{A}^n) = \alpha^n, \qquad (r \leqq n).$$

Now $\{\alpha^n\}$ is a decreasing sequence of positive numbers and bounded from below by Theorem 2. Hence, $\lim_{n\to\infty} \alpha^n = \alpha$, say. It is clear from Theorem 4, that we may assume that for each $r$, $(r = 1, 2, \ldots)$, the sequences $\{s_k^n\}$, $(n = r, r+1, \ldots)$, are uniformly summed by $A^r$. We may construct a sequence $\{s_k\}$ by using functions $\lambda(m)$ and $\mu(m)$ that are appropriate for all the matrices $A^r$, $(r = 1, 2, \ldots)$, (as in (3), (4) and (5)) and using the sequences $\{s_k^n\}$, $(n = 1, 2, \ldots)$ for an interweaving as in the proof of Theorem 6. Since only finitely many terms of $\{s_k\}$ are derived from the sequences $\{s_k^n\}$, $(n < r)$, it is clear that as in Theorem 6, we have,

$$A^r\text{-lim } s = \alpha, \qquad (r = 1, 2, \ldots).$$

Thus $s$ belongs to $\bigcap_{n=1}^{\infty} \mathscr{A}^n = \mathscr{A}$, and from Theorem 1 we have $A$-lim $s = \alpha$. But $s$ is in the unit sphere, hence, $N(\mathscr{A}) \geqq \alpha$. From Theorem 2 we see that $N(\mathscr{A}) \leqq \lim_{n\to\infty} N(\mathscr{A}^n) = \alpha$ and this implies that $N(\mathscr{A}) = \lim_{n\to\infty} N(\mathscr{A}^n)$ as required.

## 3.

We are now going to consider some results which follow from Theorem 7.

DEFINITION 4. If $A$ is a matrix, a sequence $s$ in the unit sphere is called an *extremal point* of $\mathscr{A}$ if

$$A\text{-lim } s = N(\mathscr{A}).$$

Hence, Theorem 7 tells us that every bounded convergence field of a matrix has at least one extremal point.

THEOREM 10. *Suppose that $A$ is a matrix and that $s$ is an extremal point of $\mathscr{A}$. Then* $\limsup_{n\to\infty} s_n = 1$. *Further, if* $N(\mathscr{A}) > 1$, *we also have* $\liminf_{n\to\infty} s_n = -1$. Before proceeding to the proof we remark that the condition $N(\mathscr{A}) > 1$ is essential to obtain $\liminf_{n\to\infty} s_n = -1$, since any matrix with $N(\mathscr{A}) = 1$ will sum the unit sequence $\{1\}$, to the value $1 = N(\mathscr{A})$.

PROOF. Let $s$ be an extremal point of $\mathscr{A}$. Suppose that $\limsup_{n\to\infty} s_n = U < 1$. We can find an $n_0$ such that $s_n \leqq U + L/2$, for $n \geqq n_0$. Define $s' = \{s'_n\}$ by

$$s'_n = 0, \qquad (n < n_0),$$

$$s'_n = s_n + \frac{1-U}{2}, \qquad (n \geqq n_0).$$

Then $\{s'_n\}$ is the unit sphere. If $N(\mathscr{A}) = N$, then $A$-lim $s' = N + 1 - U/2 > N$. This contradiction establishes the first part of the theorem.

Now suppose that $N > 1$, and that $\liminf_{n\to\infty} s_n = L > -1$. Let $\varepsilon = (L+1)/8 > 0$. We have $s_n \geqq -1+4\varepsilon$, if $n \geqq n_1$, say. Let

$$t_n = 0, \qquad (n < n_1),$$

$$t_n = \frac{s_n - \varepsilon/N}{1-\varepsilon/N} \qquad (n \geqq n_1).$$

Then $t_n \leqq \dfrac{1-\varepsilon/N}{1-\varepsilon/N} = 1$, also,

$$t_n \geqq \frac{-1+4\varepsilon-\varepsilon/N}{1-\varepsilon/N} \geqq \frac{-1+3\varepsilon/N}{1-\varepsilon/N} \geqq -1.$$

Thus, $\{t_n\}$, is in the unit sphere. Also,

$$A\text{-lim } t = \frac{N-\varepsilon/N}{1-\varepsilon/N} = N + \frac{\varepsilon-\varepsilon/N}{1-\varepsilon/N} > N, \text{ since } N > 1.$$

This contradiction establishes that $L = -1$, and the theorem is proved.

THEOREM 11. *Let $\mathscr{A}$ be a summability method such that $N(\mathscr{A}) > 1$. If the norm of $\mathscr{A}$ is attained, there is an extremal point of $\mathscr{A}$, $\{s_n\}$, such that $s_n = 1$, or $s_n = -1$, $(n = 1, 2, \ldots)$.*

PROOF. Without loss of generality we may assume that $A$ is a matrix with $h(A) = N(\mathscr{A})$. Let $s$ be an extremal point of $\mathscr{A}$. Then $\|s\|=1$, $A$-lim $s = N(\mathscr{A})$ and $\liminf_{n\to\infty} s_n = -1$, $\limsup_{n\to\infty} s_n = 1$. We define $s' = \{s'_n\}$ by

$$s'_n = 1 \text{ if } s_n \geqq 0; \quad s'_n = -1 \quad \text{if} \quad s_n < 0, \qquad (n = 1, 2, \ldots).$$

We shall show that $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{m,n} s'_n = N(\mathscr{A}) = N$. Now, $h(A) = N$, so that if $\{t_n\}$ is in the unit sphere, then,

$$(6) \qquad |\sum_{n=1}^{\infty} a_{m,n} t_n| \leqq N, \qquad (m = 1, 2, \ldots).$$

Suppose that

$$\liminf_{m\to\infty} \sum_{n=1}^{\infty} a_{m,n} s'_n = N_1 < N,$$

then,

$$\limsup_{m\to\infty} \sum_{n=1}^{\infty} a_{m,n}(2s_n - s'_n) = 2N - N_1 > N.$$

However, $|2s_n - s'_n| \leqq 1$, $(n = 1, 2, \ldots)$, so in view of (6) we must have,

$$\liminf_{m\to\infty} \sum_{n=1}^{\infty} a_{m,n} s'_n \geqq N.$$

But (6) is also applicable to $s'$, so we have

$$\limsup_{m\to\infty} \sum_{n=1}^{\infty} a_{m,n} s'_n \leqq N.$$

Hence, $A$ sums $s'$ to $N$, and $s'$ is the required sequence.

## 4.

We now consider some examples and applications. Firstly, it is possible that a summability method does not attain its norm. Consider the matrix, $A$, which transforms the sequence $\{s_n\}$ into the sequence $\{t_n\}$, as follows:

$$t_{2n-1} = 6s_{4n-3} - 5s_{4n-2}, \quad t_{2n} = 2s_{4n-1} - s_{4n}, \quad (n = 1, 2, \ldots).$$

If $\{s_n\}$ is the sequence with $s_{4n-3} = \frac{1}{2}$, $s_{4n-2} = 0$, $s_{4n-1} = 1$, $s_{4n} = -1$, then $A$ sums $\{s_n\}$ to the value 3. Hence $N(\mathscr{A}) \geqq 3 > 1$, but the matrix defined by the even numbered transformations is $b$-stronger than $A$ and has a matrix norm 3. From Theorems 2 and 3 we see that $N(\mathscr{A}) = 3$. It is clear that no sequence $\{s'_n\}$ of 1's and $-1$'s with $\limsup_{n\to\infty} s_n = 1$, $\liminf_{n\to\infty} s_n = -1$ can be summed by $A$. In view of Theorem 11, we conclude that $\mathscr{A}$ does not attain its norm. Further, if $B$ is any matrix such that $\mathscr{B} \subset \mathscr{A}$, and $N(\mathscr{B}) > 1$, the above argument shows us that $\mathscr{B}$ cannot attain its norm. Thus $A$ is a matrix such that no method weaker than $\mathscr{A}$ which has field norm strictly greater than unity, can attain its norm.

A slightly different example is the matrix $A$ defined by the transformations,

$$t_n = 2s_{2n} - s_{2n+1}, \qquad (n = 1, 2, \ldots).$$

Here, $N(\mathscr{A}) = 3$ and the reader can show by arguments similar to the above that no weaker matrix attains its norm; however, in this case $h(A) = N(\mathscr{A})$.

The following result which is proved in [6], Theorem 4, is relevant here. In the terminology of this paper it is as follows:

THEOREM 12. *For each summability method $\mathscr{B}$, there is a method $\mathscr{B}^{\Delta}$, with $N(\mathscr{B}) = N(\mathscr{B}^{\Delta})$ such that the norm of $\mathscr{B}^{\Delta}$ is not attained.*

We remark that, as is clear from the proof given in [6], and was meant to be included in the statement of the theorem, *the method $\mathscr{B}^{\Delta}$ is b-stronger than $\mathscr{B}$.*

This result means that the converse of Theorem 11 is not true in all cases. In fact, let $A$ be a matrix which sums a sequence $s = \{s_n\}$ with $s_n = 1$ or $s_n = -1$, $(n = 1, 2, \ldots)$, to the value $N(\mathscr{A})$. Suppose that the norm of $\mathscr{A}$ is attained. Choose a matrix $A^{\Delta}$ such that $N(\mathscr{A}) = N(\mathscr{A}^{\Delta})$, and such that the norm of $\mathscr{A}^{\Delta}$ is not attained. Then $A^{\Delta}$ sums $s$ to $N(\mathscr{A}) = N(\mathscr{A}^{\Delta})$. Thus every sequence which is an extremal point for some method is an extremal point for a method for which the norm is not attained.

We now turn to another application. We recall the following theorem, see [8], Theorem 2.

THEOREM 13. *Let $A$ be a matrix. Then there exists a positive sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty} x_n = \infty$, $x_n \searrow 0$, and such that a bounded sequence $\{s_n\}$ is summed to zero by $A$ if and only if, every sequence of the form $\{\xi_n s_n\}$ is a member of $\mathscr{A}$, where $\{\xi_n\}$ is bounded and $\xi_n - \xi_{n-1} = 0(x_n)$.*

In fact a glance at the proof shows that the sequence $\{x_n\}$ which is constructed has the property that $\{\xi_n s_n\}$ is not only summed by $A$, but is summed to zero by $A$.

Now suppose that we have a matrix $A$, and a sequence $s = \{s_n\}$ summed to zero by $A$ in $\mathscr{A}$, where $\limsup_{n\to\infty} s_n = \alpha$, $\liminf_{n\to\infty} s_n = \beta$, with $\alpha > \beta > 0$. Then we can find sequences $\{n_k\}$ and $\{p_k\}$ of natural numbers, strictly increasing for which $\lim_{k\to\infty} s_{n_k} = \alpha$, $\lim_{k\to\infty} s_{p_k} = \beta$. Suppose that $\alpha' > \beta' > 0$, define $\alpha_0 = \alpha'/\alpha$, $\beta_0 = \beta'/\beta$. We may clearly construct a sequence $\{\xi_n\}$, as in Theorem 13, for which $A$-lim $\xi_n s_n = 0$,

$$\limsup_{k\to\infty} \{\xi_{n_k}\} = \alpha_0, \quad \liminf_{k\to\infty} \{\xi_{p_k}\} = \beta_0$$

and also $\alpha_0 \geqq \xi_n \geqq \beta_0$, $(n = 1, 2, \ldots)$. Then we have

$$\limsup_{n\to\infty} \xi_n s_n = \alpha', \quad \liminf_{n\to\infty} \xi_n s_n = \beta'.$$

THEOREM 14. *Suppose that $A$ is a matrix, and that $N(\mathscr{A}) > 1$. Then for each $a$, $1 < a < N(\mathscr{A})$, there exists a sequence $t = \{t_n\}$ in $\mathscr{A}$ with $A$-$\lim t = a$, $\limsup_{n\to\infty} t_n = 1$, $\liminf_{n\to\infty} t_n = -1$ and $||t|| = 1$.*

PROOF. Let $s = \{s_n\}$ be an extremal point of $\mathscr{A}$, so that $A$-lim $s = N(\mathscr{A}) = N$, $\limsup_{n\to\infty} s_n = 1$, $\liminf_{n\to\infty} s_n = -1$. Let $s'_n = N - s_n > 0$, $(n = 1, 2, \ldots)$. Then $\limsup_{n\to\infty} s'_n = N+1$, $\liminf_{n\to\infty} s'_n = N-1$, $A$-lim $s'_n = 0$, and

$$N+1 \geqq s'_n \geqq N-1, \qquad (n = 1, 2, \ldots).$$

As indicated above, we may find a sequence $\{\xi_n\}$ with $A$-lim $\{\xi_n s'_n\} = 0$,

$$\limsup_{n\to\infty} \xi_n s'_n = N+1, \quad \liminf_{n\to\infty} \xi_n s'_n = (N+1)\frac{a-1}{a+1},$$

and $1 \geqq \xi_n \geqq (N+1)(a-1)(N-1)^{-1}(a+1)^{-1}$, $(n = 1, 2, \ldots)$.
Define,

$$t_n = \frac{\dfrac{a}{a+1}(N+1) - \xi_n s'_n}{(N+1)\left(1 - \dfrac{a}{a+1}\right)}, \qquad (n = 1, 2, \ldots),$$

$$= \frac{a(N+1) - (a+1)\xi_n s'_n}{(N+1)}.$$

Now,

$$\limsup_{n\to\infty} t_n = \frac{a(N+1) - (a-1)(N+1)}{N+1} = 1,$$

and

$$\liminf_{n\to\infty} t_n = \frac{a(N+1) - (a+1)(N+1)}{N+1} = -1,$$

also $-1 \leqq t_n \leqq 1$, $(n = 1, 2, \ldots)$. Further, $A$-lim $t_n = a$, and thus $\{t_n\}$ is the required sequence.

THEOREM 15. *If $A$ is a matrix, and $N(\mathscr{A}) > 1$, then for each $a$, $1 < a < N(\mathscr{A})$, there exists a matrix $B$, with $N(\mathscr{A}) = a$, and such that $A$ is b-stronger than $B$.*

PROOF. Let $s = \{s_n\}$ be a sequence with $||s|| = 1$, $\liminf_{n\to\infty} s_n = -1$, $\limsup_{n\to\infty} s_n = 1$, and $A$-lim $s = a$. We may construct two disjoint sequences of natural numbers, $\{n_k\}$ and $\{p_k\}$, strictly increasing, such that,

$$\lim_{k\to\infty} s_{n_k} = 1, \quad \lim_{k\to\infty} s_{p_k} = -1.$$

Define the matrix $C = (c_{m,n})$ by,

$$c_{k,n_k} = \frac{1+a}{2}, \quad c_{k,p_k} = \frac{1-a}{2}, \qquad (k = 1, 2, \ldots)$$

and

$$c_{k,n} = 0, \quad n \neq n_k \text{ or } p_k.$$

Then,

$$N(\mathscr{C}) \leqq h(C) = a,$$

also $C$-lim $s = a$. We may then define a matrix $B = (b_{m,n})$ by the equations,

$$b_{2k,n} = a_{k,n}, \quad b_{2k-1,n} = c_{k,n}, \qquad (k, n = 1, 2, \ldots).$$

Then we have $\mathscr{B} \subset \mathscr{A} \cap \mathscr{C}$. Thus, by Theorem 2, $N(\mathscr{B}) \leqq N(\mathscr{C}) \leqq a$, but on the other hand $B$-lim $s = a$, so that $||\mathfrak{B}|| = N(\mathscr{B}) = a$, and $B$ is the required matrix.

As a final application, we prove the following:

THEOREM 16. *If $\mathscr{A}$ is a summability method, and if the norm of $\mathscr{A}$ is attained, there is a matrix $B$ belonging to the method, such that,*

i) $h(B) = N(\mathscr{A})$,

ii) *in every column of $B$, all the non-zero elements have the same sign.*

PROOF. We assume that $h(A) = N(\mathscr{A})$, (if $\lim_{m\to\infty} \sum_{n=1}^{\infty} |a_{m,n}| = N(\mathscr{A})$ but $h(A) \neq N(\mathscr{A})$, this result may be achieved by multiplying the rows of $A$ by the appropriate factors). Now there is an extremal point, $s = \{s_n\}$, of $\mathscr{A}$ for which $s_n = 1$ or $s_n = -1$, $(n = 1, 2, \ldots)$. We define the matrix $B$, by

$$b_{m,n} = |a_{m,n}| \operatorname{sign} s_n, \qquad (n, m = 1, 2, \ldots).$$

This matrix clearly satisfies condition ii, also $h(B) = h(A) = N(\mathscr{A})$. Thus, we only have to verify that $\mathscr{B} = \mathscr{A}$. Let $\omega_m$ be the set of indices, $n$, for which $s_n a_{m,n} < 0$, $(m = 1, 2, \ldots)$. Then, if $t = \{t_n\}$ is any bounded sequence,

$$\left|\sum_{n=1}^{\infty} (b_{m,n} - a_{m,n}) t_n\right| = \left|\sum_{\omega_m} (b_{m,n} - a_{m,n}) t_n\right|.$$

Hence, we have,

$$(7) \qquad |\sum_{n=1}^{\infty}(b_{m,n}-a_{m,n})t_n| \leqq \sum_{\omega_m}|b_{m,n}-a_{m,n}|\,||t||,$$

and,

$$(8) \qquad \sum_{\omega_m}|b_{m,n}-a_{m,n}| = 2\sum_{\omega_m}|a_{m,n}| = |\sum_{n=1}^{\infty}(b_{m,n}-a_{m,n})s_n|.$$

Now,

$$\sum_{n=1}^{\infty} b_{m,n}s_n = \sum_{n=1}^{\infty}|b_{m,n}| = \sum_{n=1}^{\infty}|a_{m,n}| \geqq \sum_{n=1}^{\infty} a_{m,n}s_n.$$

However,

$$|\sum_{n=1}^{\infty} b_{m,n}s_n| \leqq h(B) = N(\mathscr{A})$$

and it follows that, since

$$\lim_{m\to\infty}\sum_{n=1}^{\infty} a_{m,n}s_n = N(\mathscr{A}),$$

then

$$\lim_{m\to\infty}\sum_{n=1}^{\infty} b_{m,n}s_n = N(\mathscr{A}).$$

Hence, $B$ sums $s$ and

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}(b_{m,n}-a_{m,n})s_n = 0;$$

with (7) and (8), this implies that

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}(b_{m,n}-a_{m,n})t_n = 0$$

and that $\mathscr{A} = \mathscr{B}$ as required.

## 5.

We wish to make some brief comments on '*diagonalization*'. By this latter term, we mean the following: Suppose $\alpha = \{A^1, A^2, \ldots, A^n, \ldots\}$ is a countably infinite set of matrices. In [1] and [2], we have obtained necessary and sufficient conditions for the existence of a matrix $A$, which is $b$-stronger than, (and $b$-consistent with) every matrix in $\alpha$. However, these conditions did not indicate a method of construction of such a matrix. When we speak of a *diagonalization* we shall mean that there are sequences $\{n(k)\}$, and $\{p(k)\}$, $n(k) \nearrow \infty$ and $p(k) \nearrow \infty$, of natural numbers so that the $n(k)$th row of the matrix $A^{p(k)}$ is the $k$th row of $A$. It is clear that in general, such a diagonalization does not give a matrix $b$-stronger than every member of $\alpha$.

We recall the following definition, see [2]. If $\alpha = \{A^1, A^2, \ldots, A^n, \ldots\}$ and $\beta = \{B^1, B^2, \ldots, B^n, \ldots\}$ are countable infinite sets of matrices, we shall say that $\alpha$ and $\beta$ are *b-equivalent* if $A^n$ is $b$-equivalent to $B^n$, $(n = 1, 2, \ldots)$.

The existence of a matrix $A$, $b$-stronger than all members of $\alpha$ was shown in [2] to depend on the properties of the sets of matrices which are $b$-equivalent to $\alpha$.

THEOREM 17. *Suppose that $\alpha$ is a countably infinite set of matrices, and that there exists a matrix $A$ which is b-stronger than each member of $\alpha$. Then there exists a set of matrices, $\beta$, b-equivalent to $\alpha$, such that every diagonalization from $\beta$ gives a matrix b-stronger than each member of $\alpha$. In fact every such diagonalization will be b-stronger than $A$.*

PROOF. Choose a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, $(n = 1, 2, 3, \ldots)$ such that

$$\lim_{n\to\infty} \varepsilon_n h(A^n) = 0.$$

We define the matrix $B^n$, $(n = 1, 2, \ldots)$, by the equations,

$$b^n_{2m,k} = a_{m,k}, \quad b^n_{2m-1,k} = (1-\varepsilon_n)a_{m,k} + \varepsilon_n a^n_{m,k}, \qquad (m, k = 1, 2, \ldots).$$

It is clear that if $A$ is $b$-stronger than $A^n$, $B^n$ is $b$-equivalent to $A^n$. Let $B$ be a diagonalization from $\beta$, it is clear that $B$ is $b$-stronger than $A$ and hence $b$-stronger than every member of $\alpha$. This completes the proof of the theorem.

This problem has also been discussed by Brudno, [5].

## 6.

We close by stating two important questions which arise in this paper.

1. We have seen that if the norm of a method is attained, there is necessarily a sequence of 1's and $-1$'s summed by the method to $N(\mathscr{A})$. If the norm is not attained, is there necessarily a sequence in the unit sphere, $\{s_n\}$ which is summed by the method to $N(\mathscr{A})$ and such that

$$\liminf_{n\to\infty} |s_n| = L < 1?$$

We have seen of course, that if this is the case, the norm is not attained.

2. A question proposed by Brudno also remains. For each method, is there necessarily a stronger method of the same norm which attains the norm?

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