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Certain Factors Constructed as Infinite Tensor Products*

by

D. J. C. Bures

The purpose of this paper is to determine the type of certain factors which are infinite tensor products of factors of type $\mathbf{I}_n$ on n^2-dimensional Hilbert spaces.[1]) In the course of this paper we obtain some results of a more general nature: in particular we show that any tensor product of maximal abelian von Neumann algebras is maximal abelian (proposition 3.1, below); and we show how certain [2]) tensor products of von Neumann algebras obtainable by a construction of the kind of [RO III][3]) can themselves be obtained by such a construction (proposition 4.1, below).

Our results regarding the types of the factors in question may be summarized as follows. Suppose that J is an infinite indexing set, and that, for each $\alpha \in J$, n_α is an integer $\geqq 2$. Let $H_{(\alpha, 0)}$ and $H_{(\alpha, 1)}$ be n_α-dimensional Hilbert spaces, and let $H_\alpha = H_{(\alpha, 0)} \otimes H_{(\alpha, 1)}$. Let $\mathscr{A}_\alpha$ be the factor $\mathscr{L}(H_{(\alpha, 0)}) \otimes 1$ on H_α. Suppose that, for each $\alpha \in J$, f_α is a vector of H_α with $||f_\alpha|| = 1$. Then it is possible to choose orthonormal bases $(\varphi^i_{(\alpha, \delta)})_{i=1,2,\ldots n_\alpha}$ for the $H_{(\alpha, \delta)}$ in such a way that

$$f_\alpha = \sum_{i=1}^{n_\alpha} a_i^\alpha \, \varphi^i_{(\alpha, 0)} \otimes \varphi^i_{(\alpha, 1)}$$

*) Many of the results of this paper were included in the author's doctoral dissertation at Princeton University. The author is indebted to Professor W. Feller and Professor I. Halperin for their suggestions.

1) In chapter 7 of [IDP], J. von Neumann considered countable tensor products of factors of type I_2 on Hilbert spaces of dimension 4. He showed that the types of certain of these tensor products are I_∞, II_1, and II_∞ respectively. In a later paper, [RO III], von Neumann asserted that in certain cases the type is III_∞. However, the proof was not published.

2) In particular, any finite tensor product.

3) See [RO III], Chapter III. Actually, to avoid restricting ourselves to the separable case, we use the generalization of J. Dixmier ([1], pp. 127—136). In the remainder of the paper we refer to a von Neumann algebra which can be obtained by such a construction as "a constructible algebra" (see definition 1.1, below).

where $a_1^\alpha \geqq a_2^\alpha \geqq \dots \geqq a_{n_\alpha}^\alpha \geqq 0$.

Let $\mathscr{A}$ be the tensor product of $(\mathscr{A}_\alpha)_{\alpha \in I}$ relative to $(f_\alpha)_{\alpha \in I}$.[4]) Then $\mathscr{A}$ is a factor, and:

1. $\mathscr{A}$ is of type I if and only if $\sum_{\alpha \in J}(1 - a_1^\alpha) < \infty$, in which case it is of type $\mathrm{I}_{\bar{\bar{J}}}$.[5])

2. If the n_α are bounded, $\mathscr{A}$ is of type II_1 if and only if

$$\sum_{\alpha \in J} (1 - n_\alpha^{-\frac{1}{2}} \sum_{i=1}^{n_\alpha} a_i^\alpha) < \infty.$$

In the general case, we have not been able to find a necessary and sufficient condition for $\mathscr{A}$ to be of type II_1; however,

$$\sum_{\alpha \in J} (1 - n_\alpha^{-\frac{1}{2}} \sum_{i=1}^{n_\alpha} a_i^\alpha) < \infty$$

is a sufficient condition, and

$$\sum_{\alpha \in J} \left[\sum_{i=1}^{n_\alpha} \left((a_i^\alpha)^2 - \frac{1}{n_\alpha} \right)^2 \right] < \infty$$

is a necessary condition.

3. Suppose that there exists an infinite subset K of J, such that, for some $\varepsilon > 0$ and some integers p_α, q_α with $1 \leqq p_\alpha, q_\alpha \leqq n_\alpha$ the following holds:

$$a_{p_\alpha}^\alpha,\ a_{q_\alpha}^\alpha \geqq \varepsilon$$

and

$$a_{p_\alpha}^\alpha / a_{q_\alpha}^\alpha \geqq 1 + \varepsilon \qquad \text{for all } \alpha \in K.$$

Then $\mathscr{A}$ is of type $\mathrm{III}_\aleph$ where $\aleph$ is the larger of $\aleph_0$ and the cardinality of the set of α in J for which some $a_i^\alpha = 0$.

For the most part we use the notation of [1]. However, our terminology regarding infinite tensor products needs some explanation.

Suppose that $(H_\alpha)_{\alpha \in I}$ is a family of Hilbert spaces, and that, for each $\alpha \in I$, $f_\alpha \in H_\alpha$ with $\|f_\alpha\| = 1$. Using the terminology of [IDP], let $\mathfrak{C}$ be the equivalence class of C_0 – sequences containing the C_0-sequence $(f_\alpha)_{\alpha \in I}$; and let H be the Hilbert space which is the $\mathfrak{C}$-adic incomplete direct product of the H_α, that is $\otimes_{\alpha \in I}^{\mathfrak{C}} H_\alpha$.

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[4]) See the explanation on terminology below.

[5]) The dimension function on factors on non-separable Hilbert spaces is discussed in § 2 (below), and in [3].

We shall consistently refer to H as the tensor product of $(H_\alpha)_{\alpha\in I}$ relative to $(f_\alpha)_{\alpha\in I}$, and denote it by $\otimes_{\alpha\in I}^{(f_\alpha)} H_\alpha$. We now state a result of [IDP] which gives us a working definition of the tensor product in terms of orthonormal bases. We shall use this result in the following form throughout the paper.

PROPOSITION 0.1 (lemma 4.1.4 of [IDP]). Suppose that $(H_\alpha)_{\alpha\in I}$ is a family of Hilbert spaces and that $f_\alpha \in H_\alpha$ with $\|f_\alpha\| = 1$. Choose an orthonormal basis $(\varphi_\alpha^j)_{j\in J_\alpha}$ for each H_α in such a way that $0 \in J_\alpha$ and $\varphi_\alpha^0 = f_\alpha$. Then define [6] $J = \coprod_{\alpha\in I} J_\alpha = \{(j_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} J_\alpha : j_\alpha = 0$ for all but a finite number of the α in $I\}$. If $j = (j_\alpha)_{\alpha\in I}$ is in J let $\varphi^j = \otimes_{\alpha\in I} \varphi_\alpha^{j_\alpha}$. Then $(\varphi^j)_{j\in J}$ is an orthonormal basis for $\otimes_{\alpha\in I}^{(f_\alpha)} H_\alpha$.

Suppose now that $(H_\alpha)_{\alpha\in I}$ and $(f_\alpha)_{\alpha\in I}$ are as above. Let H be the tensor product of $(H_\alpha)_{\alpha\in I}$ relative to $(f_\alpha)_{\alpha\in I}$. There is a canonical *-isomorphism [7] ϕ_β from $\mathscr{L}(H_\beta)$ into $\mathscr{L}(H)$: if $x_\alpha \in H_\alpha$ and if $x_\alpha = f_\alpha$ for all but a finite number of the α in I, then

$$(\phi_\beta T)(\otimes_{\alpha\in I} x_\alpha) = (\otimes_{\alpha\in I-\{\beta\}} x_\alpha) \otimes T x_\beta \quad \text{for}$$

all $T \in \mathscr{L}(H_\beta)$. We shall call $\phi_\beta T$ the extension of T from H_β to H and denote it by $\tilde{T}$ provided that this does not lead to confusion. If $\mathscr{A}_\alpha$ is a von Neumann algebra on H_α, then $\phi_\alpha(\mathscr{A}_\alpha)$ is a von Neumann algebra on H [8]), which we shall denote by $\overline{\mathscr{A}_\alpha}$.

If $(\mathscr{A}_\alpha)_{\alpha\in I}$ is a family of von Neumann algebras, each $\mathscr{A}_\alpha$ on H_α, then we shall write $\otimes_{\alpha\in I}^{(f_\alpha)} \mathscr{A}_\alpha$ for the von Neumann algebra generated by the $\overline{\mathscr{A}_\alpha}$ on $H = \otimes_{\alpha\in I}^{(f_\alpha)} H_\alpha$. We shall refer to $\otimes_{\alpha\in I}^{(f_\alpha)} \mathscr{A}_\alpha$ as the tensor product of $(\mathscr{A}_\alpha)_{\alpha\in I}$ relative to $(f_\alpha)_{\alpha\in I}$.

§ 1. Constructible Algebras

We summarize here the construction of [RO III] [9]) in the form given in [1]. [10]) We shall use the notation of this section throughout the paper.

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[6]) We shall use this notation for groups also. Suppose that, for each $\alpha \in I$, G_α is a group with identity e_α. Then $G = \coprod_{\alpha\in I} G_\alpha$ will mean the subgroup of $\prod_{\alpha\in I} G_\alpha$ consisting of elements $(g_\alpha)_{\alpha\in I}$ for which $g_\alpha = e_\alpha$ for all but a finite number of the α in I.

[7]) See [IDP], lemma 5.1.1.

[8]) See [IDP], lemma 5.2.3.

[9]) [RO III], Chapter III.

[10]) [1], pp. 127—136.

Suppose that $\mathscr{M}$ is a maximal abelian von Neumann algebra on the Hilbert space H; and suppose that G is a discrete group with identity e and a unitary representation $g \to \mathsf{U}_g$ on H. Assume that $\mathsf{U}_g \mathscr{M} \mathsf{U}_g^* = \mathscr{M}$ for all $g \in G$.

We shall say that the system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is:

(i) free, if $\mathsf{U}_g \mathscr{M} \cap \mathscr{M} = \{0\}$ for all $g \in G - \{e\}$;

(ii) ergodic, if $\mathscr{M} \cap \{\mathsf{U}_g : g \in G\}' = \mathbf{C}_H$.

Let $\hat{H}$ be the Hilbert space with orthonormal basis $(\hat{g})_{g \in G}$, and let $\bar{H} = H \otimes \hat{H}$. Define a unitary representation of G on $\hat{H}$, $g \to V_g$, by $V_g(\hat{h}) = \widehat{gh}$. Then $g \to \mathsf{U}_g \otimes V_g$ is a unitary representation of G on $\bar{H}$. Define $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ to be the von Neumann algebra on $\bar{H}$ generated by $\mathscr{M} \otimes 1$ and the $\mathsf{U}_g \otimes V_g$.

Definition 1.1. We shall call $\mathscr{A}$ a constructible algebra provided that $\mathscr{A}$ is a von Neumann algebra that is spatially isomorphic to $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ for some *free* system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$. Here $\mathscr{M}$ is a maximal abelian von Neumann algebra on H, G is a discrete group with unitary representation $g \to \mathsf{U}_g$, and $\mathsf{U}_g \mathscr{M} \mathsf{U}_g^* = \mathscr{M}$ for all $g \in G$.

Proposition 1.1 ([RO III] and [1]).

Suppose that the system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is free and let

$$\mathscr{A} = \mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g].$$

1. $\mathscr{A}$ is a factor if and only if the system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is ergodic.

2. Suppose that $\mathscr{A}$ is a factor.

(a) $\mathscr{A}$ is of type I if and only if $\mathscr{M}$ contains a minimal projection.

(b) $\mathscr{A}$ is finite (of type I_n or II_1) if and only if there exists a finite [11]) normal trace φ on $\mathscr{M}$ which satisfies $\varphi(\mathsf{U}_g M \mathsf{U}_g^*) = \varphi(M)$ for all $g \in G$ and all $M \in \mathscr{M}^+$.

(c) $\mathscr{A}$ is of type III if and only if there exists no semi-finite [12]) normal trace φ on $\mathscr{M}$ which satisfies $\varphi(\mathsf{U}_g M \mathsf{U}_g^*) = \varphi(M)$ for all $g \in G$ and all $M \in \mathscr{M}^+$.

3. Define $W \in \mathscr{L}(\bar{H})$ by $W(x \otimes \hat{g}) = (\mathsf{U}_g^* x) \otimes \hat{g}^{-1}$, for all $x \in H$ and all $g \in G$. Then W is a unitary involution on $\bar{H}$, and $W \mathscr{A} W = \mathscr{A}'$.

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[11]) By a finite trace we mean a trace φ with $0 < \varphi(1) < \infty$.

[12]) By a semi-finite trace on a factor $\mathscr{A}$ we mean a trace φ with $0 < \varphi(E) < \infty$ for some projection E of $\mathscr{A}$.

§ 2. The Dimension Function on Factors on Non-Separable Hilbert Spaces

DEFINITION 2.1. Suppose that $\mathscr{A}$ is a von Neumann algebra and that $\aleph$ is an infinite cardinal. A projection E of $\mathscr{A}$ is $\aleph$-decomposable (in $\mathscr{A}$) if any family $(E_i)_{i\in I}$ of mutually orthogonal projections of $\mathscr{A}$ with $0 < E_i \leqq E$ has cardinality $\overline{\overline{I}} \leqq \aleph$.

Notice that if $\mathscr{A}$ is a von Neumann algebra on a Hilbert space of dimension $\aleph$ then any projection E of $\mathscr{A}$ is $\aleph$-decomposable in $\mathscr{A}$.

DEFINITION 2.2. Suppose that $\mathscr{A}$ is a von Neumann algebra. If E is a projection of $\mathscr{A}$, the (decomposability) type of E (in $\mathscr{A}$) is the least cardinal $\aleph$ such that E is $\aleph$-decomposable in $\mathscr{A}$. The (decomposability) type of $\mathscr{A}$ is defined to be the decomposability type of 1 in $\mathscr{A}$.

It is clear that the decomposability type of $\mathscr{A}$ is invariant under *-isomorphism and that if $E \approx F$ in $\mathscr{A}$ then the type of E equals the type of F.

LEMMA 2.1. Any cyclic projection of the von Neumann algebra $\mathscr{A}$ is of type $\aleph_0$.

PROOF: Suppose that $\mathscr{A}$ is a von Neumann algebra on the Hilbert space H. If E is a cyclic projection of $\mathscr{A}$ then [13]) $E = \text{pr}\,[\mathscr{A}'x]$ for some $x \in H$. Notice that, if a projection F of $\mathscr{A}$ satisfies $F \leqq E$ and $Fx = 0$, then $FA'x = A'Fx = 0$ for all $A' \in \mathscr{A}'$ so that $F = FE = 0$.

Suppose that $(E_i)_{i\in I}$ is a family of mutually orthogonal projections of $\mathscr{A}$ with $0 < E_i \leqq E$. Then $\sum_{i\in I} E_i \leqq E$, and hence $\sum_{i\in I}||E_i x||^2 \leqq ||Ex||^2 < \infty$. Now $E_i x \neq 0$; therefore I is countable. That demonstrates that E is of type $\aleph_0$.

LEMMA 2.2. If $E = \sum_{i\in I} E_i$ where each E_i is a non-zero cyclic projection of $\mathscr{A}$, then the type of E in $\mathscr{A}$ is $\overline{\overline{I}}$, if I is infinite, and $\aleph_0$, if I is finite.

PROOF: The type of E in $\mathscr{A}$ is certainly $\geqq \overline{\overline{I}}$. We need to prove, then, only that E is $\aleph$-decomposable in $\mathscr{A}$, where $\aleph$ is the larger of $\overline{\overline{I}}$ and $\aleph_0$.

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[13]) We write $[S]$ to mean the subspace (closed linear manifold) determined by the subset S of H; pr $[S]$ denotes the orthogonal projection onto the subspace $[S]$.

Suppose that $E_i = \mathrm{pr}[\mathscr{A}'x_i]$. Notice that, if F is a projection of $\mathscr{A}$ with $Fx_i = 0$ for all $i \in I$ and $F \leqq E$, then $F = 0$.

Suppose that $(F_j)_{j \in J}$ is a family of mutually orthogonal projections of $\mathscr{A}$ with $0 < F_j \leqq E$. Then $\sum_{j \in J} ||F_j x_i||^2 \leqq ||Ex_i||^2 < \infty$ for each $i \in I$. If $i \in I$ is fixed, then, $F_j x_i = 0$ for all but a countable number of the j in J. Now every F_j satisfies $F_j x_i \neq 0$ for some $i \in I$, because $0 < F_j \leqq E$. Therefore $\bar{\bar{J}} \leqq \aleph_0 \cdot \bar{\bar{I}} = \aleph$. This demonstrates that E is $\aleph$-decomposable.

PROPOSITION 2.1. Suppose that $\mathscr{A}$ is a von Neumann algebra. Any projection E of $\mathscr{A}$ can be expressed as $\sum_{i \in I} E_i$ where each E_i is a non-zero cyclic projection of $\mathscr{A}$. The decomposability type of E in $\mathscr{A}$ is then the larger of $\bar{\bar{I}}$ and $\aleph_0$.

PROOF: Using Zorn's lemma select a maximal family $(E_i)_{i \in I}$ of mutually orthogonal cyclic projections of $\mathscr{A}$ with $0 < E_i \leqq E$. Let $F = \sum_{i \in I} E_i$. Then F is in $\mathscr{A}$ and $F \leqq E$. We shall show that $F = E$.

Suppose that $F < E$. Then there is an $x \in H$ with $(E - F)x = x \neq 0$. $G = \mathrm{pr}\,[\mathscr{A}'x]$ is a cyclic projection of $\mathscr{A}$ which satisfies $0 < G \leqq E$ and is orthogonal to all the E_i. This contradicts the maximality of the family $(E_i)_{i \in I}$; therefore $E = F = \sum_{i \in I} E_i$.

The rest of the proposition follows from lemma 2.1.

COROLLARY. If $E = \sum_{i \in I} E_i$ where each E_i is $\aleph_0$-decomposable and non-zero, then the type of E is the larger of $\bar{\bar{I}}$ and $\aleph_0$.

Suppose that $\mathscr{A}$ is a factor on a separable Hilbert space. Then [RO] there exists a dimension function d from the projections of $\mathscr{A}$ to the extended positive reals such that:

1. $d(E) = 0$ if and only if $E = 0$. $d(E) < \infty$ if and only if E is finite.
2. $E \approx F$ if and only if $d(E) = d(F)$.
3. If $(E_i)_{i \in I}$ is a family of mutually orthogonal projections of $\mathscr{A}$, then $d(\sum_{i \in I} E_i) = \sum_{i \in I} d(E_i)$.

In the non-separable case the same procedure as that of [RO] is valid but the resulting dimension fails to have property 2, above; it has properties 1 and 3 and:

2'. $E \approx F$ implies $d(E) = d(F)$.

If, however, we allow infinite cardinals as values, a dimension function with properties 1, 2 and 3 exists.

Proposition 2.2. Suppose that $\mathscr{A}$ is a factor. Define the function d' from the projections of $\mathscr{A}$ to the positive reals and infinite cardinals by: $d'(E) = d(E)$ for E finite, and $d'(E) =$ the decomposability type of E in $\mathscr{A}$ for E infinite. Then d' has properties 1, 2 and 3.

Proof: It is clear that properties 1 and 2' hold for d'. Property 3 follows from proposition 1.1. Property 2 must hold for finite projections, so that there remains to be proved only that, if E and F are infinite projections of the same type, then $E \approx F$.

First let us show that if $\mathscr{A}$ contains any infinite projection, then there is a projection E_0 of $\mathscr{A}$ which is infinite and $\aleph_0$-decomposable. Let G be a cyclic projection of $\mathscr{A}$. By lemma 2.1 G is $\aleph_0$-decomposable. If G is infinite, take E_0 to be G; if G is finite, choose[14]) a sequence $(E_n)_{n=1, 2, \ldots}$ of mutually orthogonal projections of $\mathscr{A}$ with each $E_n \leqq E$ and each $E_n \approx G$, and take E_0 to be $\sum_{n=1}^{\infty} E_n$.

Suppose that E and F are $\aleph_0$-decomposable infinite projections of $\mathscr{A}$. By the comparability of projections in a factor we may assume $E \approx F_1 \leqq F$. Then (by the argument in the proof of lemma 7.2.1 of [RO]), $F_1 \approx F$ and hence $E \approx F$.

Suppose that E and F are infinite projections of $\mathscr{A}$ of decomposability type $\aleph > \aleph_0$. As in the proof of lemma 7.1.2 of [RO], by comparability of projections and an exhaustion argument, $E = \sum_{i \in I} E_i$ and $F = \sum_{j \in J} F_j$ where each $E_i \approx E_0$ and each $F_j \approx E_0$. The corollary to proposition 2.1 shows that $\overline{\overline{I}} = \aleph = \overline{\overline{J}}$. Therefore $E \approx F$.

Corollary (c.f. theorem VIII of [RO]).

If $\mathscr{A}$ is an infinite factor of decomposability type $\aleph$, the range of the dimension function consists of certain finite real numbers and all infinite cardinals a satisfying $\aleph_0 \leqq a \leqq \aleph$. We shall say that the type of such a factor is $\mathrm{I}_{\aleph}$, $\mathrm{II}_{\aleph}$, or $\mathrm{III}_{\aleph}$ instead of the usual I_{∞}, II_{∞}, or III_{∞}.

§ 3. Tensor Products of Maximal Abelian von Neumann Algebras

Proposition 3.1. If $\mathscr{M} = \otimes_{i \in I}^{(f_i)} \mathscr{M}_i$ and each $\mathscr{M}_i$ is a maximal abelian von Neumann algebra, then $\mathscr{M}$ is also maximal abelian.

In the proof of proposition 3.1 we shall use several lemmas.

[14]) Here we use the comparability of projections in the factor $\mathscr{A}$ as well as the fact that the sum of a finite number of finite projections is finite.

Lemma 3.1. An abelian von Neumann algebra with cyclic vector is maximal abelian. (See [9], corollary 1.1, or [1], pg. 109 number 5).

Lemma 3.2. If, for each $\alpha \in I$, $\mathscr{M}^\alpha$ is a maximal abelian von Neumann algebra on H^α, then $\prod_{\alpha \in I} \mathscr{M}^\alpha$ is a maximal abelian von Neumann algebra on $\oplus_{\alpha \in I} H^\alpha$.

Proof: $(\prod_{\alpha \in I} \mathscr{M}^\alpha)' = \prod_{\alpha \in I} (\mathscr{M}^\alpha)' = \prod_{\alpha \in I} \mathscr{M}^\alpha$ provided that each $\mathscr{M}^\alpha$ is maximal abelian.

Lemma 3.3. Suppose that $(H_i)_{i \in I}$ is a family of Hilbert spaces.

Suppose that each $H_i = \oplus_{\alpha \in A_i} H_i^\alpha$ with each A_i containing 0. Let f_i be a vector of H_i^0 with $||f_i|| = 1$.

For each i and $\alpha \in A_i - \{0\}$ choose any vector f_i^α of H_i^α with $||f_i^\alpha|| = 1$; let $f_i^0 = f_i$. Define A to be $\prod_{i \in I} A_i$ and, for $\alpha = (\alpha_i)_{i \in I}$ in A, define $H^\alpha = \otimes_{i \in I}^{(f_i^{\alpha_i})} H_i^{\alpha_i}$.

Then $H = \otimes_{i \in I}^{(f_i)} H_i$ has direct-sum decomposition $H = \oplus_{\alpha \in A} H^\alpha$.

Proof: This is a direct consequence of proposition 0.1.

Lemma 3.4. Suppose that $\mathscr{A}_i$ is a von Neumann algebra on H_i and that f_i is in H_i with $||f_i|| = 1$. Let $H = \otimes_{i \in I}^{(f_i)} H_i$ and let $\mathscr{A} = \otimes_{i \in I}^{(f_i)} \mathscr{A}_i$. Suppose that $\varphi_i \in H_i$ with $||\varphi_i|| = 1$, and that φ_i is a cyclic vector for $\mathscr{A}_i$. Then, provided that $\sum_{i \in I} |1 - (f_i, \varphi_i)| < \infty$, $\varphi = \otimes_{i \in I} \varphi_i$ is a cyclic vector for $\mathscr{A}$.

Proof;[15]) Since $\sum_{i \in I} |1 - (f_i, \varphi_i)| < \infty$, $\varphi = \otimes_{i \in I} \varphi_i$ is definable in H, and $H = \otimes_{i \in I}^{(f_i)} H_i$ is also $\otimes_{i \in I}^{(\varphi_i)} H_i$. We may assume, then, that $\varphi_i = f_i$.

We have to prove that $[\mathscr{A}\varphi] = H$. By proposition 0.1, it is sufficient to show that, if $x_i \in H_i$ with $||x_i|| = 1$ and $x_i = f_i$ for all but a finite number of $i \in I$, then $\otimes_{i \in I} x^i \in [\mathscr{A}\varphi]$.

Suppose that such a family $(x_i)_{i \in I}$ is given along with $\varepsilon > 0$. We are going to produce an operator $T \in \mathscr{A}$ such that

$$||\otimes_{i \in I} x_i - T\varphi|| < \varepsilon.$$

Let $F = \{i \in I : x_i \neq f_i)$. Then F has a finite number of elements, say n. For each $i \in F$ let $T_i \in \mathscr{A}_i$ be such that

$$||x_i - T_i f_i|| < \frac{\varepsilon}{n 2^n}, \ 1.$$

Then let $T = \prod_{i \in F} T_i$.

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15) c.f. the proof of lemma 4.1.4 of [IDP].

We have

$$||\otimes_{i\in I} x_i - T\varphi|| =$$
$$||\otimes_{i\in F} x_i - \otimes_{i\in F} T_i \varphi_i|| \leqq$$
$$[\sum_{i\in F}||x_i - T_i f_i||] \cdot \prod_{i\in F} \max (1, ||T_i f_i||) \leqq$$
$$n \cdot \frac{\varepsilon}{n2^n} \prod_{i\in F} (1 + ||x_i||) = \varepsilon.$$

This completes the proof.

PROOF of PROPOSITION 3.1:

Suppose that, for each $i \in I$, $\mathscr{M}_i$ is a maximal abelian von Neumann algebra on the Hilbert space H_i. Suppose that f_i is a vector of H_i with $||f_i|| = 1$. Let $\mathscr{M} = \otimes_{i\in I}^{(f_i)} \mathscr{M}_i$ and let $H = \otimes_{i\in I}^{(f_i)} H_i$. We want to show that $\mathscr{M}$ is maximal abelian on H.

The idea behind the proof is to find a direct sum decomposition $H = \oplus_{\alpha\in A} H^\alpha$ such that:

1. Each H^α belongs to $\mathscr{M}$; that is, pr $[H^\alpha] \in \mathscr{M}$.
2. Each $\mathscr{M}|_{H^\alpha}$ has a cyclic vector.[16])

Then lemmas 3.1 and 3.2 together prove that $\mathscr{M}$ is maximal abelian.

First we make a direct sum decomposition of each H_i. By Zorn's lemma there exists a family $(f_i^\alpha)_{\alpha\in A_i}$ of vectors of H_i such that $0 \in A_i$ with $f_i^0 = f_i$ and such that $\sum_{\alpha\in A_i} \mathrm{pr}[\mathscr{M}_i f_i^\alpha] = 1$.

Let $H_i^\alpha = [\mathscr{M}_i f_i^\alpha]$. Then $H_i = \oplus_{\alpha\in A_i} H_i^\alpha$. Furthermore each H_i^α belongs to $\mathscr{M}_i$, for $(\mathscr{M}_i)' = \mathscr{M}_i$. It is clear that f_i^α is a cyclic vector for $\mathscr{M}_i|_{H_i^\alpha}$.

Now, as in lemma 3.3, let $A = \prod_{i\in I} A_i$ and, for $\alpha = (\alpha_i)_{i\in I}$ in $\mathscr{A}$, let $H^\alpha = \otimes_{i\in I}^{(f_i^{\alpha_i})} H_i^{\alpha_i}$. Then $H = \oplus_{\alpha\in A} H^\alpha$. Clearly each H^α belongs to $\mathscr{M}$. $\mathscr{M} = \mathscr{R}_H (\overline{\mathscr{M}_i} : i \in I)$; therefore $\mathscr{M}|_{H^\alpha} = \mathscr{R}_{H^\alpha}(\overline{\mathscr{M}_i}|_{H^\alpha} : i \in I) = \mathscr{R}_{H^\alpha}(\overline{(\mathscr{M}_i|_{H_i^{\alpha_i}})} : i \in I) = \otimes_{i\in I}^{(f_i)}(\mathscr{M}_i|_{H_i^{\alpha_i}})$.

We conclude that $\mathscr{M}|_{H^\alpha}$ has a cyclic vector (lemma 3.4).

This completes the proof.

Lemma 3.4 leads to an easy proof of the following results of [IDP].

PROPOSITION 3.2 (contained in theorem IX of [IDP]).

Suppose that $(H_i)_{i\in I}$ is a family of Hilbert spaces, and that f_i is in H_i with $||f_i|| = 1$. Then $\mathscr{L}(\otimes_{i\in I}^{(f_i)} H_i) = \otimes_{i\in I}^{(f_i)} \mathscr{L}(H_i)$.

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[16]) If $\mathscr{M}$ is a von Neumann algebra on the Hilbert space H, and H' is a subspace of H which reduces $\mathscr{M}$, then $\mathscr{M}|_{H'}$ denotes the von Neumann algebra on H' consisting of the restrictions of operators of $\mathscr{M}$ to H'.

PROOF:

Let $H = \otimes_{i \in I}^{(f_i)} H_i$ and $\mathscr{A} = \otimes_{i \in I}^{(f_i)} \mathscr{L}(H_i)$. We have to show that $\mathscr{A} = \mathscr{L}(H)$. It is sufficient to show that every non-zero vector $x \in H$ is a cyclic vector for $\mathscr{A}$.[17])

By proposition 0.1, choose an orthonormal basis $(\varphi_j)_{j \in J}$ for H in such a way that each $\varphi_j = \otimes_{i \in I} x_j^i$ for some $x_j^i \in H_i$. Then each φ_j is a cyclic vector for $\mathscr{A}$ (lemma 3.4). Furthermore $\mathrm{pr}\,[\varphi_j] = \prod_{i \in I} \mathrm{pr}\,[x_j^i] \in \mathscr{A}$. Suppose that x is a non-zero vector of H. For some $j \in J$, $(\mathrm{pr}\,[\varphi_j])\, x \neq 0$ since $(\varphi_j)_{j \in J}$ is an orthonormal basis for H. It follows that $(\mathrm{pr}\,[\varphi_j])x$ is a cyclic vector for $\mathscr{A}$, because φ_j is a cyclic vector for A. Finally, since $\mathrm{pr}\,[\varphi_j] \in \mathscr{A}$, x must be a cyclic vector for $\mathscr{A}$.

COROLLARY:

$\otimes_{i \in I}^{(f_i)} \mathscr{A}_i$ is a factor provided that each $\mathscr{A}_i$ is a factor.

PROOF: Let $\mathscr{A} = \otimes_{i \in I}^{(f_i)} \mathscr{A}_i$ and $H = \otimes_{i \in I}^{(f_i)} H_i$, where $\mathscr{A}_i$ is a factor on H_i. Then [18])

$$\mathscr{R}_H(\mathscr{A}, \mathscr{A}') \supset \mathscr{R}_H(\overline{\mathscr{A}_i}, \overline{\mathscr{A}'_i} : i \in I) = \mathscr{R}_H(\overline{\mathscr{L}(H_i)} : i \in I) = \mathscr{L}(H).$$

Therefore $\mathscr{A} \cap \mathscr{A}' = (\mathscr{A}', \mathscr{A})' = (\mathscr{L}(H))' = \mathbf{C}$.

§ 4. Tensor Products of Constructible Algebras

For each $i \in I$, suppose that $\mathscr{M}^i$ is a maximal abelian von Neumann algebra on the Hilbert space H^i, and suppose that G^i is a discrete group with identity e^i and unitary representation $g \to \mathsf{U}_g^i$ on H^i. Assume that $\mathsf{U}_g^i \mathscr{M}^i (\mathsf{U}_g^i)^* = \mathscr{M}^i$ for all $g \in G^i$. Let $\mathscr{A}^i = \mathscr{A}[\mathscr{M}^i, H^i, G^i, g \to \mathsf{U}_g^i]$.

Suppose that $f^i \in H^i$ with $\|f^i\| = 1$. Define the Hilbert space H to be the tensor product of $(H^i)_{i \in I}$ relative to $(f^i)_{i \in I}$; define the abelian von Neumann algebra $\mathscr{M}$ to be the tensor product of $(\mathscr{M}^i)_{i \in I}$ relative to $(f^i)_{i \in I}$. By proposition 3.1, $\mathscr{M}$ is maximal abelian on H.

Let the group G be $\coprod_{i \in I} G^i$. Denote the identity $(e^i)_{i \in I}$ of G by e. For $g = (g^i)_{i \in I}$ in G let $\mathsf{U}_g = \prod_{i \in I} \overline{\mathsf{U}_{g^i}^i}$. (Notice that this is a finite product). Then $g \to \mathsf{U}_g$ is a unitary representation of G on H.

---

[17]) Suppose that every non-zero vector of H is a cyclic vector for $\mathscr{A}$. Then if E is a non-zero projection of $\mathscr{A}'$, $Ex = x \neq 0$ for some $x \in H$, and hence $E \geqq \mathrm{pr}\,[\mathscr{A}x] = 1$. Thus $\mathscr{A}' = \mathbf{C}$ and $\mathscr{A} = \mathscr{L}(H)$.

[18]) We write $\mathscr{R}_H(\mathscr{S})$ to denote the von Neumann algebra on H generated by the subset $\mathscr{S}$ of $\mathscr{L}(H)$.

LEMMA 4.1. $\mathsf{U}_g \mathcal{M} \mathsf{U}_g^* = \mathcal{M}$ for all $g \in G$.

PROOF: It is sufficient to show that $\mathsf{U}_g M \mathsf{U}_g^* \in \mathcal{M}$ for all $g \in G$ and all $M \in \mathcal{M}$. For that to hold, it is enough that $\mathsf{U}_g M \mathsf{U}_g^* \in \mathcal{M}$ for all $g \in F$ and all $M \in \mathcal{S}$, where F generates the group G and $\mathcal{S}$ generates the von Neumann algebra $\mathcal{M}$ with $\mathcal{S}^* = \mathcal{S}$.

Since $\mathcal{M} = \otimes_{i \in I} \mathcal{M}^i = \mathcal{R}(\overline{\mathcal{M}^i} : i \in I)$ take $\mathcal{S} = \{\overline{M^i} : M^i \in \mathcal{M}^i, i \in I\}$. Take $F = \{g = (g^i)_{i \in I}$ in $G : g^i = e^i$ for all except one of the $i \in I\}$. Notice that, if $g \in F$, then $\mathsf{U}_g = \overline{\mathsf{U}^i_{g^i}}$ for some $i \in I$ and some $g^i \in G^i$. Suppose then that $i, j \in I$, $g^i \in G^i$ and $M^j \in \mathcal{M}^j$: $\overline{\mathsf{U}^i_{g^i}}\, \overline{M^j}\, (\overline{\mathsf{U}^i_{g^i}})^* =$

$$\begin{cases} \overline{M^j} \text{ if } i \neq j \\ \overline{\mathsf{U}^i_{g^i} M^i (\mathsf{U}^i_{g^i})^*} \text{ if } i = j \end{cases}$$

and in either case is in $\mathcal{M}$. This completes the proof.

We would like to be able to show that if the systems $(\mathcal{M}^i, H^i, G^i, g \to \mathsf{U}^i_g)$ are free then so is $(\mathcal{M}, H, G, g \to \mathsf{U}_g)$. For the special cases (§ 5, below) we are interested in, however, a weaker result suffices.

DEFINITION 4.1. The system $(\mathcal{M}, H, G, g \to \mathsf{U}_g)$ has property (F) if, for every $g \in G - \{e\}$, there exists a family $(E_\alpha)_{\alpha \in A}$ of projections of $\mathcal{M}$ such that $\sum_{\alpha \in A} E_\alpha = 1$ and $E_\alpha(\mathsf{U}_g^* E_\alpha \mathsf{U}_g) = 0$ for all $\alpha \in A$.

LEMMA 4.2. If $(\mathcal{M}, H, G, g \to \mathsf{U}_g)$ has property (F), then it is free.

PROOF: Suppose that $(\mathcal{M}, H, G, g \to \mathsf{U}_g)$ has property (F). Let $g \in G - \{e\}$ and let $(E_\alpha)_{\alpha \in A}$ be a family of projections of $\mathcal{M}$ with $\sum_{\alpha \in A} E_\alpha = 1$ and $E_\alpha(\mathsf{U}_g^* E_\alpha \mathsf{U}_g) = 0$ for all $\alpha \in A$.

Suppose that $M \in \mathcal{M}$ and that $\mathsf{U}_g M \in \mathcal{M}$. Then, for each $\alpha \in A$, $E_\alpha(\mathsf{U}_g M) = (\mathsf{U}_g M)\, E_\alpha$ so that $(\mathsf{U}_g^* E_\alpha \mathsf{U}_g)M = M E_\alpha = E_\alpha M$. Hence $E_\alpha M = E_\alpha(E_\alpha M) = E_\alpha(\mathsf{U}_g^* E_\alpha \mathsf{U}_g)M = 0$. Therefore $M = \sum_{\alpha \in I} E_\alpha M = 0$. We have shown that $\mathsf{U}_g \mathcal{M} \cap \mathcal{M} = \{0\}$ for any $g \in G - \{e\}$; that is, that the system is free.

LEMMA 4.3. If each $(\mathcal{M}^i, H^i, G^i, g \to \mathsf{U}^i_g)$ has property (F), then so does $(\mathcal{M}, H, G, g \to \mathsf{U}_g)$.

PROOF: Suppose that $g = (g^i)_{i \in I}$ is in $G - \{e\}$. Then for some $i \in I$, $g^i \neq e^i$. Provided that $(\mathcal{M}^i, H^i, G^i, g \to \mathsf{U}^i_g)$ has property (F), there exists a family $(E_\alpha)_{\alpha \in A}$ of projections of $\mathcal{M}^i$ such that $\sum_{\alpha \in A} E_\alpha = 1$ and $E_\alpha(\mathsf{U}^i_{g^i})^* E_\alpha \mathsf{U}^i_{g^i} = 0$ for all $\alpha \in A$. Consider the family $(\overline{E_\alpha})_{\alpha\in A}$ of projections of $\mathscr{M}$. $\sum_{\alpha\in A}\overline{E_\alpha} = 1$, for the canonical map $T \to \overline{T}$ from $\mathscr{L}(H^i)$ to $\mathscr{L}(H)$ is bicontinuous in the ultrastrong topology. For any $\alpha \in A$, $\overline{E_\alpha}\, \mathsf{U}_g^* \,\overline{E_\alpha}\, \mathsf{U}_g = \overline{E_\alpha (\mathsf{U}^i_{g^i})^* E_\alpha \mathsf{U}^i_{g^i}} = 0$. Therefore $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ has property (F).

Proposition 4.1.
$\mathscr{A} = \otimes_{i\in I}^{(f^i\otimes \hat{e}^i)} \mathscr{A}^i$ is spatially isomorphic to $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$.

$$\mathscr{A}' = \otimes_{i\in I}^{(f^i\otimes \hat{e}^i)} (\mathscr{A}^i)'.$$

If each system $(\mathscr{M}^i, H^i, G^i, g \to \mathsf{U}^i_g)$ has property (F), then the system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is free. The system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is ergodic if each $(\mathscr{M}^i, H^i, G^i, g \to \mathsf{U}^i_g)$ is ergodic.

Proof: The third statement is a direct result of lemmas 4.2 and 4.3. The fourth statement follows from the first statement, from part 1 of proposition 1.1, and from the corollary to proposition 3.2. We now proceed to prove the first statement.

$$\mathscr{A} = \otimes_{i\in I}^{(f^i\otimes \hat{e}^i)} \mathscr{A}^i$$

is a von Neumann algebra on the Hilbert space

$$\underline{H} = \otimes_{i\in I}^{(f^i\otimes \hat{e}^i)} (H^i \otimes \hat{H}^i).$$

$\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ is a von Neumann algebra on the Hilbert space $\overline{H} = H \otimes \hat{H}$. Now $\hat{H}$ is defined as the Hilbert space with orthonormal basis $(\hat{g})_{g\in G}$. Since $G = \coprod_{i\in I} G^i$, $\hat{H}$ is isomorphic to $\otimes_{i\in I}^{(\hat{e}^i)} \hat{H}^i$ by the mapping γ, defined by $\gamma(\hat{g}) = \otimes_{i\in I} \hat{g}^i$ for all $g = (g^i)_{i\in I}$ in G. Therefore $\overline{H}$ is isomorphic to $(\otimes_{i\in I}^{(f^i)} H^i) \otimes (\otimes_{i\in I}^{(\hat{e}^i)} \hat{H}^i)$ by the mapping $1 \otimes \gamma$.
Now there is an associative isomorphism ([IDP], Theorem VI) from

$$(\otimes_{i\in I}^{(f^i)} H^i) \otimes (\otimes_{i\in I}^{(\hat{e}^i)} \hat{H}^i) \text{ to } \otimes_{i\in I}^{(f^i\otimes \hat{e}^i)} (H^i \otimes \hat{H}^i) = \underline{H}.$$

Denote this isomorphism by δ; then $\delta((\otimes_{i\in I} x^i) \otimes (\otimes_{i\in I} y^i)) = \otimes_{i\in I} (x^i \otimes y^i)$, for all $\otimes_{i\in I} x^i \in H$ with each $x^i \in H^i$ and all $\otimes_{i\in I} y^i \in \otimes_{i\in I}^{(\hat{e}^i)} \hat{H}^i$ with each $y^i \in \hat{H}^i$.

Let η be the isomorphism $1 \otimes \gamma$ followed by the isomorphism δ. η is then an isomorphism of $\overline{H}$ with $\underline{H}$. It is an easy calculation to show that:

1. If $T \in \mathscr{L}(H^i)$ then $\eta^{-1}(\overline{T \otimes 1})\eta = \overline{T} \otimes 1$.
2. If $g \in G^i$, then $\eta^{-1}(\mathsf{U}^i_g \otimes V^i_g)\eta = \mathsf{U}_{\Delta^i_g} \otimes V_{\Delta^i_g}$, where $\Delta^i_g = (\delta^h)_{h\in I}$ with $\delta^h = e^h$ if $h \in I - \{i\}$ and $\delta^i = g$.

Now $\mathscr{A} = \mathscr{R}_{\underline{H}}(\overline{\mathscr{A}^i} : i \in I)$. Each $\mathscr{A}^i = \mathscr{R}_{\overline{H^i}}(M \otimes 1, \mathsf{U}^i_g \otimes V^i_g : M \in \mathscr{M}^i, g \in G^i)$. Because the canonical map $\mathscr{A}^i \to \overline{\mathscr{A}^i}$ is bicontinuous in the ultrastrong topology, $\overline{\mathscr{A}^i} = \mathscr{R}_{\underline{H}}(\overline{M \otimes 1}, \overline{\mathsf{U}^i_g \otimes V^i_g} : M \in \mathscr{M}^i, g \in G^i)$. Therefore the isomorphism η^{-1} carries $\mathscr{A}$ into
$\mathscr{B} = \mathscr{R}_{\bar{H}}(\bar{M} \otimes 1, \mathsf{U}_{\Delta^i_g} \otimes V_{\Delta^i_g} : M \in \mathscr{M}^i, g \in G^i, i \in I)$. Again because the map $\mathscr{L}(H) \to \mathscr{L}(\bar{H})$ is bicontinuous in the ultrastrong topology, $\mathscr{R}_{\bar{H}}(\bar{M} \otimes 1 : M \in \mathscr{M}^i, i \in I) = \mathscr{R}_{H}(\bar{M} : M \in \mathscr{M}^i, i \in I) \otimes 1 = \mathscr{M} \otimes 1$. Finally, then, $\mathscr{B} = \mathscr{R}_{\bar{H}}(\mathscr{M} \otimes 1, \mathsf{U}_g \otimes V_g : g \in G) = \mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$. This completes the proof that $\mathscr{A}$ and $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ are spatially isomorphic.

For each, $i \in I$, let $W^i \in \mathscr{L}(\overline{H^i})$ be defined by $W^i(x \otimes \hat{g}) = (\mathsf{U}^i_g)^* x \otimes \hat{g}^{-1}$ for all $x \in H^i$ and all $g \in G^i$. According to proposition 1.1, W^i is a unitary involution and $W^i \mathscr{A}^i W^i = (\mathscr{A}^i)'$. Define $W \in \mathscr{L}(\bar{H})$ by $W(x \otimes \hat{g}) = \mathsf{U}^*_g x \otimes \hat{g}^{-1}$ for all $x \in H$ and all $g \in G$. Then W is a unitary involution and $W\mathscr{B}W = \mathscr{B}'$ where $\mathscr{B} = \mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$. Define $Y \in \mathscr{L}(\underline{H})$ by $Y = \eta W \eta^{-1}$. Then Y is a unitary involution and $Y\mathscr{A}Y = \mathscr{A}'$. It is an easy calculation to show that $Y\bar{T}Y = \overline{W^i T W^i}$ for $T \in \mathscr{L}(\overline{H^i})$. Hence $\mathscr{A}' = Y\mathscr{A}Y$

$$
\begin{aligned}
&= \mathscr{R}_{\underline{H}}(Y\overline{\mathscr{A}^i}Y : i \in I) \\
&= \mathscr{R}_{\underline{H}}(\overline{W^i \mathscr{A}^i W^i} : i \in I) \\
&= \mathscr{R}_{\underline{H}}(\overline{(\mathscr{A}^i)'} : i \in I) \\
&= \otimes_{i \in I}^{(f^i \otimes \hat{e}^i)} (\mathscr{A}^i)'.
\end{aligned}
$$

§ 5. Infinite Tensor Products of Factors of Type I_n on n^2-Dimensional Hilbert Spaces

Suppose that $\mathscr{A}$ is a factor of type I_n on an n^2-dimensional Hilbert space $\underline{H}$. Then $\underline{H} = H_0 \otimes H_1$ and $\mathscr{A} = \mathscr{L}(H_0) \otimes 1$ where H_0 and H_1 are n-dimensional Hilbert spaces.

Lemma 5.1. If f is a vector of $\underline{H}$, it is possible to choose orthonormal bases $(\varphi^i_\delta)_{i \in \mathbf{Z}_n}$ for H_δ, $\delta = 0$ and 1, in such a way that:

$$
f = \sum_{i \in \mathbf{Z}_n} a_i \varphi^i_0 \otimes \varphi^i_1
$$

and $a_0 \geqq a_1 \geqq \ldots \geqq a_{n-1} \geqq 0$.

PROOF: We omit the proof. It is based on the fact an $n \times n$ matrix A may be expressed as UDV where U and V are $n \times n$ unitary matrices and D is an $n \times n$ positive diagonal matrix. The proof is given in detail for the case $n = 2$ in [IDP], pgs. 69—70.

In the following lemma. we show explicitly how a factor of type I_n on an n^2-dimensional Hilbert space is a constructible algebra in the sense of § 1.

LEMMA 5.2. Suppose that H_0 and H_1 are n-dimensional Hilbert spaces, that $\mathscr{A} = \mathscr{L}(H_0) \otimes 1$, and that $f \in \underline{H} = H_0 \otimes H_1$ with $||f|| = 1$.

Let H be the Hilbert space with orthonormal basis $(\psi^i)_{i \in \mathbf{Z}_n}$. Let $\mathscr{M}$ be the abelian von Neumann algebra on H generated by the pr $[\psi^i]$ for $i \in \mathbf{Z}_n$. Let $G = \mathbf{Z}_n$ and define a unitary representation $g \to \mathsf{U}_g$ of G on H by $\mathsf{U}_g(\psi^i) = \psi^{i-g}$ for all $i \in \mathbf{Z}_n$.

Then $\mathscr{M}$ is maximal abelian and $\mathsf{U}_g \mathscr{M} \mathsf{U}_g^* = \mathscr{M}$ for all $g \in G$. The system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is free and ergodic. Let $\bar{H} = H \otimes \hat{H}$ where $\hat{H}$ is the Hilbert space with orthonormal basis $(\hat{g})_{g \in G}$. Then $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ is a factor on $\bar{H}$. There is an isomorphism γ from $\underline{H}$ to $\bar{H}$ which takes $\mathscr{A}$ onto $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ and takes f into $\varphi = (\sum_{i \in \mathbf{Z}_n} a_i \psi^i) \otimes 0$, where $a_0 \geqq a_1 \geqq \ldots \geqq a_{n-1} \geqq 0$ and $\sum_{i \in \mathbf{Z}_n} (a_i)^2 = 1$.

PROOF: It is clear that $\mathscr{M}$ is maximal abelian, that $\mathsf{U}_g \mathscr{M} \mathsf{U}_g^* = \mathscr{M}$ for all $g \in G$, and that the system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is free and ergodic.

By lemma 5.1 select bases $(\varphi_\delta^i)_{i \in \mathbf{Z}_n}$ for H_δ, $\delta = 0$ and 1, in such a way that $f = \sum_{i \in \mathbf{Z}_n} a_i \varphi_0^i \otimes \varphi_1^i$ and $a_0 \geqq a_1 \geqq \ldots \geqq a_{n-1} \geqq 0$. Since $||f|| = 1$, $\sum_{i \in \mathbf{Z}_n} (a_i)^2 = 1$.

Now define γ by $\gamma(\varphi_0^i \otimes \varphi_1^j) = \psi^i \otimes (\widehat{j-i})$, for all $i, j \in \mathbf{Z}_n$. Then γ is an isomorphism from $\underline{H}$ to $\bar{H}$.

$\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ is the von Neumann algebra on $\bar{H}$ generated by the pr $[\psi^i] \otimes 1$ for $i \in \mathbf{Z}_n$ and the $\mathsf{U}_g \otimes V_g$ for $g \in \mathbf{Z}_n$. Let us calculate the operators on $\underline{H}$ corresponding to these operators under the isomorphism γ^{-1}. First we deal with pr $[\psi^i] \otimes 1$. This is the projection onto the subspace $[\psi^i] \otimes \hat{H}$; $\gamma^{-1}[\psi^i \otimes \hat{H}] = \gamma^{-1}[\psi^i \otimes \hat{g} : g \in \mathbf{Z}_n] = [\varphi_0^i \otimes \varphi_1^{g+i} : g \in \mathbf{Z}_n] = [\varphi_0^i] \otimes H_1$. Therefore γ^{-1} (pr $[\psi^i] \otimes 1)\gamma = $ pr $[\varphi_0^i] \otimes 1$. Secondly consider $\gamma^{-1}(\mathsf{U}_g \otimes V_g)\gamma$. If $i, j \in \mathbf{Z}_n$, then $[\gamma^{-1}(\mathsf{U}_g \otimes V_g)\gamma](\varphi_0^i \otimes \varphi_1^j)$ $= \gamma^{-1}(\mathsf{U}_g \otimes V_g)(\psi^i \otimes (\widehat{j-i})) = \gamma^{-1}(\psi^{i-g} \otimes (\widehat{j-i+g})) = \varphi_0^{i-g} \otimes \varphi_1^j$. Hence $\gamma^{-1}(\mathsf{U}_g \otimes V_g)\gamma = Y_g \otimes 1$ where Y_g is a unitary operator on H_0 defined by $Y_g(\varphi_0^i) = \varphi_0^{i-g}$ for all $i, g \in \mathbf{Z}_n$.

We have shown that the isomorphism γ^{-1} from $\bar{H}$ to $\underline{H}$ takes $\mathscr{A}[\mathscr{M}, G, H, g \to \mathsf{U}_g]$ onto $\mathscr{R}_{\underline{H}}(\mathrm{pr}[\varphi_0^i] \otimes 1,\ Y_g \otimes 1 : i \in \mathbf{Z}_n, g \in \mathbf{Z}_n) = \mathscr{L}(H_0) \otimes 1 = \mathscr{A}$. Finally $\gamma(f) = \gamma(\sum_{i\in\mathbf{Z}_n} a_i \varphi_0^i \otimes \varphi_1^i) = (\sum_{i\in\mathbf{Z}_n} a_i \psi^i) \otimes 0$.

We are now going to consider the most general infinite tensor product of factors of type I_n on n^2-dimensional Hilbert spaces. Let J be an infinite indexing set. For $\alpha \in J$, let n_α be an integer $\geqq 2$, and let $H_{(\alpha,\delta)}$ for $\delta = 0$ and 1 be the n_α-dimensional Hilbert space with orthonormal basis $(\varphi^i_{(\alpha,\delta)})_{i\in\mathbf{Z}_{n_\alpha}}$. Let f_α be a vector of $\underline{H_\alpha} = H_{(\alpha,0)} \otimes H_{(\alpha,1)}$ with $\|f_\alpha\| = 1$; by lemma 5.1 we might as well assume that $f_\alpha = \sum_{i\in\mathbf{Z}_{n_\alpha}} a_i^\alpha \varphi^i_{(\alpha,0)} \otimes \varphi^i_{(\alpha,1)}$ with $a_0^\alpha \geqq a_1^\alpha \geqq \ldots \geqq a^\alpha_{n_\alpha-1} \geqq 0$ and $\sum_{i\in\mathbf{Z}_{n_\alpha}} (a_i^\alpha)^2 = 1$. Let the factor $\mathscr{A}_\alpha$ on $\underline{H_\alpha}$ be defined as $\mathscr{L}(H_{(\alpha,0)}) \otimes 1$. Let $\mathscr{A}$ be the tensor product of $(\mathscr{A}_\alpha)_{\alpha\in J}$ relative to $(f_\alpha)_{\alpha\in J}$.

For the remainder of the paper, $\mathscr{A}$ will be as it is defined above. $\mathscr{A}$ depends on the indexing set J, the family of integers $(n_\alpha)_{\alpha\in J}$, and the families of real numbers $(a_i^\alpha)_{i\in\mathbf{Z}_{n_\alpha}}$. We always assume that J is infinite, that each $n_\alpha \geqq 2$, and that $a_0^\alpha \geqq a_1^\alpha \geqq \ldots \geqq a^\alpha_{n_\alpha-1} \geqq 0$ and $\sum_{i\in\mathbf{Z}_{n_\alpha}} (a_i^\alpha)^2 = 1$ for all $\alpha \in J$.

Proposition 5.1. $\mathscr{A}$ is a factor. $\mathscr{A}$ is a constructible algebra; specifically, $\mathscr{A}$ is spatially isomorphic to $\mathscr{A}[\mathscr{M}, H, G, g \to \mathsf{U}_g]$ where: $H = \otimes^{(\varphi_\alpha)}_{\alpha\in J} H_\alpha$ where H_α is the Hilbert space with orthonormal basis $(\psi_\alpha^i)_{i\in\mathbf{Z}_{n_\alpha}}$ and $\varphi_\alpha = \sum_{i\in\mathbf{Z}_{n_\alpha}} a_i^\alpha \psi_\alpha^i$;
$\mathscr{M} = \mathscr{R}_H(\overline{E_\alpha^i} : i \in \mathbf{Z}_{n_\alpha}, \alpha \in J)$ where $E_\alpha^i = \mathrm{pr}\,[\psi_\alpha^i]$; $G = \coprod_{\alpha\in J} \mathbf{Z}_{n_\alpha}$; and, for $g = (g_\alpha)_{\alpha\in J}$ in G, $\mathsf{U}_g = \overline{\prod_{\alpha\in J} \mathsf{U}^\alpha_{g\alpha}}$, where $\mathsf{U}^\alpha_{g\alpha} \in \mathscr{L}(H_\alpha)$ is defined by $\mathsf{U}^\alpha_{g\alpha}(\psi_\alpha^i) = \psi_\alpha^{i-g\alpha}$.

Proof: $\mathscr{A}$ is a factor by the corollary to proposition 3.2. The rest results from lemma 5.2 and proposition 4.1. Proposition 4.1 also shows that the following is true:

Corollary

$$\mathscr{A}' = \otimes^{(f\alpha)}_{\alpha\in J} \mathscr{A}'_\alpha$$
$$= \otimes^{(f\alpha)}_{\alpha\in J} \big(1 \otimes \mathscr{L}(H_{(\alpha,1)})\big).$$

We now determine the decomposability type of $\mathscr{A}$.

Proposition 5.2. Let $K = \{\alpha \in J : a_i^\alpha = 0 \text{ for some } i \in \mathbf{Z}_{n_\alpha}\}$. Then the decomposability type of $\mathscr{A}$ is the larger of $\aleph_0$ and $\overline{\overline{K}}$.

Proof: We are going to find a family $(E_\lambda)_{\lambda\in L}$ of non-zero cyclic projections of $\mathscr{A}$ with $\sum_{\lambda\in L} E_\lambda = 1$. Then lemma 2.2 will show that the decomposability type of $\mathscr{A}$ is the larger of $\aleph_0$ and $\overline{\overline{L}}$.

Let us fix our attention on a particular $\alpha \in J$. We have $\mathscr{A}_\alpha = \mathscr{L}(H_{(\alpha,0)}) \otimes 1$ and $\mathscr{A}'_\alpha = 1 \otimes \mathscr{L}(H_{(\alpha,1)})$. Let $L_\alpha = \{0\} \cup \{i \in \mathbf{Z}_{n_\alpha} : a_i^\alpha = 0\}$. Notice that this is a disjoint union and that $L_\alpha = \{0\}$ if and only if $\alpha \in J - K$. For $i \in L_\alpha$, define θ_α^i to be f_α for $i = 0$ and $\varphi^i_{(\alpha,0)} \otimes \varphi^i_{(\alpha,1)}$ for $i \neq 0$. Let $F^i_\alpha = \mathrm{pr}\,[\mathscr{A}'_\alpha \theta^i_\alpha]$, then each F^i_α is a non-zero cyclic projection of $\mathscr{A}$, and $\sum_{i \in L_\alpha} F^i_\alpha = 1$.

Let $L = \prod_{\alpha \in J} L_\alpha$, and, for $\lambda = (i_\alpha)_{\alpha \in J}$ in L, let $\theta^\lambda = \otimes_{\alpha \in J} \theta^{i_\alpha}_\alpha$ and $E_\lambda = \prod_{\alpha \in J} F^{i_\alpha}_\alpha$. Then each $E_\lambda \in \mathscr{A}$. By lemma 3.4, each $E_\lambda = \mathrm{pr}\,[\mathscr{A}'\,\theta^\lambda]$, so that each E_λ is a non-zero cyclic projection of $\mathscr{A}$. As in lemma 3.3, $\sum_{\lambda \in L} E_\lambda = 1$.

We have proved that the decomposability type of $\mathscr{A}$ is the larger of $\aleph_0$ and $\overline{\overline{L}}$. Recall that $L = \prod_{\alpha \in J} L_\alpha$ where each L_α is finite and is $\{0\}$ if and only if $\alpha \in J - K$. Therefore J is finite if K is finite, and $\overline{\overline{J}} = \overline{\overline{K}}$ if K is infinite. That completes the proof.

We now give a necessary and sufficient condition for $\mathscr{A}$ to be of type I.

Proposition 5.3. $\mathscr{A}$ is of type I if and only if $\sum_{\alpha \in J}(1 - a_0^\alpha) < \infty$, and in that case is of type $\mathrm{I}_{\overline{\overline{J}}}$.

Proof: Notice that $|1 - (f_\alpha, \varphi^0_{(\alpha,0)} \otimes \varphi^0_{(\alpha,1)})| = 1 - a_0^\alpha$. This means that, if $\sum(1 - a_0^\alpha) < \infty$, then the C_0-sequences $(f_\alpha)_{\alpha \in J}$ and $(\varphi^0_{(\alpha,0)} \otimes \varphi^0_{(\alpha,1)})_{\alpha \in J}$ are equivalent, and hence define identical $\mathscr{A}$'s. To prove the sufficiency of the condition, then, we need to show only that $\mathscr{A}$ is of type I provided that each $a_0^\alpha = 1$.

Suppose then that $a_0^\alpha = 1$ for all $\alpha \in J$. By an associative transformation (theorem VI of [IDP]),

$$\mathscr{A} = \otimes_{\alpha \in J}^{(\varphi^0(\alpha,0) \otimes \varphi^0(\alpha,1))} (\mathscr{L}(H_{(\alpha,0)}) \otimes 1)$$

is spatially isomorphic to $(\otimes_{\alpha \in J}^{(\varphi^0(\alpha,0))} \mathscr{L}(H_{(\alpha,0)})) \otimes (\otimes_{\alpha \in J}^{(\varphi^0(\alpha,1))} 1)$ $= \mathscr{L}(H) \otimes 1$.

Therefore $\mathscr{A}$ is of type I. This completes the proof of the sufficiency.[19]) Since H has dimension $\overline{\overline{J}}$, $\mathscr{A}$ is of type $\mathrm{I}_{\overline{\overline{J}}}$.

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[19]) For more detail, compare [IDP], pp. 71—72.

Conversely, suppose that $\mathscr{A}$ is of type I. Then, according to propositions 5.1 and 1.1, $\mathscr{M}$ has a minimal projection E_0. For each $\alpha \in J$, $\sum_{i \in \mathbf{Z}_{n_\alpha}} E_0 \overline{E^i_\alpha} = E_0$. Therefore, since E_0 is minimal and $\mathscr{M}$ is abelian, for each $\alpha \in J$ there is an $i(\alpha) \in \mathbf{Z}_{n_\alpha}$ such that $\overline{E^{i(\alpha)}_\alpha} \geqq E_0$. Let $F = \prod_{\alpha \in J} \overline{E^{i(\alpha)}_\alpha}$; then F is a projection of $\mathscr{M}$ and $F \geqq E_0 > 0$.

By proposition 0.1, there exists an orthonormal basis for H consisting of vectors of the form $\otimes_{\alpha \in J} y_\alpha$, where each $y_\alpha \in H_\alpha$ with $||y_\alpha|| = 1$, and $y_\alpha = \varphi_\alpha$ for all but a finite number of the α in J. Since $F > 0$, there is some vector y of the above form, $y = \otimes_{\alpha \in J} y_\alpha$, with $||Fy|| > 0$. Let $J_0 = \{\alpha \in J : y_\alpha = \varphi_\alpha\}$; then $J - J_0$ is finite.

$$||Fy|| = \prod_{\alpha \in J} ||E^{i(\alpha)}_\alpha y_\alpha|| \leqq \prod_{\alpha \in J_0} a^\alpha_{i(\alpha)} \leqq \prod_{\alpha \in J_0} a^\alpha_0.$$

Therefore $\prod_{\alpha \in J_0} a^\alpha_0 > 0$, so that $\sum_{\alpha \in J_0} (1 - a^\alpha_0) < \infty$. Finally, because $J - J_0$ is finite, $\sum_{\alpha \in J} (1 - a^\alpha_0) < \infty$.

We now state a sufficient condition for $\mathscr{A}$ to be of type II_1. If the n_α are bounded, the below condition is also necessary for $\mathscr{A}$ to be of type II_1 (proposition 5.6, below).

Proposition 5.4

$\mathscr{A}$ is of type II_1 if

$$\sum_{\alpha \in J} [1 - n_\alpha^{-\frac{1}{2}} \sum_{i \in \mathbf{Z}_{n_\alpha}} a^\alpha_i] < \infty.$$

Proof: Suppose that the above series converges. For each $\alpha \in J$, let $\theta_\alpha = n_\alpha^{-\frac{1}{2}} \sum_{i \in \mathbf{Z}_{n_\alpha}} \psi^i_\alpha$; then $||\theta_\alpha|| = 1$ and, because $|1 - (\theta_\alpha, \varphi_\alpha)| = 1 - n_\alpha^{-\frac{1}{2}} \sum_{i \in \mathbf{Z}_{n_\alpha}} a^\alpha_i$, the C_0-sequences $(\theta_\alpha)_{\alpha \in J}$ and $(\varphi_\alpha)_{\alpha \in J}$ are equivalent. Therefore $\theta = \otimes_{\alpha \in J} \theta_\alpha$ is definable in $H = \otimes^{(\varphi_\alpha)}_{\alpha \in J} H_\alpha$.

$\mathscr{M}$ is an abelian von Neumann algebra on H; thus $\omega(M) = (M\theta, \theta)$ for $M \in \mathscr{M}^+$ defines a finite normal trace ω on $\mathscr{M}$. For all $g \in \mathbf{Z}_{n_\alpha}$ and all $\alpha \in J$, $\overline{\mathsf{U}^\alpha_g}\theta = \theta$. Therefore $\mathsf{U}_g\theta = \theta$ for all $g \in G$, and $\omega(\mathsf{U}^*_g M \mathsf{U}_g) = (M\mathsf{U}_g\theta, \mathsf{U}_g\theta) = (M\theta, \theta) = \omega(M)$ for all $g \in G$ and all $M \in \mathscr{M}$. By propositions 5.1 and 1.1, then, $\mathscr{A}$ is finite. The possibility $\mathbf{I}_n$ for an integer n is ruled out by proposition 5.3 (recall our assumption that J is infinite). Consequently $\mathscr{A}$ is of type II_1.

Suppose that J is a disjoint union of two infinite sets, J_1 and J_2. Then $\mathscr{A} = \mathscr{A}^1 \otimes \mathscr{A}^2$ where $\mathscr{A}^\delta = \otimes^{(f\alpha)}_{\alpha \in J\delta} \mathscr{A}_\alpha$ for $\delta = 1$

and 2. The tensor product of a factor of type I_a with a factor of type II_1 is a factor of type II_a.[20]) Hence $\mathscr{A}$ is of type $II_{\bar{J}_1}$ provided that

$$\sum_{\alpha\in J_1}(1-a_0^\alpha)<\infty$$

$$\text{and} \sum_{\alpha\in J_2}[1-n_\alpha^{-\frac{1}{2}}\sum_{i\in \mathbf{Z}_{n_\alpha}} a_i^\alpha]<\infty.$$

We are now going to show that $\mathscr{A}$ can be of type III, that, in fact, $\mathscr{A}$ is almost always of type III. In the course of the proof, we shall need to use a Radon-Nikodym theorem for the maximal abelian von Neumann algebra $\mathscr{M}$. We state the theorem which we need as lemma 5.3. It may be considered to be a special case of the Radon-Nikodym theorem for finite von Neumann algebras [2]. It may be proved also by a straightforward transfer of the classical Radon-Nikodym theorem for a localizable measure space.[21])

Lemma 5.3. Suppose that $\mathscr{M}$ is a maximal abelian von Neumann algebra, and that ω is a faithful semi-finite normal trace on $\mathscr{M}$.

If ν is a semi-finite normal trace on $\mathscr{M}$, then there exists a unique resolution of the identity in $\mathscr{M}$, $E(\lambda)$, such that: $\nu(T)=\int_0^\infty \lambda d\omega(E(\lambda)T)$ for all $T\in\mathscr{M}^+$. (By a resolution of the identity in $\mathscr{M}$, we mean a monotone function from $[0,\infty)$ to the projections of $\mathscr{M}$ with $\lim_{\lambda\to\infty}E(\lambda)=1$ and $\lim_{\lambda\to\lambda_0^+}E(\lambda)=E(\lambda_0)$ for all $\lambda_0\in[0,\infty)$.)

The vector $\varphi=\otimes_{\alpha\in J}\varphi_\alpha$ in H defines a finite normal trace ν on $\mathscr{M}$, $\nu(M)=(M\varphi,\varphi)$ for all $M\in\mathscr{M}^+$.

Lemma 5.4. Suppose that E is a projection of $\mathscr{M}$ with $E\leqq\overline{E_\alpha^i}$, where $i\in\mathbf{Z}_{n_\alpha}$ and $\alpha\in J$. Then, for $g\in\mathbf{Z}_{n_\alpha}$:

$$(a_i^\alpha)^2\,\nu((\overline{\mathsf{U}_g^\alpha})^*\,E(\overline{\mathsf{U}_g^\alpha}))=(a_{i+g}^\alpha)^2\,\nu(E).$$

Proof: Such an E can be expressed as $\overline{E_\alpha^i}F$, where $F\in\mathscr{R}_H(\overline{E_\beta^i}:i\in\mathbf{Z}_{n_\beta},\beta\in J-\{\alpha\})$. Then $\nu((\overline{\mathsf{U}_g^\alpha})^*E(\overline{\mathsf{U}_g^\alpha}))=\nu(\overline{E_\alpha^{i+g}}F)=\nu(\overline{E_\alpha^{i+g}})\,\nu(F)=(a_{i+g}^\alpha)^2\,\nu(F)$. Similarly $\nu(E)=\nu(\overline{E_\alpha^i})\,\nu(F)=(a_i^\alpha)^2\,\nu(F)$.

[20]) [RO IV], chapter II.

[21]) A maximal abelian von Neumann algebra is spatially isomorphic to $L^\infty(\mathscr{S})$ acting by multiplication on $L^2(\mathscr{S})$ for some localizable measure space $\mathscr{S}$. ([9], theorem 1).

LEMMA 5.5. Suppose that $p \to \alpha(p)$ is a one-to-one map from the positive integers into J. Suppose that, for each positive integer p, $i(p) \in \mathbf{Z}_{n_{\alpha(p)}}$. If $a_{i(p)}^{\alpha(p)}$ tends to a limit as $p \to \infty$, say $\lim_{p\to\infty} a_{i(p)}^{\alpha(p)} = a$; then

$$\lim_{p\to\infty} \nu(M\overline{E_{\alpha(p)}^{i(p)}}) = a^2\, \nu(M) \qquad \text{for all } M \in \mathscr{M}.$$

PROOF: The lemma clearly holds for M a finite product of projections $\overline{E_\alpha^i}$. Hence by linearity it holds for all M in $\mathscr{S}$, the *-algebra generated by the $\overline{E_\alpha^i}$. $\mathscr{S}$ is strongly dense in $\mathscr{M}$. Given $M \in \mathscr{M}$, then, and $\varepsilon > 0$, there exists $T \in \mathscr{S}$ such that $||T\varphi - M\varphi|| < \frac{1}{3}\varepsilon$. Then:

$$|a^2\,\nu(M) - \nu(M\overline{E_{\alpha(p)}^{i(p)}})| \leqq |a^2\,\nu(M) - a^2\,\nu(T)| + |a^2\,\nu(T) - \nu(T\overline{E_{\alpha(p)}^{i(p)}})| + |\nu(T\overline{E_{\alpha(p)}^{i(p)}}) - \nu(M\overline{E_{\alpha(p)}^{i(p)}})| = a^2|((M-T)\varphi, \varphi)| + |a^2\nu(T) - \nu(T\overline{E_{\alpha(p)}^{i(p)}})| + |((T-M)\varphi, \overline{E_{\alpha(p)}^{i(p)}}\varphi)| \leqq \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon$$

provided that p is large enough. That completes the proof.

PROPOSITION 5.5.

Suppose that there exists an infinite subset K of J, such that, for some $\varepsilon > 0$ and some p_α, $q_\alpha \in \mathbf{Z}_{n_\alpha}$:

$$a_{p_\alpha}^\alpha,\ a_{q_\alpha}^\alpha \geqq \varepsilon$$

and $a_{p_\alpha}^\alpha / a_{q_\alpha}^\alpha \geqq 1 + \varepsilon$ for all $\alpha \in K$. Then $\mathscr{A}$ is of type III.

PROOF: [22]) Suppose that $\mathscr{A}$ is not of type III. Then, by propositions 5.1 and 1.1, there exists on $\mathscr{M}$ a semi-finite normal trace ω which satisfies $\omega(\overline{\mathsf{U}_g^\alpha} M (\overline{\mathsf{U}_g^\alpha})^*) = \omega(M)$ for all $M \in \mathscr{M}^+$ and all $g \in \mathbf{Z}_{n_\alpha}$, $\alpha \in J$. This condition and the fact that the system $(\mathscr{M}, H, G, g \to \mathsf{U}_g)$ is ergodic imply that ω is faithful: for the largest projection of $\mathscr{M}$ on which ω is 0 has to be invariant under the $\overline{\mathsf{U}_g^\alpha}$, and thus equal to 0 or 1.

By lemma 5.3, there exists a resolution of the identity in $\mathscr{M}$, $E(\lambda)$, such that

$$\nu(T) = \int_0^\infty \lambda\, d\omega(E(\lambda)T)$$

for all $T \in \mathscr{M}^+$.

[22]) c.f. [8], pp. 140—141.

Fix, for the moment, α in J and i, j in $\mathbf{Z}_{n_\alpha}$. Assume that a_i^α, $a_j^\alpha \neq 0$ and let $r = (a_i^\alpha/a_j^\alpha)^2$. Write U for $\overline{\mathsf{U}_{j-i}^\alpha}$. Suppose that E is a projection of $\mathscr{M}$ satisfying $E \leqq \overline{E_\alpha^i}$. By lemma 5.4,

$$\nu(E) = r\nu(\mathsf{U}^*E\mathsf{U}) = r\int_0^\infty \lambda d\omega(E(\lambda)\mathsf{U}^*E\mathsf{U}) = r\int_0^\infty \lambda d\omega(\mathsf{U}E(\lambda)\mathsf{U}^*E)$$
$$= \int_0^\infty \lambda\, d\omega(\mathsf{U}E(\lambda/r)\mathsf{U}^*E).$$

On the other hand, $\nu(E) = \int_0^\infty \lambda d(E(\lambda)E)$. Therefore, by the uniqueness part of lemma 5.3 applied to $\mathscr{M}\big|_{\overline{E_\alpha^i}}$,
$\overline{E_\alpha^i}E(\lambda) = \overline{E_\alpha^i}\,\mathsf{U}\,E(\lambda/r)\mathsf{U}^*$ for all $\lambda \in [0, \infty)$. Then $\nu(\overline{E_\alpha^i}E(\lambda)) =$

$$r\,\nu(\mathsf{U}^*\overline{E_\alpha^i}\,\mathsf{U}\,E(\lambda/r)\mathsf{U}^*\mathsf{U}) \text{ by lemma 5.4, or}$$
$$\nu(\overline{E_\alpha^i}E(\lambda)) = r\,\nu(\overline{E_\alpha^j}\,E(\lambda/r)).$$

Assume now the hypothesis of the theorem. Then there exist maps from the positive integers, $p \to \alpha(p)$, $p \to i(p)$, and $p \to j(p)$, such that:

1. $p \to \alpha(p)$ is a one-to-one map from $\mathbf{N}$ into J.
2. For each $p \in \mathbf{N}$, $i(p)$, $j(p) \in \mathbf{Z}_{n_{\alpha(p)}}$.
3. $\lim_{p\to\infty} a_{i(p)}^{\alpha(p)} = a > 0$,
   $\lim_{p\to\infty} a_{j(p)}^{\alpha(p)} = b > 0$,

and

$$a_{i(p)}^{\alpha(p)}/a_{j(p)}^{\alpha(p)} > 1 + \varepsilon \qquad \text{for all } p \in \mathbf{N}.$$

Let $r_p = [a_{i(p)}^{\alpha(p)}/a_{j(p)}^{\alpha(p)}]^2$, and let $E_p = \overline{E_{\alpha(p)}^{i(p)}}$ and $F_p = \overline{E_{\alpha(p)}^{j(p)}}$. By the preceding paragraph, $\nu(E_pE(\lambda)) = r_p\,\nu(F_pE(\lambda/r_p))$. Since each $r_p > 1 + \varepsilon$, $\nu(E_pE(\lambda)) \leqq r_p\nu(F_pE(\lambda/1 + \varepsilon))$. Taking the limit as $p \to \infty$, and using lemma 5.5, we get:
$a^2\nu(E(\lambda)) \leqq (\lim_{p\to\infty} r_p)b^2\nu(E(\lambda/1 + \varepsilon)) = a^2\nu(E(\lambda/1 + \varepsilon))$. That is, $\nu(E(\lambda)) \leqq \nu(E(\lambda/1 + \varepsilon))$ for all $\lambda \in [0, \infty)$. We conclude that $\nu(E(0)) = 1$; this contradiction completes the proof that $\mathscr{A}$ is of type III.

We now examine the situation when $\mathscr{A}$ is of type II_1. Our final result is proposition 5.6 (below), which states that a certain series converges if $\mathscr{A}$ is of type II_1. The proof is based on the Kolmogoroff criterion for the convergence almost everywhere of a sequence of independent functions. Since we choose to carry out our reasoning on the maximal abelian von Neumann algebra $\mathscr{M}$, rather than on an appropriate probability space, we must translate the classical Kolmogoroff theorems into the language of von Neumann algebras. We state the result that we need as lemma 5.6 (below).

We need a few preliminary definitions. Suppose that $\mathscr{M}$ is an abelian von Neumann algebra on H, and that ω is a normal trace on $\mathscr{M}$ satisfying $\omega(1) = 1$. ω extends to a weakly continuous functional on $\mathscr{M}$, which we denote by ω also.

DEFINITION 5.1. A family $(E_i)_{i\in I}$ of projections of $\mathscr{M}$ is p-independent (with respect to ω) if, for every finite subset I' of I, $\omega(\prod_{i\in I'} E_i) = \prod_{i\in I'} \omega(E_i)$. A family $(T_i)_{i\in I}$ of hermitian operators of $\mathscr{M}$ is p-independent if, for every function $f : I \to \mathbf{R}$, $(E_i(f(i)))_{i\in I}$ is a p-independent family of projections. Here we have denoted by $E_i(\lambda)$ the spectral resolution of T_i.

DEFINITION 5.2. If T is a hermitian operator of $\mathscr{M}$, the variance of T (with respect to ω) is $\sigma^2(T) = \omega((T - \omega(T))^2)$.

DEFINITION 5.3.[23]) A linear subset L of H is essentially dense (with respect to $\mathscr{M}$ and ω), if there is an increasing sequence of closed subspaces $(M_n)_{n=1,2,\ldots}$ such that $\mathrm{pr}\,[M_n] \in \mathscr{A}$, $\lim_{n\to\infty} \omega(\mathrm{pr}\,[M_n]) = 1$, and each $M_n \subset L$.

LEMMA 5.6. Suppose that $(T_n)_{n=1,2,\ldots}$ is a p-independent sequence of hermitian operators of M, and that $\|T_n\| \leqq K < \infty$ for $n = 1, 2, \ldots$. Then $\sum_{n=1}^{\infty} T_n$ converges weakly on an essentially dense (with respect to $\mathscr{M}$ and ω) subset of H if and only if:

1. $\sum_{n=1}^{\infty} \omega(T_n)$ converges,

and 2. $\sum_{n=1}^{\infty} \sigma^2(T_n)$ converges.

PROPOSITION 5.6.

Let $\lambda_i^\alpha = (a_i^\alpha)^2 - 1/n_\alpha$. If $\mathscr{A}$ is of type II_1, then $\sum_{\alpha\in J} [\sum_{i\in \mathbf{Z}_{n_\alpha}} (\lambda_i^\alpha)^2] < \infty$.

PROOF: Assume that $\mathscr{A}$ is of type II_1. Then, by propositions 5.1 and 1.1, there exists on $\mathscr{M}$ a finite normal trace ω which satisfies $\omega(\mathsf{U}_g^* M \mathsf{U}_g) = \omega(M)$ for all $g \in G$ and all $M \in \mathscr{M}$. We may assume that $\omega(1) = 1$. As in the proof of proposition 5.5, ω must be faithful. It is clear that

$$\omega\Big(\prod_{\alpha\in F} \overline{E_\alpha^{i(\alpha)}}\Big) = \prod_{\alpha\in F} \frac{1}{n_\alpha}$$

for any finite subset F of J and for each $i(\alpha) \in \mathbf{Z}_{n_\alpha}$. Therefore,

[23]) If ω is faithful, this definition agrees with definition 16.2.1 of [RO].

if $(i(\alpha))_{\alpha\in J}$ is in $\prod_{\alpha\in J} \mathbf{Z}_{n_\alpha}$, then $(\overline{E^{i(\alpha)}_\alpha})_{\alpha\in J}$ is a p-independent (with respect to ω) family of projections of $\mathscr{M}$.

There is another finite normal trace on $\mathscr{M}$, namely the trace ν defined by $\nu(M) = (M\varphi, \varphi)$ for all $M \in \mathscr{M}$. Notice that $\nu(1) = 1$. A short calculation shows that, for $(i(\alpha))_{\alpha\in J}$ in $\prod_{\alpha\in J} \mathbf{Z}_{n_\alpha}$, $(\overline{E^{i(\alpha)}_\alpha})_{\alpha\in J}$ is a p-independent (with respect to ν) family of projections of $\mathscr{M}$.

Let K be a countable subset of J. Let X be the real Hilbert space of sequences of real numbers $(x^\alpha)^{\alpha\in K}_{i\in\mathbf{Z}_{n_\alpha}}$ satisfying $\sum_{i\in\mathbf{Z}_{n_\alpha}} x^\alpha_i = 0$ for all $\alpha \in K$ and $\sum_{\alpha\in K}\sum_{i\in\mathbf{Z}_{n_\alpha}} (x^\alpha_i)^2 < \infty$: the inner product in X is to be defined by

$$((x^\alpha_i), (y^\alpha_i)) = \sum_{\alpha\in K}\sum_{i\in\mathbf{Z}_{n_\alpha}} (x^\alpha_i y^\alpha_i).$$

For each $(x^\alpha_i) \in X$ define a p-independent (with respect to both ω and ν) sequence of hermitian operators of $\mathscr{M}$, $(T_\alpha)_{\alpha\in K}$, by

$$T_\alpha = \sum_{i\in\mathbf{Z}_{n_\alpha}} x^\alpha_i \overline{E^i_\alpha}.$$

Then, for each $\alpha \in K$, $||T_\alpha|| \leqq \sup |x^\alpha_i| \leqq ||(x^\alpha_i)||$.
Also $\omega(T_\alpha) = 1/n_\alpha \sum_{i\in\mathbf{Z}_{n_\alpha}} x^\alpha_i = 0$, and $\sigma^2(T_\alpha) =$
$1/n_\alpha \sum_{i\in\mathbf{Z}_{n_\alpha}} (x^\alpha_i)^2 \leqq \sum_{i\in\mathbf{Z}_{n_\alpha}} (x^\alpha_i)^2$. Therefore $\sum_{\alpha\in K} \sigma^2(T_\alpha) < \infty$.

By lemma 5.6, then, $\sum_{\alpha\in K} T_\alpha$ converges weakly on an essentially dense (with respect to $\mathscr{M}$ and ω) linear subset of H. But ω is faithful; therefore $\sum_{\alpha\in K} T_\alpha$ converges weakly on an essentially dense (with respect to $\mathscr{M}$ and ν) linear subset of H. Hence, by lemma 5.6 in the other direction, $\sum_{\alpha\in K} \nu(T_\alpha)$ converges. That is, $\sum_{\alpha\in K}\sum_{i\in\mathbf{Z}_{n_\alpha}} (a^\alpha_i)^2 x^\alpha_i$ converges. Substituting $\lambda^\alpha_i = (a^\alpha_i)^2 - 1/n_\alpha$, and using the fact that $\sum_{i\in\mathbf{Z}_{n_\alpha}} x^\alpha_i = 0$, we find that $\sum_{\alpha\in K}\sum_{i\in\mathbf{Z}_{n_\alpha}} \lambda^\alpha_i x^\alpha_i$ converges.

We have defined a linear functional ϕ on X: $\phi((x^\alpha_i)) = \sum_{\alpha\in K}\sum_{i\in\mathbf{Z}_{n_\alpha}} \lambda^\alpha_i x^\alpha_i$. It is clear that ϕ is closed; therefore, by the closed-graph theorem, ϕ is bounded. Therefore, for all $x \in X$, $\phi(x) = (x, y)$ for some $y = (y^\alpha_i)$ in X. Then each y^α_i must be λ^α_i. Hence

$$\sum_{\alpha\in K}\sum_{i\in\mathbf{Z}_{n_\alpha}} (\lambda^\alpha_i)^2 = ||y|| < \infty.$$

Since K can be any countable subset of J,

$$\sum_{\alpha\in J}\sum_{i\in\mathbf{Z}_{n_\alpha}} (\lambda^\alpha_i)^2 < \infty.$$

COROLLARY. If the n_α are bounded, $\mathscr{A}$ is of type II_1 if and only if $\sum_{\alpha \in J} \sum_{i \in Z_{n_\alpha}} (\lambda_i^\alpha)^2 < \infty$.

PROOF: The corollary will follow from propositions 5.6 and 5.4, provided that we can show that $\sum_{\alpha \in J} \sum_{i \in Z_{n_\alpha}} (\lambda_i^\alpha)^2 < \infty$ implies that $\sum_{\alpha \in J} [1 - n_\alpha^{-\frac{1}{2}} \sum_{i \in Z_{n_\alpha}} a_i^\alpha] < \infty$ when the n_α are bounded.

Suppose that each $n_\alpha \leqq N$. Choose $\delta > 0$ such that $|(1+x)^{\frac{1}{2}} - (1 + \frac{1}{2}x - \frac{1}{8}x^2)| < x^2$ for $|x| < \delta$. Write x_i^α for $\lambda_i^\alpha n_\alpha$. Then, provided that $|\lambda_i^\alpha| < \delta/N$, $|1 - n_\alpha^{-\frac{1}{2}} \sum_{i \in Z_{n_\alpha}} a_i^\alpha| =$

$$\begin{aligned} | \textstyle\sum_{i \in Z_{n_\alpha}} \frac{1}{n_\alpha} [1 - (1 + x_i^\alpha)^{\frac{1}{2}}] \,| &\leqq | \textstyle\sum_{i \in Z_{n_\alpha}} \frac{1}{2} x_i^\alpha - \frac{1}{8} (x_i^\alpha)^2 | \\ &\quad + \textstyle\sum_{i \in Z_{n_\alpha}} (x_i^\alpha)^2 \\ &\leqq (\tfrac{1}{8} + 1) \textstyle\sum_{i \in Z_{n_\alpha}} (x_i^\alpha)^2 \\ &\leqq \tfrac{9N^a}{8} \textstyle\sum_{i \in Z_{n_\alpha}} (\lambda_i^\alpha)^2. \end{aligned}$$

That completes the proof.

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