CH. PISOT

R. SALEM

Distribution modulo 1 of the powers of real numbers larger than 1

Compositio Mathematica, tome 16 (1964), p. 164-168

<http://www.numdam.org/item?id=CM_1964__16__164_0>

Distribution modulo 1 of the powers of real numbers larger than 1 *

by

Ch. Pisot and R. Salem †

1. A class of algebraic integers

Since the fundamental work of H. Weyl in 1916 on the theory of uniform distribution modulo 1, exhaustive studies have been made about the distribution of $f(n)$ when $f(n)$ is a polynomial in n, or when $f(n)$ increases slower than a polynomial in n (e.g. $f(n) = n^\alpha$, $(\alpha > 0)$ or $f(n) = \log^\alpha n$).

In comparison, very little is known on the distribution of the simpler functions increasing faster than any power of n, in particular of the distribution of θ^n when θ is a real number larger than one. Whereas Koksma [1] proved in 1935 that θ^n is uniformly distributed except for a set of θ of measure zero, nothing is known for the simplest individual values of θ, such as, e.g. $\theta = e$, or $\theta = \frac{3}{2}$.

Pisot, in 1938 ([2]) concentrated his work on the study of those values of θ, for which, far from being uniformly distributed, θ^n is, in a certain sense, as badly distributed as possible. It had been known, for a time, that if we denote by S the class of algebraic integers whose conjugates lie all (except θ itself) *inside* the unit circle then θ^n (mod. 1) tends to zero for $n = \infty$. This is true also for $\lambda\theta^n$ when λ is any algebraic integer of the field $K(\theta)$ generated by θ over the rationals.

Let us denote by $||a||$ the distance between a and the nearest integer. Pisot [2] proved that if $\theta > 1$ is such that there exists a real λ with the property that

$$\sum ||\lambda\theta^n||^2 < \infty \tag{1}$$

then θ belongs to the class S and λ is an algebraic number of the field $K(\theta)$.

The question whether this important theorem is true if we replace (1) by the weaker condition

* Nijenrode lecture.

† Professor Raphaël Salem died suddenly on the 20th of June 1963.

$$||\lambda\theta^n|| \to 0 \qquad (n = \infty) \tag{2}$$

is open, except if we know beforehand that θ is algebraic [3], in which case we can again assert that $\theta \in S$ and $\lambda \in K(\theta)$.

In other words, the open problem can be stated as follows: Do there exist transcendental θ with the property that, for some λ, (2) holds?

2. Another remarkable class of algebraic integers

Instead of considering the class S of algebraic integers θ such that all the conjugates of θ (except θ itself) have moduli strictly inferior to 1, Salem [4] has introduced the class T of algebraic integers $\tau > 1$ such that the conjugates α_K of τ have their moduli $|\alpha_K| \leqq 1$, the equality being permitted.

One sees immediately, (since if $|\alpha_K| = 1$, $\overline{\alpha_K} = 1/\alpha_K$), that if τ does not belong to S, i.e. if there exist actually conjugates with moduli 1, τ is the root of an irreducible reciprocal equation of even degree, whose roots lie all on the circumference of the unit circle, the only conjugate lying inside the unit circle being $1/\tau$.

The distribution modulo 1 of τ^m has interesting properties:

1°) *The numbers τ^m reduced modulo 1 are everywhere dense.*

The proof of this result is based on the following lemma. Let $2K$ be the degree of τ and $\alpha_j = e^{2\pi i\omega_j}$, $\bar{\alpha}_j = e^{-2\pi i\omega_j}$ $(j = 1, \ldots, K-1)$ be the conjugates of τ having moduli 1. *Then $\omega_1, \ldots, \omega_{K-1}$ and 1 are linearly independent.* It will be enough to prove that, if the A_j are rational integers,

$$\alpha_1^{A_1} \ldots \alpha_{K-1}^{A_{K-1}} = 1 \tag{3}$$

is an impossible equality.

Since the equation having the root τ is irreducible, its Galois group is transitive, i.e. there exists an automorphism σ of the Galois group sending, e.g. the root α_1 into the root τ. This automorphism cannot send α_j $(j \neq 1)$ into $1/\tau$ for since $\sigma(\alpha_1) = \tau$, $\sigma(1/\alpha_1) = 1/\tau$, and thus this would imply $\alpha_j = 1/\alpha_1$ which is not the case. Thus the automorphism applied to (3) would give

$$\tau^{A_1}\alpha_2'^{A_2} \ldots \alpha_{K-1}'^{A_{K-1}} = 1$$

if $\sigma(\alpha_j) = \alpha_j'$ $(j \neq 1)$. This is clearly impossible since $\tau > 1$ and $|\alpha_s'| = 1$. Hence the proof of the linear independence of $\alpha_1 \ldots \alpha_{K-1}$ and 1.

Now, we have modulo 1

$$\tau^m + \frac{1}{\tau^m} + \sum_{j=1}^{K-1} (e^{2\pi i m\omega_j} + e^{-2\pi i m\omega_j}) \equiv 0$$

or

$$\tau^m + 2\sum_{j=1}^{K-1} \cos 2\pi m\omega_j \to 0 \text{ (mod. 1)} \tag{4}$$

as $m \to \infty$. But, by Kronecker's theorem and the linear independence just proved we can determine m arbitrarily large, such that

$$2\sum_{j=1}^{K-1} \cos 2\pi m\omega_j$$

be arbitrarily close to any number given in advance (mod. 1). Take, e.g. m such that

$$|m\omega_1 - \delta| < \varepsilon \quad \text{(mod. 1)}$$
$$|m\omega_j - \tfrac{1}{4}| < \varepsilon \quad \text{(mod. 1)} \quad (j = 2, 3, \ldots, K-1).$$

Thus $\{\tau^m\}$ (mod 1) is everywhere dense.
(The same argument, applied to $\lambda\tau^m$, λ being an integer of $K(\tau)$ shows that $\{\lambda\tau^m\}$ is everywhere dense in a certain interval).

2°) *The powers τ^m of a number τ of the class T, although everywhere dense modulo* 1, *are not uniformly distributed modulo* 1.

We prove first the following lemma.

LEMMA. If the p-dimensional vector $\{u_n^j\}_{n=1}^{\infty}$ $(j = 1, 2, \ldots, p)$ is uniformly distributed (mod. 1) in the p-dimensional euclidean space R^p, the sequence

$$v_n = \omega(u_n^1) + \omega(u_n^2) + \ldots + \omega(u_n^p),$$

where $\omega(x)$ is continuous with period 1, is uniformly distributed if and only if the condition

$$\int_0^1 e^{2\pi i h\omega(x)}\, dx = 0$$

is satisfied for all integers $h \neq 0$.

(A different form of this condition is that the distribution function of $\omega(x)$ (mod. 1) should be linear.)

To prove the lemma, remark that $\{v_n\}$ is uniformly distributed (mod. 1) if and only if

$$\lim_{N=\infty} \frac{1}{N} \sum_1^N e^{2\pi i h v_n} = 0 \qquad (h \text{ integer} \neq 0).$$

But

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h\{\omega(u_n^1) + \ldots + \omega(u_n^p)\}} \to \left\{\int_0^1 e^{2\pi i h\omega(x)}\, dx\right\}^p.$$

Now in our case, in view of (4) we have to prove that the sequence

$$v_m = 2 \cos 2\pi m\omega_1 + \ldots + 2 \cos 2\pi m\omega_{K-1}$$

is not uniformly distributed (mod. 1).

In view of the lemma it is enough to prove that

$$\int_0^1 e^{4\pi i h \cos 2\pi x}\, dx = J_0(4\pi h),$$

J_0 being the Bessel function of order zero, does not vanish for all integers $h \neq 0$, which is true.

(If we use the second form of the lemma, we would have to show that the distribution function (mod. 1) of $2 \cos 2\pi x$ is not linear, which can be proved by direct computation.)

Unsolved problems.

Besides the unsolved problem quoted at the end of § 1, we quote the following one:

I) Salem has proved [5] that the set of all numbers θ of the class S is closed.

Little is known about the set of numbers τ of the class T, except [4] that every number θ of the class S is a limit point (on both sides) for numbers of T. It is not known what are the other limit points of T, if any.

II) Pisot [6] has proved that if there exists $\lambda \geqq 1$ such that

(5) $$||\lambda\theta^n|| \leqq \varepsilon \quad \text{for } all \quad n = 0, 1, \ldots,$$

where

$$\varepsilon = \frac{1}{2e\theta(\theta+1)(1+\log \lambda)}$$

then $\theta \in S \cup T$.

It is still an open question whether there exists a theorem including both results (1) and (5).

BIBLIOGRAPHY

J. F. Koksma

[1] Comp. Math. 2 (1935), 250—258.

Ch. Pisot

[2] Ann. Sc. Norm. Sup. Pisa (Ser. II) 7 (1938), 205—248.

G. H. Hardy

[3] J. Ind. Math. Soc. 11 (1919), 162—166.

R. Salem
[4] Duke Math. J. 12 (1945), 153—172.
R. Salem
[5] Duke Math. J. 11 (1944), 103—108.
Ch. Pisot
[6] Comm. Math. Helv. 19 (1946), 153—159.

(Oblatum 29-5-63).