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Fourier series for Meijer's G-functions

by

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The object of this paper is to establish the following two Fourier series expansions for the Meijer's G-functions.

$$\begin{aligned}(1.1)\quad &\sum_{r=0}^{\infty} G_{p+2,\,q+2}^{m+1,\,n+1}\left(z\middle|{}_{3/2,\,b_1,\cdots,\,b_q,\,1}^{1-r,\,a_1,\cdots,\,a_p,\,2+r}\right)\sin\,(2r+1)\theta \\ &\qquad = \sqrt{\pi}/2\,\sin\theta\; G_{p,\,q}^{m,\,n}\left(z/\sin^2\theta\middle|{}_{b_1,\cdots,\,b_q}^{a_1,\cdots,\,a_p}\right),\end{aligned}$$

where $0 \leqq \theta \leqq \pi$, $|\arg z| < (m+n-p/2-q/2)\pi$.

$$\begin{aligned}(1.2)\quad &G_{p+1,\,q+1}^{m+1,\,n}\left(z\middle|{}_{\frac12,\,b_1,\cdots,\,b_q}^{a_1,\cdots,\,a_p,\,1}\right)+2\sum_{r=0}^{\infty} G_{p+2,\,q+2}^{m+1,\,n+1}\left(z\middle|{}_{\frac12,\,b_1,\cdots,\,b_q,\,1}^{1-r,\,a_1,\cdots,\,a_p,\,1+r}\right)\cos r\theta \\ &\qquad = \sqrt{\pi}\; G_{p,\,q}^{m,\,n}\left(z/\sin^2(\theta/2)\middle|{}_{b_1,\cdots,\,b_q}^{a_1,\cdots,\,a_p}\right),\end{aligned}$$

where $0 < \theta \leqq \pi$, $|\arg z| < (m+n-p/2-q/2)\pi$.

The Meijer's G-function is sum of hypergeometric functions each of which is usually an entire function. It is defined [**1**, p. 207] by a Mellin-Barnes type integral

$$(1.3)\quad G_{p,\,q}^{m,\,n}\left(z\middle|{}_{b_1,\cdots,\,b_q}^{a_1,\cdots,\,a_p}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, z^s\, ds,$$

where m, n, p, q are integers with $q \geqq 1$, $0 \leqq n \leqq p$, $0 \leqq m \leqq q$. The parameters a_j and b_j are such that no pole of $\Gamma(b_j-s)$, $j = 1, \ldots, m$ coincides with any pole of $\Gamma(1-a_j+s)$, $j = 1, \ldots, n$. The poles of the integrand must be simple and those of $\Gamma(b_j-s)$, $j = 1, \ldots, m$ lie on one side of the contour L and those of $\Gamma(1-a_j+s)$, $j = 1, \ldots, n$ must lie on the other side.

To prove (1.1) and (1.2) whose conditions of validity are given in section 2, we require the following Fourier series established by McRobert [**2**, p. 79 and **3**, p. 143].

$$(1.4)\quad \frac{\sqrt{\pi}\,\Gamma(2-s)}{2\Gamma(\frac32-s)}(\sin\theta)^{1-2s} = \sum_{r=0}^{\infty}\frac{(s;\,r)}{(2-s;\,r)}\sin\,(2r+1)\theta,$$

where $0 \leqq \theta \leqq \pi$, $R(s) \leqq \frac{1}{2}$. Here $(s; 0) = 1$, $(s; r) = s(s+1) \ldots (s+r-1)$, $r = 1, 2, 3, \ldots$

$$\frac{\sqrt{\pi}\Gamma(1-s)}{\Gamma(\frac{1}{2}-s)} (\sin (\theta/2))^{-2s} = 1+2\sum_{r=0}^{\infty} \frac{(s; r)}{(1-s; r)} \cos r\theta \tag{1.5}$$

where $0 < \theta \leqq \pi$.

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From (1.3), the expression on the left of (1.1) can be written as

$$\sum_{r=0}^{\infty} \frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{3}{2}-s) \prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)\Gamma(r+s)}{\Gamma(s) \prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)\Gamma(2+r-s)} z^s ds \sin (2r+1)\theta.$$

Here the path L of integration runs from $c-i\infty$ to $c+i\infty$. The conditions

$$0 < c < \tfrac{3}{2}$$
$$Rl(b_j) > c, \qquad j = 1, \ldots, m$$
$$Rl(a_j) < c+1, \qquad j = 1, \ldots, n$$

ensure that all the poles of $\Gamma(\frac{3}{2}-s)$ and $\Gamma(b_j-s)$, $j = 1, \ldots, m$ lie to the right of L and those of $\Gamma(r+s)$ and $\Gamma(1-a_j+s)$, $j = 1, \ldots, n$ lie to the left of L, as required for the definition of the G-function on the left side of (1.1). The integral converges if $p+q < 2(m+n)$ and $|\arg z| < (m+n-p/2-q/2)\pi$. On changing the order of integration and summation, which is easily seen to be justified, the above expression becomes

$$\frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{3}{2}-s) \prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\Gamma(2-s) \prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} \cdot \left\{\sum_{r=0}^{\infty} \frac{(s; r)}{(2-s; r)} \sin (2r+1)\theta\right\} z^s ds$$

and on using (1.4), it takes the form

$$\sqrt{\pi}/2 \sin \theta \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} \left(\frac{z}{\sin^2 \theta}\right)^s ds$$

which is just the expression on the right of (1.1).

(1.1) is the Fourier sine series for the G-functions. The Fourier cosine series (1.2) is proved in an analogous manner by using (1.3) and (1.5). The conditions of validity of (1.2) are

$$0 < c < \tfrac{1}{2}$$
$$Rl(b_j) > c, \qquad j = 1, \ldots, m$$
$$Rl(a_j) < c+1, \qquad j = 1, \ldots, n.$$

If we make use of the relation [1, p. 215]

$$G^{q,1}_{p+1,q}\left(z\middle|{}^{1,\, a_1, \cdots, a_p}_{b_1, \cdots, b_q}\right) = E\left({}^{b_1, \cdots, b_q : z}_{a_1, \cdots a_p}\right),$$

where $E(\cdot)$ denotes McRobert's E-function [1, p. 203], the formulae (1.1) and (1.2) reduce to the Fourier series for E-functions obtained by McRobert [2, p. 79, eqns (1) and (2)].

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From (1.1) and (1.2), we easily deduce the integrals

$$\int_0^\pi \sin\,(2r+1)\theta \sin\theta\, G^{m,n}_{p,q}\left(z/\sin^2\theta\middle|{}^{a_1, \cdots, a_p}_{b_1, \cdots, b_q}\right)d\theta$$
$$= \sqrt{\pi}\, G^{m+1,\,n+1}_{p+2,\,q+2}\left(z\middle|{}^{1-r,\, a_1, \cdots, a_p,\, 2+r}_{\frac{3}{2},\, b_1, \cdots, b_q,\, 1}\right),$$

and

$$\int_0^\pi \cos r\theta\, G^{m,n}_{p,q}\left(z/\sin^2(\theta/2)\middle|{}^{a_1, \cdots, a_p}_{b_1, \cdots, b_q}\right)d\theta$$
$$= \sqrt{\pi}\, G^{m+1,\,n+1}_{p+2,\,q+2}\left(z\middle|{}^{1-r,\, a_1, \cdots, a_p,\, 1+r}_{\frac{1}{2},\, b_1, \cdots, b_q,\, 1}\right),$$

where $p+q < 2(m+n)$, $|\arg z| < (m+n-p/2-q/2)\pi$ and $r = 0, 1, 2, \ldots$.

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