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On the order and type of integral functions of several complex variables

by

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1.

Let [1])

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n, \tag{1.1}$$

be a function of two complex variables z_1 and z_2, where the coefficients a_{mn} are complex numbers. The series (1.1) represents an integral function of two variables z_1, z_2, if it converges absolutely for all values of $|z_1| < \infty$ and $|z_2| < \infty$. M. M. Dzrbasyan (1, p. 1) has shown that the necessary and sufficient condition for the series (1.1) to represent an integral function of variables z_1 and z_2, is

$$\limsup_{m+n\to\infty} (|a_{mn}|)^{1/(m+n)} = 0. \tag{1.2}$$

Let $\bar{G}_r$ be the family of closed polycircular domains in space (z_1, z_2) dependent on parameter $r > 0$ and possess the property that $(z_1, z_2) \in \bar{G}_r$, if and only if $(z_1/r, z_2/r) \in \bar{G}_1$. The maximum modulus of the integral function $f(z_1, z_2)$ is denoted by

$$M_G(r, f) = \max_{(z_1, z_2)\in \bar{G}_r} |f(z_1, z_2)|$$

and the function will be called G — order and G — type respectively, if

$$\rho_G = \limsup_{r\to\infty} \frac{\log\log M_G(r, f)}{\log r} \tag{1.3}$$

$$T_G = \limsup_{r\to\infty} \frac{\log M_G(r, f)}{r^{\rho_G}}. \tag{1.4}$$

Denote

$$\phi_G(m, n) = \max_{(z_1, z_2)\in \bar{G}_r} |z_1|^m |z_2|^n.$$

A. A. Goldberg (2, p. 146) has proved the following theorems.

[1] For simplicity we consider only two variables, though the results can easily be extended to several complex variables.

THEOREM A. All orders ρ_G be equal and

(1.5) $$\rho = \rho_G = \limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{-\log|a_{mn}|}.$$

THEOREM B. G-type T_G satisfies the correlation

(1.6) $$(e\rho T_G)^{1/\rho} = \limsup_{m+n\to\infty} [(m+n)^{1/\rho}\{\phi_G(m,n)|a_{mn}|\}^{1/(m+n)}].$$

In this paper we shall obtain the relations between two or more integral functions and study the relations between the coefficients in the Taylor expansion of integral functions and their orders and types.

2.

THEOREM 1. *Let $f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ and $f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{mn} z_1^m z_2^n$ be two integral functions of non-zero finite orders ρ_1 and ρ_2 respectively. Then the function*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n,$$

where [2]

$$|c_{mn}| \sim |a_{mn}|\,|b_{mn}|$$

is an integral function such that

$$1/\rho \geqq 1/\rho_1 + 1/\rho_2,$$

where ρ is the order of $f(z_1, z_2)$.

PROOF: Since $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ are integral functions, therefore, using (1.2), we have

$$\limsup_{m+n\to\infty} |a_{mn}|^{1/(m+n)} = \limsup_{m+n\to\infty} |b_{mn}|^{1/(m+n)} = 0.$$

Also, $|c_{mn}| \sim |a_{mn}|\,|b_{mn}|$, therefore,

$$\limsup_{m+n\to\infty} |c_{mn}|^{1/(m+n)} \leqq \limsup_{m+n\to\infty} |a_{mn}|^{1/(m+n)} \limsup_{m+n\to\infty} |b_{mn}|^{1/(m+n)}.$$

Hence $f(z_1, z_2)$ is an integral function.

Now using (1.5) for the functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$, we have

$$\limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{-\log|a_{mn}|} = \rho_1$$

and

[2] By $|c_{mn}| \sim |a_{mn}|\,|b_{mn}|$, we mean $\lim_{m+n\to\infty}\{|c_{mn}|/|a_{mn}|\,|b_{mn}|\} = 1$.

$$\limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{-\log|b_{mn}|} = \rho_2.$$

Therefore, for an arbitrary $\varepsilon > 0$, we get

(2.1) $-\log|a_{mn}| > (1/\rho_1-\varepsilon/2)(m+n)\log(m+n)$, for $m+n > k_1$

and

(2.2) $-\log|b_{mn}| > (1/\rho_2-\varepsilon/2)(m+n)\log(m+n)$, for $m+n > k_2$.

Thus, for $(m+n) > k = \max.(k_1, k_2)$

$$-\log(|a_{mn}|\,|b_{mn}|) > (1/\rho_1+1/\rho_2-\varepsilon)(m+n)\log(m+n)$$

or,

$$\liminf_{m+n\to\infty} \frac{-\log(|a_{mn}|\,|b_{mn}|)}{(m+n)\log(m+n)} \geqq 1/\rho_1+1/\rho_2.$$

Therefore, if $|c_{mn}| \sim |a_{mn}|\,|b_{mn}|$, we get

$$\liminf_{m+n\to\infty} \frac{-\log|c_{mn}|}{(m+n)\log(m+n)} \geqq 1/\rho_1+1/\rho_2.$$

Hence

$$1/\rho \geqq 1/\rho_1+1/\rho_2.$$

Corollary. Let $f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$, where $s = 1, 2, \ldots$, p be p integral functions of non-zero finite orders $\rho_1, \rho_2, \ldots, \rho_p$ respectively. Then the function

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n, \text{ where } |a_{mn}| \sim \prod_{s=1}^{p} |a_{mn}^{(s)}|$$

is an integral function such that

$$1/\rho \geqq \sum_{s=1}^{p} 1/\rho_s,$$

where ρ is the order of $f(z_1, z_2)$.

Theorem 2. *Let* $f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *and* $f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{mn} z_1^m z_2^n$ *be two integral functions of finite non-zero orders* ρ_1 *and* ρ_2 *respectively. Then the function* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$, *where*

$$\log(1/|c_{mn}|) \sim |\{\log(1/|a_{mn}|)\log(1/|b_{mn}|)\}^{\frac{1}{2}}|$$

is an integral function, such that

$$\rho \leqq (\rho_1\rho_2)^{\frac{1}{2}},$$

where ρ *is the order of* $f(z_1, z_2)$.

PROOF: Since $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ are integral functions, therefore, using (1.2), we have for an arbitrary $\varepsilon > 0$ and large R

$$1/|a_{mn}| > (R-\varepsilon)^{m+n}, \text{ for } m+n > k_1$$

and

$$1/|b_{mn}| > (R-\varepsilon)^{m+n}, \text{ for } m+n > k_2.$$

Therefore, for $m+n > k = \max(k_1, k_2)$

$$|\{\log (1/|a_{mn}|) \log (1/|b_{mn}|)\}^{\frac{1}{2}}| > (m+n) \log (R-\varepsilon).$$

Thus, if

$$\log (1/|c_{mn}|) \sim |\{\log (1/|a_{mn}|) \log (1/|b_{mn}|)\}^{\frac{1}{2}}|$$

then, for large $m+n$

$$\log (1/|c_{mn}|) > (m+n) \log (R-\varepsilon),$$

or,

$$\limsup_{m+n\to\infty} |c_{mn}|^{1/(m+n)} = 0.$$

Hence $f(z_1, z_2)$ is an integral function.

Now, from (2.1) and (2.2), we have for sufficiently large $(m+n)$

$$|\{\log (1/|a_{mn}|) \log (1/|b_{mn}|)\}^{\frac{1}{2}}| > \{(1/\rho_1-\varepsilon/2)(1/\rho_2-\varepsilon/2)\}^{\frac{1}{2}} (m+n) \log (m+n),$$

or,

$$\liminf_{m+n\to\infty} \frac{|\{\log (1/|a_{mn}|) \log (1/|b_{mn}|)\}^{\frac{1}{2}}|}{(m+n) \log (m+n)} \geqq \left(\frac{1}{\rho_1\rho_2}\right)^{\frac{1}{2}}.$$

Thus, if $\log (1/|c_{mn}|) \sim |\{\log (1/|a_{mn}|) \log (1/|b_{mn}|)\}^{\frac{1}{2}}|$

$$1/\rho = \liminf_{m+n\to\infty} \frac{\log (1/|c_{mn}|)}{(m+n) \log (m+n)} \geqq \left(\frac{1}{\rho_1\rho_2}\right)^{\frac{1}{2}}$$

or,

$$\rho \leqq (\rho_1\rho_2)^{\frac{1}{2}}.$$

COROLLARY: Let $f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$, where $s = 1, 2, \ldots, p$ be p integral functions of finite non-zero orders $\rho_1, \rho_2, \ldots, \rho_p$ respectively. Then the function

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n,$$

where

$$\log (1/|a_{mn}|) \sim \left|\left\{\prod_{s=1}^{p} \log (1/|a_{mn}^{(s)}|)\right\}^{1/p}\right|$$

is an integral function such that

$$\rho \leqq \left(\prod_{s=1}^{p} \rho_s\right)^{1/p},$$

where ρ is the order of $f(z_1, z_2)$.

THEOREM 3. *Let* $f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *and* $f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{mn} z_1^m z_2^n$ *be two integral functions of finite non-zero orders* ρ_1, ρ_2 *and finite non-zero types*[3] T_1, T_2 *respectively. Then the function*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n,$$

where

$$|c_{mn}| \sim \left|\{|a_{mn}|\,|b_{mn}|\}^{\frac{1}{2}}\right|$$

is an integral function such that

$$(\rho T)^{2/\rho} \leqq (\rho_1 T_1)^{1/\rho_1} (\rho_2 T_2)^{1/\rho_2},$$

where ρ *and* T *are the order and type of* $f(z_1, z_2)$ *respectively and* $2/\rho = 1/\rho_1 + 1/\rho_2$.

PROOF: We can prove, as in the proof of Theorem 1 that $f(z_1, z_2)$ is an integral function, when

$$|c_{mn}| \sim \left|\{|a_{mn}|\,|b_{mn}|\}^{\frac{1}{2}}\right|.$$

Further, using (1.6) for the functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$, we have

$$\text{(2.3)} \quad \limsup_{m+n\to\infty} [(m+n)^{1/\rho_1} \{\phi(m, n)|a_{mn}|\}^{1/(m+n)}] = (e\rho_1 T_1 e)^{1/\rho_1}$$

and

$$\text{(2.4)} \quad \limsup_{m+n\to\infty} [(m+n)^{1/\rho_2} \{\phi(m, n)|b_{mn}|\}^{1/(m+n)}] = (e\rho_2 T_2 e)^{1/\rho_2}.$$

From (2.3) and (2.4), we get for an arbitrary $\varepsilon > 0$

$$(m+n)^{1/\rho_1} \{\phi(m, n)|a_{mn}|\}^{1/(m+n)} < \{e\rho_1 (T_1+\varepsilon) e\}^{1/\rho_1},$$

for $m+n > k_1$ and

$$(m+n)^{1/\rho_2} \{\phi(m, n)|b_{mn}|\}^{1/(m+n)} < \{e\rho_2 (T_2+\varepsilon) e\}^{1/\rho_2},$$

for $m+n > k_2$.

Thus, for $(m+n) > k = \max(k_1, k_2)$ and $2/\rho = 1/\rho_1 + 1/\rho_2$

$$[(m+n)^{1/\rho} \{\phi(m, n)\left|(|a_{mn}|\,|b_{mn}|)^{\frac{1}{2}}\right|\}^{1/(m+n)}]^2 < \{e\rho_1 (T_1+\varepsilon) e\}^{1/\rho_1} \{e\rho_2 (T_2+\varepsilon) e\}^{1/\rho_2}.$$

[3] The types T_1 and T_2 correspond to the same family of closed polycircular domains $\bar{G}_r$.

Therefore, if $|c_{mn}| \sim |(|a_{mn}|\,|b_{mn}|)^{\frac{1}{2}}|$, we obtain

$$\limsup_{m+n\to\infty} [(m+n)^{1/\rho}\{\phi(m,n)|c_{mn}|\}^{1/(m+n)}] \leqq (e^{\rho_1 T_1 e})^{\frac{1}{2}\rho_1}(e^{\rho_2 T_2 e})^{\frac{1}{2}\rho_2}$$

or,

$$(e^{\rho T})^{1/\rho} \leqq (e^{\rho_1 T_1 e})^{\frac{1}{2}\rho_1}(e^{\rho_2 T_2 e})^{\frac{1}{2}\rho_2},$$

where ρ, T are respectively the order and type of $f(z_1, z_2)$. Hence,

$$(\rho T)^{2/\rho} \leqq (\rho_1 T_1)^{1/\rho_1}(\rho_2 T_2)^{1/\rho_2}.$$

COROLLARY 1. Let $f_s(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(s)} z_1^m z_2^n$, where $s = 1, 2, \ldots, p$ be p integral functions of finite non-zero orders $\rho_1, \rho_2, \ldots, \rho_p$ and finite non-zero types $T_1, T_2, \cdots, T_p$ respectively. Then the function $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$, where $|c_{mn}| \sim |(\prod_{s=1}^{p} |a_{mn}^{(s)}|)^{1/p}|$ is an integral function such that

$$(\rho T)^{p/\rho} \leqq \prod_{s=1}^{p} (\rho_s T_s)^{1/\rho_s},$$

where ρ and T are the order and type of $f(z_1, z_2)$ respectively and $p/\rho = \sum_{s=1}^{p} 1/p_s$.

COROLLARY 2. If in the above theorem the functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ are of the same finite non-zero order, then

$$T \leqq (T_1 T_2)^{\frac{1}{2}}.$$

COROLLARY 3. The result of the corollary 2 can be extended to p integral functions.

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