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On Special Pairs of Monotonic Series

by

A. Ziv

1. All numbers considered here are real and all series and sequences consist of positive (non-zero) terms.

By a Monotonic Couple, or M. C., we mean a pair of monotonic not increasing and divergent series $\sum a_n$; $\sum b_n$ for which

$$\sum_{n=1}^{\infty} \min\,\{a_n, b_n\} < \infty$$

An example of such M. C. was first constructed by J. P. C. Miller and is the following:

$$\begin{array}{lllll}
\frac{1}{2^2}+\cdots+\frac{1}{2^2} & +\frac{1}{2^5}+\cdots+\frac{1}{2^5} & +\frac{1}{2^5}+\cdots+\frac{1}{2^5} & +\frac{1}{2^{12}}+\cdots+\frac{1}{2^{12}} & +\cdots \\
\underbrace{\frac{1}{2^3}+\cdots+\frac{1}{2^3}}_{2^2 \text{ terms}} & \underbrace{+\frac{1}{2^3}+\cdots+\frac{1}{2^3}}_{2^3 \text{ terms}} & \underbrace{+\frac{1}{2^8}+\cdots+\frac{1}{2^8}}_{2^5 \text{ terms}} & \underbrace{+\frac{1}{2^8}+\cdots+\frac{1}{2^8}}_{2^8 \text{ terms}} & \underbrace{+\cdots}_{2^{12} \text{ terms}}
\end{array}$$

Another simple M. C. is:

$$\begin{array}{llllll}
\frac{1}{1^2}+ & \frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2} & +\frac{1}{6^2}+\frac{1}{7^2}+\cdots+\frac{1}{41^2} & +\frac{1}{42^2}+\frac{1}{42^2}+\cdots+\frac{1}{42^2} & +\cdots \\
\underbrace{\frac{1}{1^2}+} & \underbrace{\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}}_{2^2 \text{ terms}} & \underbrace{+\frac{1}{6^2}+\frac{1}{6^2}+\cdots+\frac{1}{6^2}}_{6^2 \text{ terms}} & \underbrace{+\frac{1}{42^2}+\frac{1}{43^2}+\cdots+\frac{1}{1805^2}}_{42^2 \text{ terms}} & \underbrace{+\cdots}
\end{array}$$

It is easy to see that if $\sum a_n$; $\sum b_n$ is a M. C. then $d_n = \max\{a_n, b_n\}$ $(n = 1, 2, \ldots)$ is monotonic not increasing and $\sum_{n=1}^{\infty} d_n = \infty$. But there exist series $\sum d_n$ with those properties for which there is no M. C. $\sum a_n$; $\sum b_n$ such that $d_n = \max\{a_n, b_n\}$. In fact we shall prove (theorem 1) that a necessary (and sufficient) condition for the existence of such M. C. is $\underline{\lim}_{n\to\infty} nd_n = 0$. The same condition will be shown (theorem 3) to be necessary and sufficient for $\sum a_n$ to be a series of a M. C.

THEOREM 1: Given a monotonic and divergent series $\sum d_n$ a necessary and sufficient condition for the existence of a M. C. $\sum a_n$; $\sum b_n$ such that $d_n = \max\{a_n, b_n\}$ $(n = 1, 2, \ldots)$ is $\underline{\lim}_{n\to\infty} nd_n = 0$.

2. We prove first:

LEMMA: If $\sum a_n$; $\sum b_n$ is a M. C. then each of the two index sets

$$(1) \qquad A = \{n | a_n < b_n\}, \quad B = \{n | b_n < a_n\}$$

is infinite:

PROOF: Suppose e.g. that B is finite, then there exists n_0 for which $n > n_0$ implies $n \notin B$ or in other words $\min\{a_n, b_n\} = a_n$. Denoting

$$\min\{a_n, b_n\} = c_n \quad (n = 1, 2, \ldots)$$

we get

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{n_0} c_n + \sum_{n=n_0+1}^{\infty} a_n.$$

By the definition of M. C. $\sum_{n=1}^{\infty} a_n = \infty$ and therefore $\sum_{n=1}^{\infty} c_n = \infty$. This contradicts the definition of M. C. thus proving our Lemma.

3. PROOF OF THEOREM 1: We begin with the proof of the necessity.

Suppose $\sum a_n$; $\sum b_n$ is M. C.; we denote

$$d_n = \max\{a_n, b_n\}, \quad c_n = \min\{a_n, b_n\}, \quad (n = 1, 2, \ldots).$$

Considering the two sets (1), we define by induction two monotonically increasing sequences of integers $\{m_i\}$ and $\{n_i\}$: m_1 will be any fixed element of A and then n_i is the least element of B, greater than m_i and m_{i+1} is any element of A greater than n_i.

Consider now the sequence $\{n_i\}$. Our lemma assures us that $\{n_i\}$ is infinite; by definition $n_i \in B$, $n_i - 1 \notin B$ $(i = 1, 2, \ldots)$ so that

$$(2) \qquad a_{n_i} = d_{n_i}, \quad a_{n_i-1} = c_{n_i-1} \quad (i = 1, 2, \ldots).$$

Since $\sum_{n=1}^{\infty} c_n < \infty$ and — as can be seen easily — $\{c_n\}$ is monotonic we have by a well known theorem

$$(3) \qquad nc_n \to 0 \quad (n \to \infty).$$

Recalling (2) we get

$$n_i d_{n_i} = n_i a_{n_i} \leqq n_i a_{n_i-1} = n_i c_{n_i-1} = (n_i - 1)c_{n_i-1} + c_{n_i-1}$$

now using (3)

$$n_i d_{n_i} \to 0 \quad (i \to \infty)$$

and the necessity is proved.

4. Let now $\sum d_n$ be a monotonic and divergent series satisfying $\underline{\lim}_{n\to\infty} nd_n = 0$.
$\{d_n\}$ has a subsequence $\{d_{n_j}\}$ for which

$$\lim_{j\to\infty} n_j d_{n_j} = 0.$$

We may choose the integers n_j so that

$$n_j d_{n_j} < \tfrac{1}{2^j} \quad (j = 1, 2, \ldots) \tag{4}$$

and

$$\sum_{n=n_j}^{n_{j+1}-1} d_n > 1. \tag{5}$$

Let now $\{a_n\}$ and $\{b_n\}$ be defined as follows:

$$a_n = \begin{cases} d_n \\ d_n \\ d_{n_{2k+1}} \end{cases}, \quad b_n = \begin{cases} d_n & \text{for} \quad n < n_1 \\ d_{n_{2k}} & \text{for} \quad n_{2k-1} \leqq n < n_{2k} \\ d_n & \text{for} \quad n_{2k} \quad \leqq n < n_{2k+1} \end{cases}$$

Since $\{d_n\}$ is non increasing, it follows that both $\{a_n\}$ and $\{b_n\}$ are non increasing and also that max $\{a_n, b_n\} = d_n$. Thus it remains to be proved that $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} \min\{a_n, b_n\} < \infty$.
From (5) follows

$$\sum_{n=1}^{\infty} a_n \geqq \sum_{k=1}^{\infty}\left(\sum_{n=n_{2k-1}}^{n_{2k}-1} a_n\right) = \sum_{k=1}^{\infty}\left(\sum_{n=n_{2k-1}}^{n_{2k}-1} d_n\right) = \infty$$

$$\sum_{n=1}^{\infty} b_n \geqq \sum_{k=1}^{\infty}\left(\sum_{n=n_{2k}}^{n_{2k+1}-1} b_n\right) = \sum_{k=1}^{\infty}\left(\sum_{n=n_{2k}}^{n_{2k+1}-1} d_n\right) = \infty.$$

Further, using the definition of $\{a_n\}$ and $\{b_n\}$ we get

$$\min\{a_n, b_n\} = \begin{cases} d_n & \text{for} \quad n < n_1 \\ d_{n_{j+1}} & \text{for} \quad n_j \leqq n < n_{j+1} \end{cases}$$

hence by (4)

$$\sum_{n=1}^{\infty} \min\{a_n, b_n\} = \sum_{n=1}^{n_1-1} d_n + \sum_{j=1}^{\infty} (n_{j+1}-n_j) d_{n_{j+1}} \leqq \sum_{n=1}^{n_1-1} d_n + \sum_{j=1}^{\infty} n_{j+1} d_{n_{j+1}} <$$

$$\sum_{n=1}^{n_1-1} d_n + \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} < \infty$$

and the proof is completed.

5. For a further discussion of M. C. we need the following theorem:

THEOREM 2: Let $f(x)$ be a positive, monotonic and continuous function, defined for $x \geqq 1$ and such that

$$\varliminf_{x\to\infty} xf(x) = 0.$$

Given any positive number ρ, define $y = y(x)$ by $(y-x)f(x) = \rho$. Then there exists an infinite sequence $\{x_n\}$ for which both

$$\lim_{n\to\infty} x_n f(x_n) = 0 \tag{6}$$

and

$$\lim_{n\to\infty}\int_{x_n}^{y_n} f(t)dt = 0 \quad y_n \text{ being } y(x_n). \tag{7}$$

6. PROOF: Denoting $\tau = (t-x)f(x)$ we get

$$\int_x^{y(x)} f(t)dt = \int_0^\rho \frac{1}{f(x)} f\left(\frac{\tau}{f(x)} + x\right) d\tau. \tag{8}$$

We prove first that if $\{x_n\}$ satisfies

$$\lim_{n\to\infty} \frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right) = 0 \quad \text{for each } \tau > 0 \tag{9}$$

it also satisfies

$$\lim_{n\to\infty}\int_0^\rho \frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right) d\tau = 0 \tag{10}$$

which by (8) is equivalent to (7).

7. To prove this, we shall use the monotony of the integrand of (10) with respect to the variable τ from which it follows, that for any n and $\rho > \eta > 0$

$$\int_0^\eta \frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right) d\tau \leqq \int_0^\eta \frac{1}{f(x_n)} f\left(\frac{0}{f(x_n)} + x_n\right) d\tau = \eta$$

and also

$$\int_\eta^\rho \frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right)d\tau$$
$$\leqq \int_\eta^\rho \frac{1}{f(x_n)} f\left(\frac{\eta}{f(x_n)} + x_n\right)d\tau \leqq \frac{\rho}{f(x_n)} f\left(\frac{\eta}{f(x_n)} + x_n\right).$$

Given now any $\varepsilon > 0$ we substitute $\eta = \varepsilon/2$ in the preceding inequalities and add them to get

$$\int_0^\rho \frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right) d\tau \leqq \frac{\varepsilon}{2} + \frac{\rho}{f(x_n)} f\left(\frac{\frac{\varepsilon}{2}}{f(x_n)} + x_n\right).$$

Assuming that (9) holds we see that there exists such n_0 that $n > n_0$ implies

$$\int_0^\rho \frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right) d\tau < \varepsilon$$

proving that (7) follows from (9).

8. We are going to show the existence of $\{x_n\}$ which satisfies (6) and (9).

First let us verify — for a given $\tau > 0$ — the existence of a sequence $\{z_m(\tau)\}$ satisfying

$$\lim_{m\to\infty} z_m f(z_m) = 0 \tag{11}$$

and

$$\lim_{m\to\infty} \frac{1}{f(z_m)} f\left(\frac{\tau}{f(z_m)} + z_m\right) = 0. \tag{12}$$

By multiplying numerator and denominator by the same factor we get from (12)

$$\lim_{m\to\infty} \frac{\left(\frac{\tau}{f(z_m)} + z_m\right) f\left(\frac{\tau}{f(z_m)} + z_m\right)}{\tau + z_m f(z_m)} = 0$$

implying for a fixed $\tau > 0$ that (11) together with (12) is equivalent to (11) together with

$$\lim_{m\to\infty} \left(\frac{\tau}{f(z_m)} + z_m\right) f\left(\frac{\tau}{f(z_m)} + z_m\right) = 0. \tag{13}$$

We have then to find $\{z_m\}$ which satisfies (11) and (13).

9. Since $\underline{\lim}_{x\to\infty} x f(x) = 0$ there is a sequence $\{s_k\}$ ($s_k \geqq 1$; $s_k \to \infty$) for which

$$\lim_{k\to\infty} s_k f(s_k) = 0. \tag{14}$$

For an $\alpha \geqq 1$ we get $\alpha s_k f(\alpha s_k) \leqq \alpha s_k f(s_k)$ and therefore

$$\lim_{k\to\infty} (\alpha s_k) f(\alpha s_k) = 0 \quad \text{for each} \quad \alpha \geqq 1. \tag{15}$$

Since $s_k \to \infty$ and $f(x)$ is continuous there exists for each $\alpha \geqq 1$ and $k > k_0(\alpha, \tau)$ (k_0 appropriate constant) a number $z_k^{(\alpha)}(\tau)$ for which

$$\frac{\tau}{f(z_k^{(\alpha)})} + z_k^{(\alpha)} = \alpha s_k. \tag{16}$$

Using (15) it is clear that for any $\alpha \geqq 1$ $\{z_k^{(\alpha)}\}$ satisfies (13) (substituting z_m for $z_k^{(\alpha)}$ and m for k).

Define now $N_\alpha = N_\alpha(\tau)$ for each $\alpha \geqq 1$ as some limit point of the sequence

$$\left\{\frac{\alpha s_k}{z_k^{(\alpha)}}\right\} \quad (N_\alpha = \infty \text{ is not excluded})$$

by (16)

$$\frac{\alpha s_k}{z_k^{(\alpha)}} = \frac{\tau}{z_k^{(\alpha)} f(z_k^{(\alpha)})} + 1. \tag{17}$$

Now, if for some $\alpha, N_\alpha = \infty$, $\{z_k^{(\alpha)}\}$ contains a subsequence $\{z_m\}$ for which

$$\lim_{m\to\infty} \left(\frac{\tau}{z_m f(z_m)} + 1\right) = \infty.$$

This subsequence obviously satisfies (11) and (13). It remains to consider the case that N_α is always finite. Suppose that for some α, $N_\alpha < \alpha$. In this case denote by $\{z_m\}$ some subsequence of $\{z_k^{(\alpha)}\}$ for which

$$\lim_{m\to\infty} \frac{\alpha s'_m}{z_m} = N_\alpha < \alpha \tag{18}$$

s'_m being those terms of $\{s_k\}$ which correspond to z_m by (16). From (18) we get for every $m > m_0(\alpha, \tau)$ (m_0 some appropriate constant)

$$s'_m < z_m \quad \text{and} \quad \frac{\alpha s'_m}{z_m} > \tfrac{1}{2} N_\alpha$$

consequently

$$f(z_m) \leqq f(s'_m) \quad \text{and} \quad z_m < \frac{2\alpha}{N_\alpha} s'_m$$

and therefore for $m > m_0$

$$z_m f(z_m) < \frac{2\alpha}{N_\alpha} s'_m f(s'_m).$$

Regarding the inclusion $\{s'_m\} \subset \{s_k\}$ we get by (14) that $\{z_m\}$ satisfies (11). This implies by (18)

$$N_\alpha = \lim_{m\to\infty} \frac{\alpha s'_m}{z_m} = \lim_{m\to\infty} \left(\frac{\tau}{z_m f(z_m)} + 1\right) = \infty$$

so that the case $N_\alpha < \alpha$ cannot actually exist.

10. There remains the case

$$N_\alpha \geqq \alpha \quad \text{for any} \quad \alpha \geqq 1. \tag{19}$$

Recalling that $\{z_k^{(\alpha)}\}$ satisfies (13) for any $\alpha \geqq 1$ (after replacing z_m by $z_k^{(\alpha)}$ and m by k) and using the definition of N_α and (17), there exists a term of $\{z_k^{(\alpha)}\}$ denoted $z_\alpha(\tau)$, for which both

$$\left(\frac{\tau}{f(z_\alpha)} + z_\alpha\right) f\left(\frac{\tau}{f(z_\alpha)} + z_\alpha\right) < \frac{1}{\alpha} \tag{20}$$

and

$$\frac{\tau}{z_\alpha f(z_\alpha)} + 1 > \tfrac{1}{2} N_\alpha. \tag{21}$$

Take now $\alpha = m$. The sequence $\{z_m\}$ $(m = 1, 2, \ldots)$ thus obtained satisfies (13), as implied by (20). Furthermore, by (19) we get from (21)

$$\lim_{\alpha\to\infty} \left(\frac{\tau}{z_\alpha f(z_\alpha)} + 1\right) = \infty$$

which implies (11).

11. Thus the existence of $\{z_m(\tau)\}$ satisfying (11) and (13) or — their equivalent — (11) and (12) is proved.

Now, since $\{z_m(\tau)\}$ satisfies (11) and (12) there is a term of this sequence, denoted $x^{(\tau)}$ for which both

$$x^{(\tau)} f(x^{(\tau)}) < \tau \tag{22}$$

and

$$\frac{1}{f(x^{(\tau)})} f\left(\frac{\tau}{f(x^{(\tau)})} + x^{(\tau)}\right) < \tau. \tag{23}$$

Let now n range over the natural numbers and denote $x^{(\tau)} = x_n$ for $\tau = 1/n$. From (22) we get

$$x_n f(x_n) < \frac{1}{n} \quad (n = 1, 2, \dots)$$

which implies (6). In addition, (23) implies

$$\frac{1}{f(x_n)} f\left(\frac{\frac{1}{n}}{f(x_n)} + x_n\right) < \frac{1}{n} \quad (n = 1, 2, \dots).$$

Since f is monotonic, we get for any $\tau > 0$ that if $n > 1/\tau$ then

$$\frac{1}{f(x_n)} f\left(\frac{\tau}{f(x_n)} + x_n\right) \leqq \frac{1}{f(x_n)} f\left(\frac{\frac{1}{n}}{f(x_n)} + x_n\right) < \frac{1}{n}$$

which implies that (9) is valid for any $\tau > 0$.

Thus our theorem is proved.

12. Theorem 3: Given a monotonic and divergent series $\sum a_n$, a necessary and sufficient condition for $\sum a_n$ to be a series of a M. C. is

$$\underline{\lim}_{n\to\infty} n a_n = 0.$$

Proof: The necessity of $\underline{\lim}_{n\to\infty} n a_n = 0$ is an immediate consequence of theorem 1 since if $\sum a_n$; $\sum b_n$ is a M. C. and $d_n = \max\{a_n, b_n\}$, we have $n a_n \leqq n d_n$.

To prove the sufficiency we use theorem 2. Let $f(x)$ be any continuous monotonic function passing through the points (n, a_n) $(n = 1, 2, \dots)$ (e.g. the poligonal function obtained by connecting the points (n, a_n) by straight lines). Clearly $f(x)$ satisfies $a_n = f(n)$ and have all the required conditions of theorem 2. Hence there exists a sequence $\{x_n\}$ for which (6) and (7) are true. Now the sequence $\{x_n\}$ should be thinned out until it satisfies the following three demands:

$$y_{n-1} < x_n \quad (n = 2, 3, \dots)$$

$$\sum_{n=1}^{\infty} x_n f(x_n) < \infty \tag{24}$$

$$\sum_{n=1}^{\infty} \int_{x_n}^{y_n} f(t)dt < \infty. \tag{25}$$

Denoting $y_0 = 1$ we define a function $g(x)$:

$$g(x) = f(x_n) \quad \text{for} \quad y_{n-1} \leqq x < y_n \quad (n = 1, 2, \ldots).$$

$\{b_n\}$ is defined now by

$$b_n = g(n) \quad (n = 1, 2, \ldots).$$

It is clear that $\{b_n\}$ is monotonic.

13. To show that $\sum_{n=1}^{\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} \min\{a_n, b_n\} < \infty$ we use the integral test for convergence:

$$\int_1^{\infty} g(t)dt \geqq \sum_{n=1}^{\infty} \int_{x_n}^{y_n} g(t)dt = \sum_{n=1}^{\infty} \int_{x_n}^{y_n} f(x_n)dt = \sum_{n=1}^{\infty} (y_n - x_n) f(x_n) = \sum_{n=1}^{\infty} \rho = \infty$$

and therefore

$$\sum_{n=1}^{\infty} b_n = \infty.$$

Define now $h(x) = \min\{f(x), g(x)\}$. It is clear that $\min\{a_n, b_n\} = h(n)$. We can see that

$$h(x) = \begin{cases} f(x_n) & \text{for} \quad y_{n-1} \leqq x < x_n \\ f(x) & \text{for} \quad x_n \quad \leqq x < y_n \end{cases}$$

therefore

$$\int_1^{\infty} h(t)dt = \sum_{n=1}^{\infty} \int_{y_{n-1}}^{x_n} f(x_n)dt + \sum_{n=1}^{\infty} \int_{x_n}^{y_n} f(t)dt = \sum_{n=1}^{\infty} (x_n - y_{n-1}) f(x_n) +$$
$$+ \sum_{n=1}^{\infty} \int_{x_n}^{y_n} f(t)dt \leqq \sum_{n=1}^{\infty} x_n f(x_n) + \sum_{n=1}^{\infty} \int_{x_n}^{y_n} f(t)dt$$

by (24) and (25) we get then

$$\int_1^{\infty} h(t)dt < \infty$$

and therefore $\sum_{n=1}^{\infty} \min\{a_n, b_n\} < \infty$ proving that $\sum a_n; \sum b_n$ is a M. C.

It may be added that in each relevant statement the condition of weak monotony could be replaced by the stronger condition of strict monotony without any alteration in the results of theorems 1 and 3.

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