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A regularity condition for a class of partitioned matrices

by

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1. We consider in this note an Hermitian matrix A partitioned into blocks $A_{\mu\nu}$:

$$A = \begin{pmatrix} A_{11} \cdots A_{1n} \\ \cdots\cdots\cdots \\ A_{n1} \cdots A_{nn} \end{pmatrix}, \tag{1}$$

where each $A_{\mu\nu}$ is a square matrix of order m and generally

$$A_{\mu\nu} = A^*_{\nu\mu}. \tag{2}$$

In the case $m = 1$, that is to say if *all $A_{\mu\nu}$ are scalars,* a regularity condition for A is contained in the so-called Hadamard's theorem (see O. Tausski-Todd [1] and the extensive bibliography given there), stating that A is regular if

$$|A_{\mu\mu}| > \sum_{\nu\neq\mu} |A_{\mu\nu}| \qquad (\mu = 1, \ldots, n). \tag{3}$$

2. One possibility of generalizing this theorem to the case $m > 1$ is given by the so-called *polar decomposition* of a general $(k \times k)$-matrix A into the product

$$A = EP, \tag{4}$$

where E is a unitary matrix and P a non-negative Hermitian matrix and E and P are uniquely determined by these properties. Then $P = P(A)$ corresponds to the modulus and E to the complementary factor of a complex number, containing its argument. On the other hand, there exists in the case of Hermitian matrices a *partial ordering* based on the concept of a non-negative Hermitian matrix. We say of two Hermitian matrices H_1 and H_2 of the same order k that H_1 is *majorated by* H_2. $H_1 \leqq H_2$, if $H_2 - H_1$ is a *non-negative* Hermitian matrix, and that H_1 is *properly majorated* by H_2, $H_1 \ll H_2$, if $H_2 - H_1$ is *positive.*

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3. The relation corresponding to (3) would then be

$$P(A_{\mu\mu}) \gg \sum_{\mu \neq \nu} P(A_{\mu\nu}) \qquad (\mu = 1, \ldots, n), \tag{6}$$

and the question arises whether (6) is then sufficient for the regularity of A. However, this is not generally true. A counter-example is given for $n = m = 2$ by

$$A_{11} = A_{22} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}, \; P(A_{12}) = P = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A_{12} = EP$$

Here we have $A_{11} \gg P$, $A_{22} \gg P$; and since

$$A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix}, \; A_{21} = P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix},$$

we have

$$|A| = \begin{vmatrix} 9 & 0 & 0 & 1 \\ 0 & 4 & 6 & 0 \\ 0 & 6 & 9 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = 0.$$

4. On the other hand, we can prove that the relations (6) are indeed sufficient for the regularity of A if the corresponding E factors are *scalars*. Indeed, we are going to prove

THEOREM. *Suppose that for $N = nm$ the blocks $A_{\mu\mu}$ in* (1) *are non-negative Hermitian matrices of order m. Suppose that we have for all $\mu \neq \nu$: $A_{\mu\nu} = \varepsilon_{\mu\nu} H_{\mu\nu}$, where $H_{\mu\nu}$ are non-negative Hermitian matrices with $H_{\mu\nu} = H_{\nu\mu}$ and $\varepsilon_{\mu\nu}$ are scalars of modulus* 1 *with $\varepsilon_{\mu\nu} = \bar{\varepsilon}_{\nu\mu}$. Suppose finally that we have*

$$A_{\mu\mu} \underline{\gg} \sum_{\nu \neq \mu} H_{\mu\nu} \qquad (\mu = 1, \ldots, n). \tag{7}$$

Then the matrix A is a non-negative Hermitian matrix and we have

$$A \underline{\gg} (A_{11} - \sum_{\nu \neq 1} H_{1\nu}) \dot{+} (A_{22} - \sum_{\nu \neq 2} H_{2\nu}) \dot{+} \cdots \dot{+} (A_{nn} - \sum_{\nu \neq n} H_{n\nu}), \tag{8}$$

and in particular

$$|A| \geqq \prod_{\mu=-1}^{n} |A_{\mu\mu} - \sum_{\nu \neq \mu} H_{\mu\nu}|; \tag{9}$$

we have even, if the right-side expression in (9) *is positive,*

$$|A| > \prod_{\mu=1}^{n} |A_{\mu\mu} - \sum_{\nu \neq \mu} H_{\mu\nu}|, \tag{10}$$

unless all $A_{\mu\nu} \doteq 0$ $(\mu \neq \nu)$.

5. Before giving the proof of our theorem, we make some observations on the majoration relation for $k \times k$ matrices.

If we have $H_1 \underline{\ll} H_2$ and if the eigenvalues of H_1 and H_2, decreasingly ordered, are respectively $\lambda_\nu(H_1)$, $\lambda_\nu(H_2)(\nu = 1, \ldots, k)$, then we have by a theorem of H. Weyl

$$\lambda_\nu(H_1) \leqq \lambda_\nu(H_2) \quad (\nu = 1, \ldots, k) \tag{11}$$

and even

$$P \geqq \lambda_\nu(H_2) - \lambda_\nu(H_1) \geqq p \qquad (\nu = 1, \ldots, k), \tag{12}$$

where P is the greatest and p the smallest eigenvalue of the matrix $H_2 - H_1$, see H. Weyl [1].

6. If $H_2 - H_1$ is *semidefinite*, we have $p = 0$ and obtain no sharpening of (11) from (12). However, we can prove:

Lemma. *If* $H_1 \underline{\ll} H_2$ *and* $H_1 \neq H_2$, *then we have in one at least of the inequalities* (11) *the sign* $<$. *If further* H_1 *is positive, we have*

$$|H_1| < |H_2|. \tag{13}$$

Proof. If we add to both matrices λI, the differences $\lambda_\nu(H_2) - \lambda_\nu(H_1)$ are not changed. We can therefore without loss of generality assume, that H_1 and H_2 are both positive. But then it follows from

$$|H_1| = \lambda_1(H_1)\lambda_2(H_1) \ldots \lambda_k(H_1) \ , \quad |H_2| = \lambda_1(H_2)\lambda_2(H_2) \ldots \lambda_k(H_2)$$

and (11) that it is sufficient to prove (13). By a simultaneous transformation of co-ordinates the Hermitian forms corresponding to H_1 and H_2 can be brought into the "sum of the squares" and the matrices H_1 and H_2 into the diagonal matrices

$$D_1 = \text{Diag}\,(p_1, \ldots, p_n), \quad D_2 = \text{Diag}\,(q_1, \ldots, q_n).$$

It is then sufficient to prove that $|D_1| < |D_2|$, that is, that

$$\prod_\nu p_\nu < \prod_\nu q_\nu.$$

But we have $D_1 \underline{\ll} D_2$, and therefore

$$p_\nu \leqq q_\nu \quad (\nu = 1, \ldots, k).$$

If we had here the equality sign for every ν, then we would have $D_1 = D_2$, $H_1 = H_2$. Therefore, we have indeed (13) and our lemma is proved.

7. Assume again that H_1 and H_0 are two positive Hermitian matrices with $H_1 \underline{\ll} H_2$. By an inequality due to Minkowski we have for two arbitrary non-negative Hermitian matrices A and B of order k

$$\sqrt[k]{|A|} + \sqrt[k]{|B|} \leqq \sqrt[k]{|A + B|}. \tag{14}$$

Applying this to $A = H_1$, $B = H_2 - H_1$ we can sharpen (13) to

$$\sqrt[k]{|H_2|} \geqq \sqrt[k]{|H_1|} + \sqrt[k]{|H_2 - H_1|}. \tag{15}$$

Therefore, if p is the smallest eigenvalue of $H_2 - H_1$, we have

$$\sqrt[k]{|H_2|} \geqq \sqrt[k]{|H_1|} + p. \tag{16}$$

8. Proof of the Theorem. Put

$$R_\mu = \sum_{\nu \neq \mu} H_{\mu\nu} \qquad (\mu = 1, \ldots, n) \tag{17}$$

and denote by S the matrix obtained from A by replacing each $A_{\mu\mu}$ by the corresponding R_μ,

$$S = \begin{pmatrix} R_1 A_{12} \cdots A_{1n} \\ A_{21} R_2 \cdots A_{2n} \\ \cdots\cdots\cdots \\ A_{n1} A_{n2} \cdots R_n \end{pmatrix}. \tag{18}$$

Then we have

$$A = \sum_\mu \cdot (A_{\mu\mu} - R_\mu) + S, \tag{19}$$

and (8) follows at once if we prove that S is *non-negative*; and (9) and (10) follows from (8) immediately by the Lemma of sec. 6.

9. In order to discuss the inertia character of S, decompose the general N-dimensional vector ξ into a Cartesian sum corresponding to the decomposition (1)

$$\xi = \sum_{\nu=1}^{n} \cdot \xi_\mu \tag{20}$$

and consider the corresponding Hermitian form

$$H(\xi) = \xi S \xi'. \tag{21}$$

We have then, using (17)

$$\begin{aligned} H(\xi) &= \sum_{\mu \neq \nu} \xi_\mu A_{\mu\nu} \xi_\nu^* + \sum_{\mu=1}^{n} \xi_\mu R_\mu \xi_\mu^* \\ &= \sum_{\mu \neq \nu} \xi \varepsilon_{\mu\nu} H_{\mu\nu} \xi_\nu^* + \sum_{\mu \neq \nu} \xi_\mu H_{\mu\nu} \xi_\mu^*. \end{aligned}$$

As $H_{\mu\nu} = H_{\nu\mu}$, we re-order the terms of this sum into groups, each containing four terms with the same $H_{\mu\nu}$, $\mu > \nu$. We obtain for a general $H_{\mu\nu}$ the group

$$\begin{aligned} &\xi_\mu \varepsilon_{\mu\nu} H_{\mu\nu} \xi_\nu^* + \xi_\nu \varepsilon_{\nu\mu} H_{\mu\nu} \xi_\mu^* + \xi_\mu H_{\mu\nu} \xi_\mu^* + \xi_\nu H_{\mu\nu} \xi_\nu^* \\ &= (\xi_\mu \varepsilon_{\mu\nu} + \xi_\nu) H_{\mu\nu} \xi_\nu^* + (\xi_\nu \varepsilon_{\nu\mu} + \xi_\mu) H_{\mu\nu} \xi_\mu^*. \end{aligned}$$

If we put

$$\eta_{\mu\nu} = \xi_\mu \varepsilon_{\mu\nu} + \xi_\nu \qquad (\mu > \nu),$$

we have, using $\varepsilon_{\nu\mu} = \bar{\varepsilon}_{\mu\nu} = \dfrac{1}{\varepsilon_{\mu\nu}}$,

$$\xi_\nu \varepsilon_{\nu\mu} + \xi_\mu = \bar{\varepsilon}_{\mu\nu}(\xi_\nu + \varepsilon_{\mu\nu}\xi_\mu) = \bar{\varepsilon}_{\mu\nu}\eta_{\mu\nu}.$$

Therefore our group of four terms containing $H_{\mu\nu}$ becomes $\eta_{\mu\nu} H_{\mu\nu} \eta^*_{\mu\nu}$, and we have

$$H(\xi) = \sum_{\mu>\nu} \eta_{\mu\nu} H_{\mu\nu} \eta^*_{\mu\nu} \geqq 0,$$

as $H_{\mu\nu}$ are assumed to be non-negative. Our theorem is proved.

9. If we now assume, under the conditions of our theorem, that the relations (6) are true, that is, that we have

$$A_{\mu\mu} \gg \sum_{\nu\neq\mu} H_{\mu\nu} \qquad (\mu = 1, \ldots, n),$$

then the differences $A_{\mu\mu} - \sum_{\nu\neq\mu} H_{\mu\nu}$ $(\mu = 1, \ldots, n)$ are positive and have positive determinants. But then by (10) we have $|A| > 0$ and A is indeed regular.

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H. WEYL

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