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by

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1. Introduction

We describe here some pairs of dual spaces determined from initially prescribed normed linear spaces by means of bounded linear operators. If the prescribed spaces are function spaces, the dual pair frequently play the roles of a space of generalized functions and its space of test functions, and the construction serves as a method of embedding a given function space in an extension space of generalized functions which can be described as strong limits. The construction of a pair of dual spaces is straightforward. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and A a suitable operator on $\mathfrak{X}$ into $\mathfrak{Y}$. We can define a new norm on $\mathfrak{X}$ by writing

$$||x||_A = ||Ax||_{\mathfrak{Y}},$$

and if $\mathfrak{X}$ so normed is incomplete, embed $\mathfrak{X}$ in its completion, which we write as $\mathfrak{X}_A^+$ and call an *inflation* of $\mathfrak{X}$ by A. At the same time, the range of A in $\mathfrak{Y}$ can be normed by

$$||y||_{A^{-1}} = ||A^{-1}y||_{\mathfrak{X}} \qquad (y \in A\mathfrak{X});$$

we call this a *deflation* of $\mathfrak{Y}$, and denote it by $\mathfrak{Y}_{A^{-1}}^-$. The spaces

$$\mathfrak{X}_A^+,\ (\mathfrak{X}^*)_{A^{*-1}}^-$$

(where * denotes the adjointing operation) constitute the dual pair determined by $\mathfrak{X}$, $\mathfrak{Y}$ and A.

Consider two examples.

1°. Take $\mathfrak{X} = \mathfrak{Y} = L^2(0, \infty)$ and define A by

$$Ax(t) = t^{-k}\Gamma(k)^{-1}\int_0^t (t-u)^{k-1}x(u)du \qquad (x \in L^2,\ k \text{ a fixed positive integer}) \tag{1.1}$$

It turns out that $\mathfrak{X}_A^+$ is an extension of $L^2(0, \infty)$ whose elements have some of the properties usually associated with generalized functions. The space contains a delta function, and up to k derivatives can be defined locally for its members, though not very conveniently. The dual space $(\mathfrak{X}^*)^-_{A^{*-1}}$ determined by the adjoint operator

$$A^*y(u) = \Gamma(k)^{-1}\int_u^\infty (t-u)^{k-1}\, t^{-k}\, y(t)dt \quad (x \in L^2) \qquad (1.2)$$

is made up of L^2 functions x for which $t^k x^{(k)}(t) \in L^2$ and $x = A^*[u^k x^{(k)}(u)]$. Elements of this space possess atleast k derivatives, with certain Lipschitz properties. These spaces are discussed in detail in [5], [6] and [7]. Some other extensions of $L^2(0, \infty)$ are described in [8].

2°. Take $\mathfrak{X} = L^1(0, \infty)$, let F be a compact subset of the positive reals with non-empty interior, and consider the Laplace transformation

$$Ax(z) = \int_0^\infty e^{-zu}\, x(u)du = x^\vee(z) \qquad (1.3)$$

as a mapping of $L^1(0, \infty)$ into the space $\mathfrak{Y} = C(F)$ of continuous functions on F with the uniform norm. Then

$$||x||_A = \sup_{z\in F} |x^\vee(z)|, \qquad (1.4)$$

and by completion under this norm L^1 yields a space in which every element has a well-defined strong left derivative. We return to this example later, and obtain a generalization of it in § 7.

Other examples of the types of structures contemplated in this paper will be found in [2], [3] and [4]. A. P. Guinand in [2] describes some deflations of $L^2(0, \infty)$, and also uses deflationary processes to obtain a pair of subclasses of $L^2(0, 2\pi)$ and l^2 with a Fourier-series reciprocity property. R. R. Goldberg in [3] generalizes some deflations described by Guinand and the author.

Contents. In § 2 we specify a class of operators which give rise to inflations and deflations, and in § 3 we examine further the duality between the two spaces; § 4 is devoted to examples. § 5 discusses the partial ordering of inflations by inclusion. § 6 describes inflation of algebras.

We use Example 2° as a motivating and illustrative example in the course of the discussion, and in § 7 obtain a natural generalization by using the Gelfand representation of a commutative Banach algebra.

2. Inflating operators in Banach spaces

Starting with spaces $\mathfrak{X}$ and $\mathfrak{Y}$, we first consider the conditions which A should satisfy in order that $\mathfrak{X}^+_A$ be a workable extension of

$\mathfrak{X}$. For simplicity we suppose A linear; and although the norm of $\mathfrak{X}$ and its completeness are not necessary for the definition of the inflation, none the less we suppose both $\mathfrak{X}$ and $\mathfrak{Y}$ to be Banach spaces. We can regard $\mathfrak{X}_A^+$ as usual as the set of equivalence classes of sequences of elements of $\mathfrak{X}$ which are Cauchy with respect to the A-norm $||\cdot||_A$, and write $(x_n) \sim_A x$, $x_n \to_A x$, $x = \lim_A x_n$ if the sequence (x_n), $x_n \in \mathfrak{X}$ $(n \to \infty)$ determines x in $\mathfrak{X}_A^+$. We lay down the following requirements.

(a) $||x||_A$ be defined for all $x \in \mathfrak{X}$; i.e. $\mathfrak{D}(A) = \mathfrak{X}$.

(b) The norm of $\mathfrak{X}$ be stronger than $||\cdot||_A$, so that the limiting process in $\mathfrak{X}$ be preserved in $\mathfrak{X}_A^+$; i.e.

$$x,\ x_n \in \mathfrak{X},\ ||x_n - x|| \to 0 \ \textit{imply}\ ||x_n - x||_A \to 0.$$

For this it is necessary and sufficient that A be bounded.

(c) The norm topology of $\mathfrak{X}_A^+$ induce a Hausdorff topology on $\mathfrak{X}$; i.e.

$$x,\ x_0,\ x_n \in \mathfrak{X},\quad ||x_n - x||_A \to 0,\quad ||x_n - x_0||_A \to 0\quad \textit{imply}\quad x = x_0,$$

which is the case if $Ax = 0$, $x \in \mathfrak{X}$ imply $x = 0$. This condition also ensures that $||\cdot||_A$ has the properties of a norm.

(d) $A\mathfrak{X}$ be dense in $\mathfrak{Y}$. (If the closure $\overline{A\mathfrak{X}}$ were a proper subspace of $\mathfrak{Y}$, we could restate the theory using this subspace in place of $\mathfrak{Y}$.)

(e) $\mathfrak{X}_A^+$ be a proper extension of $\mathfrak{X}$; i.e. there exist at least one sequence (s_n), $s_n \in \mathfrak{X}$, which is Cauchy in $\mathfrak{X}_A^+$ but not in $\mathfrak{X}$.

These suggest

Definition 1. *The linear operator A from $\mathfrak{X}$ into $\mathfrak{Y}$ is called a "proper inflator" (proper inflating operator) if*

Ω_A(1) *A is bounded, with domain $\mathfrak{X}$;*

(2) *$Ax = 0$, $x \in \mathfrak{X}$ imply $x = 0$;*

(3) *the range of A is dense, but properly contained, in $\mathfrak{Y}$.*

If instead A satisfies (1), (2) *and*

(3)′ *the range of A is $\mathfrak{Y}$,*

it is called an "improper inflator".

We denote the set of inflating operators by $\mathfrak{I}(\mathfrak{X}, \mathfrak{Y})$, of proper inflators by $\mathfrak{I}_p$. A proper inflator satisfies (a) to (e). If A is an improper inflator, then $\mathfrak{X}_A^+ = \mathfrak{X}$: for A^{-1} exists by (2), and is bounded, by a well-known theorem of Banach;[1]) hence every A-Cauchy sequence is Cauchy in $\mathfrak{X}$.

As consequences of Ω(1)—(3) we note that A is closed, and its

[1]) See [1], Theorem 2.12.1. Other theorems in Chapters 1 and 2 of [1] are used below.

range is of the first category in $\mathfrak{Y}$ (by the closed graph theorem); A^{-1} is defined and closed but unbounded; A^* (the adjoint of A) exists as a bounded linear operator mapping $\mathfrak{Y}^*$ into $\mathfrak{X}^*$ (the conjugate Banach spaces), $||A^*|| = ||A||$, and $(A^*)^{-1} = (A^{-1})^*$, the operators being unbounded. We summarize the construction of $\mathfrak{X}_A^+$ in

THEOREM 1. *If A is a proper inflator in $\mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$, then $\mathfrak{X}_A^+$ is a Banach space isometrically isomorphic to $\mathfrak{Y}$, and $\mathfrak{X}$ is a dense subspace in $\mathfrak{X}_A^+$. The operator A can be extended to an operator mapping $\mathfrak{X}_A^+$ onto $\mathfrak{Y}$, with a unique inverse which is the extension of the A^{-1} determined by $\Omega_A(2)$.*

We shall not as a rule distinguish A, A^* or A^{-1} from their extensions explicitly. The following result is useful.

LEMMA 1. *If a subset $\mathfrak{U}$ of $\mathfrak{X}$ is dense in $\mathfrak{X}$, then it is dense in $\mathfrak{X}_A^+$; that is, $A\mathfrak{U}$ is dense in $\mathfrak{Y}$.*

The proof is straightforward.

A consequence of the lemma is that $\mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$ is a semigroup under operator multiplication; for if A, B, $\in \mathfrak{J}$, then AB clearly satisfies Ω_{AB} (1) and (2), and (3) follows from the lemma. $\mathfrak{J}_p$ is likewise a semi-group, and we have

$$A \in \mathfrak{J}_p,\ B \in \mathfrak{J} \textit{ imply } AB,\ BA \in \mathfrak{J}_p;$$

for if AB, for example, is improper, $(AB)^{-1}$ is bounded, and so then is $A^{-1} = B(AB)^{-1}$, implying $A \notin \mathfrak{J}_p$. We note that $I \in \mathfrak{J} - \mathfrak{J}_p$, $0 \notin \mathfrak{J}$.

THEOREM 2. *If A is an improper inflator, then so is A^*. If $A \in \mathfrak{J}_p(\mathfrak{X}, \mathfrak{Y})$ and $\mathfrak{X}$ is reflexive, then $A^* \in \mathfrak{J}_p(\mathfrak{Y}^*, \mathfrak{X}^*)$.*

PROOF. If $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$, then $\Omega_{A^*}(1)$ and (2) hold. If A is improper then $(A^*)^{-1} = (A^{-1})^*$ has domain $\mathfrak{X}^*$, and so A^* is an improper inflator. Suppose A proper; we prove $\Omega_{A^*}(3)$. Now $A^*\mathfrak{Y}^*$ is properly contained in $\mathfrak{X}^*$, for if not then A^{-1} is bounded and A is improper. Also $A^*\mathfrak{Y}^*$ is dense in $\mathfrak{X}^*$ if $\mathfrak{X}$ is reflexive. For then the closure $\overline{A^*\mathfrak{Y}^*}$ equals $[\mathfrak{N}(A)]^0$, the annihilator of the null space of A: here $\mathfrak{N}(A)$ is $\{0\}$, and hence $[\mathfrak{N}(A)]^0 = \mathfrak{X}^*$. Thus $\Omega_{A^*}(3)$ holds. [2])

The condition that $\mathfrak{X}$ be reflexive cannot be omitted. A counter-example will be found in [12], Ex. (II_2, III_2).

THEOREM 3. *When $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$, the deflation $\mathfrak{Y}_{A^{-1}}^-$ is a Banach space.*

PROOF. The deflation space is clearly linear; the proof of its

[2]) We have used [1], Theorem 2.11.15 and [14], p. 286, Theorem 2.

completeness follows directly from its definition and the assumption that $\mathfrak{X}$ is complete.

If A^* is an inflator, the set $A^*\mathfrak{Y}^*$ can in the same way be made into a Banach space $(\mathfrak{X}^*)^-_{A^{*-1}}$, a deflation of $\mathfrak{X}^*$, with the norm

$$||x^*||_{A^{*-1}} = ||A^{*-1}x^*||, \qquad x^* \in A^*\mathfrak{Y}^*.$$

We note that if $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$, $(\mathfrak{X}^*)^-_{A^{*-1}}$ is a Banach space even when $A^* \notin \mathfrak{J}(\mathfrak{Y}^*, \mathfrak{X}^*)$, i.e. when $\Omega_{A^*}(3)$ does not hold.

Let $[\cdot, \cdot]_A$ be the complex-valued bilinear function on $(\mathfrak{X}^*)^-_{A^{*-1}} \times \mathfrak{X}^+_A$ defined by

$$[x^*, x]_A = x^*(x), \quad x^* \in (\mathfrak{X}^*)^-_{A^{*-1}}, \quad x \in \mathfrak{X}^+_A.$$

With this form, the deflation and inflation become a pair of dual spaces in the sense of Rickart [10], p. 62,[3]) in fact normed dual, since

$$|[x^*, x]_A| = |(A^{*-1}x^*)(Ax)|$$
$$\leqq ||A^{*-1}x^*|| \cdot ||Ax|| = ||x^*||_{A^{*-1}} \cdot ||x||_A. \qquad (2.1)$$

We shall denote this pair of spaces briefly by $\mathfrak{X}^{*-}$, $\mathfrak{X}^+$, omitting the "A" when there is no ambiguity.

3. Conjugacy and A-weak convergence

We now look for conditions under which the duality between the spaces $\mathfrak{X}^{*-}$ and $\mathfrak{X}^+$ becomes one of conjugacy, and to this end prove Theorem 5 below. We also consider a form of weak convergence in $\mathfrak{X}^+$ under which $\mathfrak{X}^+$ may be complete. The two results show the way in which $\mathfrak{X}^{*-}$ may play the role of a space of test functions for a space $\mathfrak{X}^+$ of generalized functions. We assume in this section that $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$, but make no stipulation about A^*.

DEFINITION 2. *The sequence* (x_n), $x_n \in \mathfrak{X}$, *is called A-weakly Cauchy if* $[x^*, x_n - x_m]_A \to 0$ *as* $\min(n, m) \to \infty$, *for every* $x^* \in (\mathfrak{X}^*)^-_{A^{*-1}}$.

THEOREM 4. *Let* $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$ *and let* $\mathfrak{Y}$ *be reflexive. Then the A-weak completion of* $\mathfrak{X}$ *is* $\mathfrak{X}^+_A$, *and* $\mathfrak{X}^+_A$ *is A-weakly complete.*

PROOF. If $x \in \mathfrak{X}$, then (2.1) shows that

$$\phi(x^*) = [x^*, x]_A = x^*(x) \qquad (3.1)$$

defines a bounded linear functional ϕ in $(\mathfrak{X}^{*-})^*$, the conjugate

[3]) Cf. Lemma 2, below.

space of $(\mathfrak{X}^*)^-_{A^{*-1}}$. If (x_n) is A-weakly Cauchy, it then follows from the theorem of uniform boundedness that

$$x(x^*) = \lim_{n\to\infty} [x^*, x_n]_A$$

also defines an element of $(\mathfrak{X}^{*-})^*$. Therefore every $x^* \in \mathfrak{X}^{*-}$, a bounded linear functional in $\mathfrak{X}^*$, can be extended to a bounded linear functional in $\mathfrak{X}^{+*}$ by defining $x^*(x)$ when $x \in \mathfrak{X}^+$ to be $\lim_{n\to\infty} x^*(x_n)$, where $(x_n) \sim_A x$: the limit exists since any A-Cauchy sequence is A-weakly Cauchy, and it is independent of the sequence chosen for x. Moreover

$$|x^*(x)| \leqq ||x||_A \cdot ||x^*||_{A^{*-1}}, \tag{3.2}$$

and $||x||_A$ is the norm of this functional in $\mathfrak{X}^{+*}$. Inequality (3.2) is valid for all $x \in \mathfrak{X}^+$, $x^* \in \mathfrak{X}^{*-}$, and the definition of an A-weakly Cauchy sequence can be extended to include sequences with elements from $\mathfrak{X}^+$. We call the collection of A-weak limits of (equivalent) A-weakly Cauchy sequences from $\mathfrak{X}$ the A-weak completion of $\mathfrak{X}$.

Clearly $\mathfrak{X}^+_A$ is contained in the A-weak completion of $\mathfrak{X}$. Conversely, suppose that (x_n), $x_n \in \mathfrak{X}$, is A-weakly Cauchy. Then

$$y^*(Ax_n - Ax_m) = (A^*y^*)(x_n - x_m) = [x^*, x_n - x_m]_A \to 0$$

for all $x^* = A^*y^* \in \mathfrak{X}^{*-}$, i.e. the sequence (Ax_n) is weakly Cauchy in $\mathfrak{Y}$, and since $\mathfrak{Y}$ is reflexive [4]) converges weakly to some element $y \in \mathfrak{Y}$. By Theorem 1, $y = Ax$ for some $x \in \mathfrak{X}^+$; since $x^*(x) = y^*(Ax)$, we have

$$[x^*, x_n - x]_A = y^*(Ax_n - Ax) \to 0,$$

showing that (x_n) converges A-weakly to an element of $\mathfrak{X}^+$. The same argument shows that $\mathfrak{X}^+$ is A-weakly complete.

Lemma 2. *Let* $A \in \mathfrak{I}(\mathfrak{X}, \mathfrak{Y})$ *and* $x \in \mathfrak{X}^+_A$. *Then*

$$x^*(x) = 0, \quad \textit{all} \quad x^* \in (\mathfrak{X}^*)^-_{A^{*-1}} \tag{3.3}$$

if and only if $x = 0$.

Proof. If $x \in \mathfrak{X}$, the result is trivial. The sufficiency of $x = 0$ is also obvious. Suppose (x_n), $x_n \in \mathfrak{X}$, is a sequence for $x \in \mathfrak{X}^+$ and that (3.3) holds, i.e. $\lim_{n\to\infty} x^*(x_n) = 0$, all $x^* \in A^*\mathfrak{Y}^*$. Write $x^* = A^*y^*$; it follows that (Ax_n) is weakly Cauchy in $\mathfrak{Y}$. It is also strongly Cauchy by definition; since the strong and weak limits coincide, we have $||Ax_n|| \to 0$, that is, $x = 0$.

[4]) [14] p. 156, Theorem 2.

Theorem 5. *If* $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$ *and* $\mathfrak{Y}$ *is reflexive, then* $\mathfrak{X}_A^+$ *and* $[(\mathfrak{X}^*)^-_{A^{*-1}}]^*$ *are isometrically isomorphic.*

Proof. Let $x \in \mathfrak{X}_A^+$; we saw that ϕ in (3.1) is then an element in $(\mathfrak{X}^{*-})^*$. Conversely, any element in $(\mathfrak{X}^{*-})^*$ can be so written; let ϕ be an arbitrary bounded linear functional on $\mathfrak{X}^{*-}$, so that $|\phi(x^*)| \leqq ||\phi|| \cdot ||x^*||_{A^{*-1}}$, i.e.

$$|\phi(A^*y^*)| \leqq ||\phi|| \cdot ||y^*||, \quad \text{all} \quad y^* \in \mathfrak{Y}^*.$$

Then $\phi(A^*\cdot)$ defines a bounded linear functional y^{**} on $\mathfrak{Y}^*$, and since $\mathfrak{Y}$ is reflexive, an element $y \in \mathfrak{Y}$ such that $\phi(A^*y^*) = y^{**}(y^*) = y^*(y)$, all $y^* \in \mathfrak{Y}^*$. Since $y = Ax$ for some $x \in \mathfrak{X}^+$, $\phi(A^*y^*) = y^*(Ax) = (A^*y^*)(x)$; thus ϕ has the form (3.1) for some $x \in \mathfrak{X}^+$. Moreover

$$\begin{aligned} ||\phi|| &= \sup\{|\phi(x^*)|; \quad x^* \in \mathfrak{X}^{*-}, \quad ||x^*||_{A^{*-1}} = 1\} \\ &= \sup\{|\phi(A^*y^*)|; \quad ||y^*|| = 1\} \\ &= ||y^{**}|| = ||y|| = ||x||_A. \end{aligned}$$

The mapping $x \to \phi$ of $\mathfrak{X}^+$ onto $(\mathfrak{X}^{*-})^*$ determined by (3.1) is easily seen to be a homomorphism, in fact an isomorphism by Lemma 2, and it has been shown to be an isometry. This proves the theorem.

When A is improper, so is A^*, and the theorem takes the form $\mathfrak{X} \cong \mathfrak{X}^{**}$. Thus it may be thought of as providing a generalization of reflexivity. If $\mathfrak{Y}$ is not reflexive, we can still conclude that $\mathfrak{X}^+ \subseteqq (\mathfrak{X}^{*-})^*$.

Corollary. *If* $A \in \mathfrak{J}(\mathfrak{X}, \mathfrak{Y})$ *and* $\mathfrak{X}$ *is reflexive, then* $(\mathfrak{Y}^*)^+_{A^*}$ *and* $(\mathfrak{Y}^-_{A^{-1}})^*$ *are isometrically isomorphic.*

The proof comes by applying the theorem to $A^* \in \mathfrak{J}(\mathfrak{Y}^*, \mathfrak{X}^*)$.

4. Examples

1° (*continued*). It can be verified that the operators A and A^* of § 1, 1° are inflators, in the sense of Definition 1. Theorems 4 and 5 apply.

2° (*continued*). Let us verify that $A \in \mathfrak{J}\{L^1(0, \infty), C(F)\}$ for the operator in (1.3). Clearly $\Omega(1)$ holds. Moreover, if $x \in L^1(0, \infty)$, its Laplace transform $x^\vee(z)$ is a holomorphic function in $\Re(z) > 0$;[5]) therefore if $x^\vee(z)$ vanishes on F, it vanishes for all $\Re(z) > 0$, and so $x = 0$; thus $\Omega(2)$ holds. To prove $\Omega(3)$ we use the Stone-

[5]) [13], p. 57.

Weierstrass Theorem. Let a product in $L^1(0, \infty)$ be defined by

$$(x \cdot y)(t) = \int_0^t x(u)y(t-u)du, \tag{4.1}$$

the product induced by regarding $L^1(0, \infty)$ as the closed subalgebra $\{x: x(u) = 0 \text{ if } u < 0\}$ of the group algebra $L^1(-\infty, \infty)$ with convolution product, and consider the images of $L^1(0, \infty)$ under A. Since $x^\vee(z)y^\vee(z) = (x \cdot y)^\vee(z)$, these form an algebra. The algebra separates points; for if $x^\vee(z_1) = x^\vee(z_2)$ for all $x \in L^1$, then $e^{-z_1 u} - e^{-z_2 u}$ as an element of L^∞ defines a zero functional in $(L^1)^*$, and so $z_1 = z_2$. It follows from the complex case of the above-mentioned theorem that AL^1 is dense in $C(F)$; [6]) since it is certainly not all of $C(F)$, $\Omega(3)$ is true. Clearly $\mathfrak{X}_A^+$ is an algebra with identity.

The adjoint deflation in this case is the space of all measurable functions f on $(0, \infty)$ of the form

$$f(t) = \int_F e^{-ts}\, d\mu(s)$$

where μ is a regular countably-additive set function on the Borel sets of F, and $||f||_{A^{*-1}} = \int_F |d\mu(s)|$.

3°. Take $\mathfrak{X} = L^1(-\infty, \infty)$, and $\mathfrak{Y} = C_0(-\infty, \infty)$, the space of continuous functions on the real line which vanish at $\pm\infty$, with uniform norm, and take for A the Fourier transformation

$$Ax(t) = \int_{-\infty}^{\infty} e^{iut}\, x(u)du = x^\wedge(t). \tag{4.2}$$

$\Omega(1)$ and (2) hold, and (3) also, for it is known [7]) that the Fourier transforms of functions of $L^1(-\infty, \infty)$ are dense and of the first category in $C_0(-\infty, \infty)$. Thus A determines a proper inflation of L^1. If L^1 is made into a commutative algebra by means of the convolution product, so that $A(x \cdot y) = AxAy$, then $\mathfrak{X}_A^+$ is also an algebra; but it does not contain an identity (delta function), nor is it possible to define derivatives conveniently in it, even of all L^1 functions; thus it lacks the more useful properties of the usual generalized-function spaces.

4°. Consider the A of 3° instead as a mapping into $\mathfrak{Y} = C(F)$. In this case $\Omega(1)$ and (3) hold, but not (2). Let

$$I_F = \{x: x \in L^1(-\infty, \infty), \quad x^\wedge(t) = 0 \quad \text{for} \quad t \in F\};$$

[6]) [10], (3.2.13). AL^1 is self-adjoint on F since F is contained in the positive real axis.

[7]) See Segal, [11].

I_F is a closed ideal in L^1. Let $\tilde{A}$ be the operator induced by A which maps L^1/I_F into $C(F)$, i.e.

$$\tilde{A}\tilde{x} = x^\wedge \qquad (x \in \text{coset } \tilde{x}).$$

Then $\tilde{A}$ is a proper inflator on $\mathfrak{X} = L^1/I_F$, and the algebra $\mathfrak{X}^+_{\tilde{A}}$ is well defined. It has an identity, the element $\tilde{\delta}$ for which $\delta^\wedge(t) = 1$. $(t \in F)$. Let τ^h be the translation operator

$$(\tau^h x)(u) = x(u+h) \qquad (x \in L^1), \tag{4.3}$$

so that

$$(\tau^h x)^\wedge(t) = e^{-iht}x^\wedge(t);$$

τ^h is constant on cosets of I_F, and so $\tilde{\tau}^h(=(\tau^h)^\sim)$ is defined, mapping $\mathfrak{X}$ into $\mathfrak{X}$; and since $||\tilde{\tau}^h\tilde{x}||_{\tilde{A}} = ||\tilde{x}||_{\tilde{A}}$, $\tilde{\tau}^h$ is extendible to $\mathfrak{X}^+_{\tilde{A}}$. Consider the operator on L^1 given by $\alpha^h = h^{-1}(\tau^h-1)$, for which, $(\alpha^h x)^\wedge(t) = h^{-1}(e^{-iht}-1)x^\wedge(t)$. Now if $|t| \leqq C$ and h is small

$$h^{-1}(e^{-iht}-1) = -it+O(C^2|h|e^{C|h|}), \tag{4.4}$$

and so by appropriate choice of C we find that, for $x \in L^1$,

$$\begin{aligned} ||\alpha^h x - \alpha^k x||_A &\leqq C^2 \sup_{t \in F} |(|h|e^{C|h|}+|k|e^{C|k|})x^\wedge(t)| \\ &= (O(h)+O(k))||x||_A. \end{aligned} \tag{4.5}$$

It follows that $(x(t+h)-x(t))/h$ as $h \to 0$ is Cauchy in A-norm. In fact, it is easy to see that $(\tilde{\alpha}^h\tilde{x})$ is Cauchy in $\tilde{A}$-norm for any $\tilde{x} \in \mathfrak{X}^+_{\tilde{A}}$, and hence that derivatives are definable by strong limits in $\mathfrak{X}^+_{\tilde{A}}$, for all elements of the space.

A similar argument (without recourse to a factor algebra) justifies the assertion at the end of § 1, 2°.

To identify the adjoint deflation, notice that $\mathfrak{X}^* = (L^1/I_F)^*$ can be identified with those elements of $(L^1)^*$ which are constant on I_F, with the same norms, while $(C(F))^*$ is the space $rca(F)$ of all regular countably-additive set functions on the Borel sets of F. It can then be shown that $\tilde{A}^*$ maps $\mu \in rca(F)$ into $f(t) = \int_F e^{its}d\mu(s)$; thus $(\mathfrak{X}^*)^-_{\tilde{A}^{*-1}}$ consists of such f, with $||f||_{A^{*-1}} = \int_F |d\mu(s)|$.

5. Partial ordering of inflations

We examine conditions for different inflating operators to determine the same inflation or same deflation, and more generally, for inflations and deflations to be ordered by inclusion. Inclusion and equality for two deflations of the same space may obviously be taken to mean set inclusion and equality; and then we have

Theorem 6. *If* $A \in \mathfrak{I}(\mathfrak{X}, \mathfrak{Y})$ *and* $B \in \mathfrak{I}(\mathfrak{W}, \mathfrak{Y})$, *a necessary and sufficient condition for* $\mathfrak{Y}^{-}_{A^{-1}} \subseteqq \mathfrak{Y}^{-}_{B^{-1}}$ *is that* $B^{-1}A$ *have domain* $\mathfrak{X}$. *In this case* $B^{-1}A$ *is bounded.*

The proof is straightforward. $B^{-1}A$ is a closed operator, and therefore bounded when its domain is $\mathfrak{X}$.

In defining inclusion for two inflations of the same space $\mathfrak{X}$, we wish to preserve the individuality of the elements of the included space, and this is achieved if we regard an inflation of $\mathfrak{X}$ as a set of equivalence classes of sequences from $\mathfrak{X}$, and so as a subclass of the class $\mathfrak{S}(\mathfrak{X})$ of all sets of sequences from $\mathfrak{X}$, and understand inclusion and equality to mean set inclusion and equality in $\mathfrak{S}$. Accordingly we make

Definition 3. *If* $A \in \mathfrak{I}(\mathfrak{X}, \mathfrak{Y})$ *and* $B \in \mathfrak{I}(\mathfrak{X}, \mathfrak{Z})$ *then* $\mathfrak{X}^{+}_{A} \subseteqq \mathfrak{X}^{+}_{B}$ *shall mean that*

(*a*) *every* A*-Cauchy sequence from* $\mathfrak{X}$ *is also* B*-Cauchy,*

(*b*) *any two* A*-Cauchy sequences which are* B*-equivalent are also* A*-equivalent.*

We note that (*a*) implies that two sequences which are Cauchy and equivalent in A-norm are so in B-norm also, and that a sequence which converges to $x \in \mathfrak{X}$ in A-norm does so in B-norm also. Thus (*a*) ensures that an equivalence class in $\mathfrak{X}^{+}_{A}$ is preserved intact in $\mathfrak{X}^{+}_{B}$; (*b*) ensures that $||\cdot||_B$ imposes a Hausdorff topology on $\mathfrak{X}^{+}_{A}$, as required. It is clear that $\subseteqq$ partially orders the deflations and inflations of a given space.

A necessary and sufficient condition for (*a*) to hold is that BA^{-1} be bounded in $\mathfrak{Y}$. For BA^{-1} maps $A\mathfrak{X}$ onto $B\mathfrak{X}$; if it is bounded, then $||BA^{-1}y|| \leqq c||y||$ for all $y \in A\mathfrak{X}$, and so $||Bx|| \leqq c||Ax||$ for all $x \in \mathfrak{X}$, and (*a*) follows. Suppose conversely that (*a*) holds, and let $y_n \to y$ for y_n, $y \in A\mathfrak{X}$. Writing $y_n = Ax_n$, $y = Ax$, we have $x_n \to_A x$, and therefore $x_n \to_B x$; i.e., $BA^{-1}y_n \to BA^{-1}y$. Thus BA^{-1} is continuous on its domain, and so bounded. In this case the least bounded closure $\overline{BA^{-1}}$ exists.

Theorem 7. *If* $A \in \mathfrak{I}(\mathfrak{X}, \mathfrak{Y})$ *and* $B \in \mathfrak{I}(\mathfrak{X}, \mathfrak{Z})$, *a necessary and sufficient condition for* $\mathfrak{X}^{+}_{A} \subseteqq \mathfrak{X}^{+}_{B}$ *is that* $\overline{BA^{-1}} \in \mathfrak{X}(\mathfrak{Y}, \mathfrak{Z})$. *For* $\mathfrak{X}^{+}_{A} = \mathfrak{X}^{+}_{B}$, *it is necessary and sufficient that* $\overline{BA^{-1}}$ *be improper, i.e., that* BA^{-1} *and* AB^{-1} *are bounded.*

Proof. Suppose that $C = \overline{BA^{-1}}$ is an inflator; then C is bounded, and (*a*) holds. To prove (*b*), let (x_n^1), (x_n^2) be two A-Cauchy sequences from $\mathfrak{X}$ which are B-equivalent, and write $x_n = x_n^1 - x_n^2$, $y_n = Ax_n$. Then $Bx_n \to 0$; also $y_n \to y$ for some $y \in \mathfrak{Y}$, and so $Cy_n \to Cy$. Since $\|Cy_n\| = \|Bx_n\| \to 0$, we have $\overline{BA^{-1}}\, y = Cy = 0$, and therefore $y = 0$ by $\Omega_C(2)$. Thus $Ax_n \to 0$, and (x_n^1), (x_n^2) are A-equivalent. Hence (*b*) holds, and $\mathfrak{X}_A^+ \subseteqq \mathfrak{X}_B^+$.

To prove necessity, suppose (*a*) and (*b*) hold. By (*a*), $\overline{BA^{-1}}$ is bounded with domain $\mathfrak{Y}$, and $\Omega_C(1)$ is satisfied. Clearly $\Omega_C(3)$ or $(3)'$ holds, and it remains to prove $\Omega_C(2)$. Let $y \in \mathfrak{Y}$ be such that $\overline{BA^{-1}}\, y = 0$, $y \neq 0$. By $\Omega_A(3)$ we can find a sequence of elements $y_n = Ax_n$, $x_n \in \mathfrak{X}$, such that $y_n \to y$; the (x_n) so determined is then an A-Cauchy sequence defining some $x \in \mathfrak{X}_A^+$, and $x \neq 0$ since $y \neq 0$. On the other hand,

$$\|BA^{-1}y_n\| = \|\overline{BA^{-1}}(y_n - y)\| \leqq \|BA^{-1}\| \cdot \|y_n - y\| \to 0;$$

that is, $\|Bx_n\| \to 0$, so that (x_n) and (0) are A-Cauchy sequences which are B-equivalent but not A-equivalent, which contradicts (*b*). The first part of the theorem is proved. The second follows without difficulty.

Corollary 1. *If A and B belong to $\mathfrak{I}(\mathfrak{X}, \mathfrak{Y})$ and $\mathfrak{X}$ is reflexive, $\mathfrak{X}_A^+ \subseteqq \mathfrak{X}_B^+$ implies $(\mathfrak{X}^*)_{A^{*-1}}^- \supseteqq (\mathfrak{X}^*)_{B^{*-1}}^-$.*

Proof. The first inclusion implies that $\overline{BA^{-1}} \in \mathfrak{I}(\mathfrak{Y}, \mathfrak{Y})$, and hence that $A^{*-1}B^* = (BA^{-1})^* = (\overline{BA^{-1}})^*$ has domain $\mathfrak{Y}^*$; the result follows from Theorems 2 and 6.

Corollary 2. *If A and B belong to $\mathfrak{I}(\mathfrak{X}, \mathfrak{X})$, then $\mathfrak{X}_B^+ \subseteqq \mathfrak{X}_{AB}^+$, with equality if and only if A is improper.*

Proof. We know that $AB \in \mathfrak{I}(\mathfrak{X}, \mathfrak{X})$. Since $A = \overline{AB \cdot B^{-1}}$ is an inflator, $\mathfrak{X}_B^+ \subseteqq \mathfrak{X}_{AB}^+$. If the spaces are equal, $A^{-1} = B(AB)^{-1}$ is bounded and A is improper; conversely if A is improper, $B(AB)^{-1}$ and $AB \cdot B^{-1}$ are both bounded and so $\mathfrak{X}_B^+ = \mathfrak{X}_{AB}^+$.

Corollary 3. *Unless $\mathfrak{I}_p(\mathfrak{X}, \mathfrak{X})$ is empty, $\mathfrak{X}$ has no greatest inflation.*

Proof. If $A \in \mathfrak{I}_p(\mathfrak{X}, \mathfrak{X})$, $\mathfrak{X}_{A^n}^+ \subset \mathfrak{X}_{A^{n+1}}^+$ for $n = 1, 2, \ldots$.

The next result concerns repeated inflation.

Theorem 8. *If A and B belong to $\mathfrak{I}(\mathfrak{X}, \mathfrak{X})$ and $\mathfrak{X}_A^+ \subseteqq \mathfrak{X}_B^+$, then B has a closure $\tilde{B}$ in $\mathfrak{I}(\mathfrak{X}_A^+, \mathfrak{X}_A^+)$, and*

$$\mathfrak{X}_B^+ = (\mathfrak{X}_A^+)_{\tilde{B}}^+.$$

PROOF. Since $||Bx||_A = ||ABA^{-1} \cdot Ax||$, $\tilde{B}$ exists if and only if ABA^{-1} is bounded; but this is a consequence of $\overline{BA^{-1}} \in \mathfrak{I}(\mathfrak{X}, \mathfrak{X})$. Now suppose C is a bounded linear operator mapping $\mathfrak{X}_A^+$ into itself; it is easy to verify that $C \in \mathfrak{I}(\mathfrak{X}_A^+, \mathfrak{X}_A^+)$ if and only if $ACA^{-1} \in \mathfrak{I}(\mathfrak{X}, \mathfrak{X})$. For example, $\Omega_C(2; \mathfrak{X}_A^+)$ takes the form

$$x_n \in \mathfrak{X}, x_n - x_m \to_A 0, \ Cx_n \to_A 0 \quad \textit{imply} \quad x_n \to_A 0,$$

which by the substitution $\bar{x}_n = Ax_n$ becomes

$$\bar{x}_n \in A\mathfrak{X}, \ \bar{x}_n - \bar{x}_m \to 0, \ ACA^{-1}\bar{x}_n \to 0 \quad \textit{imply} \quad \bar{x}_n \to 0,$$

and this is equivalent to: $\bar{x} \in \mathfrak{X}$, $\overline{ACA^{-1}}\, \bar{x} = 0$ *imply* $\bar{x} = 0$. Thus it follows from $A\tilde{B}A^{-1} = \overline{ABA^{-1}} = A \cdot \overline{BA^{-1}} \in \mathfrak{I}(\mathfrak{X}, \mathfrak{X})$ that $\tilde{B} \in \mathfrak{I}(\mathfrak{X}_A^+, \mathfrak{X}_A^+)$. And $\mathfrak{X}_B^+ = (\mathfrak{X}_A^+)_{\tilde{B}}^+$. For the elements of these spaces are the classes of $\tilde{B}$-equivalent sequences of elements from $\mathfrak{X}$, $\mathfrak{X}_A^+$ respectively, and any class of $(\mathfrak{X}_A^+)_{\tilde{B}}^+$ can by the diagonal process be seen to contain a sequence from $\mathfrak{X}$. But two such sequences determine the same or distinct elements in $(\mathfrak{X}_A^+)_{\tilde{B}}^+$ according as they determine the same or distinct elements in $\mathfrak{X}_B^+$. Thus the spaces are isomorphic, and since one contains the other, they are equal.

The theorem and corollary point the distinction between $\mathfrak{X}_{A^2}^+$ and $(\mathfrak{X}_A^+)_A^+ = \mathfrak{X}_A^+$.

If the operator of (1.1) is denoted by A_k, it can be verified by using Mellin transform theory that

$$\mathfrak{X}_{A_k}^+ \subseteqq \mathfrak{X}_{A_l}^+ \quad \text{if} \quad k \leqq l,$$

in the sense of Definition 3.

Consider the dependence in Example 2° of $\mathfrak{X}_A^+$ upon the set F: write A_F for the operator in (1.3), and let F, G be two compact sets of the type described in § 1, 2°, with $F \subset G$. Clearly $||x||_{A_F} \leqq ||x||_{A_G}$ for $x \in \mathfrak{X}$; but this is not sufficient to imply that $\mathfrak{X}_{A_G}^+ \subseteqq \mathfrak{X}_{A_F}^+$. In fact, $\overline{A_F A_G^{-1}}$ satisfies $\Omega(1)$ and (3)', but not (2). The set

$$\mathfrak{N} = \{x \colon x \in \mathfrak{X}_{A_G}^+, \quad x^{\blacktriangledown}(z) = 0 \quad \text{for} \quad z \in F\}$$

is a closed ideal in $\mathfrak{X}_{A_G}^+$, and

$$\mathfrak{X}_{A_G}^+/\mathfrak{N} \cong \mathfrak{X}_{A_F}^+.$$

(On the other hand, the adjoint deflation for F is contained in that for G). At the same time there exist sequences which are Cauchy in A_F-norm but not in A_G-norm. Shrinking the set F has the effect of making the inflation less discriminating.

6. Inflation of Banach algebras

We suppose now that $\mathfrak{X}$ and $\mathfrak{Y}$ are both commutative Banach algebras over the complex field, and that A is also an isomorphism of $\mathfrak{X}$ onto a dense subset in $\mathfrak{Y}$, i.e. that it satisfies $\Omega_A(1)$ to (3) as a mapping of linear spaces, and also

$$\Omega_A(4) \quad A(x_1 \cdot x_2) = Ax_1 \cdot Ax_2, \quad \textit{all} \quad x_1, x_2 \in \mathfrak{X}.$$

Then the norms $||\cdot||_A$ and $||\cdot||_{A^{-1}}$ are algebra norms, and $\mathfrak{X}_A^+$, $\mathfrak{Y}_{A^{-1}}^-$ are likewise Banach algebras.

Let $\Phi(\mathfrak{X})$, $\Phi(\mathfrak{Y})$ denote the carrier spaces [8]) of $\mathfrak{X}$, $\mathfrak{Y}$, i.e. the subsets of $\mathfrak{X}^*$, $\mathfrak{Y}^*$ respectively whose elements are the homomorphisms from the spaces onto the complex-number field; for $\phi \in \Phi(\mathfrak{X})$ and $x \in \mathfrak{X}$ let $\hat{x}(\phi)$ denote the image of x under ϕ. The isomorphism A induces a mapping $\hat{A}$ of $\Phi(\mathfrak{Y})$ into $\Phi(\mathfrak{X})$, by the relation

$$\hat{x}(\hat{A}\phi) = (Ax)^\wedge(\phi), \quad \text{all} \quad x \in \mathfrak{X}, \quad \text{all} \quad \phi \in \Phi(\mathfrak{Y}). \tag{6.1}$$

$\hat{A}$ is a continuous mapping under the $\mathfrak{X}$- and $\mathfrak{Y}$-topologies (the weakest topologies on $\Phi(\mathfrak{X})$ and $\Phi(\mathfrak{Y})$ for which the functions $\hat{x}$, $\hat{y}$ are continuous). Because of $\Omega_A(3)$, $\hat{A}$ maps the zero homomorphism onto the zero homomorphism, and is one-to-one: for if $\hat{A}\phi = \hat{A}\psi$ for $\phi, \psi \in \Phi(\mathfrak{Y})$, then by (6.1), $\hat{y}(\phi) = \hat{y}(\psi)$ for all y in a dense set of $\mathfrak{Y}$, in fact all $y \in \mathfrak{Y}$ since $\hat{y}$ is a continuous function of y; hence $\phi = \psi$. It is clear that if $\phi \in \Phi(\mathfrak{X})$ corresponds to $\xi^* \in \mathfrak{X}^*$ under the embedding $\Phi \subseteqq \mathfrak{X}^*$, then $\hat{x}(\phi) = \xi^*(x)$ for all $x \in \mathfrak{X}$; (6.1) shows that the adjoint operator A^* coincides with $\hat{A}$ as a mapping on $\Phi(\mathfrak{Y})$.

THEOREM 9. *The carrier space of* $\mathfrak{X}_A^+$ *is* $\hat{A}\Phi(\mathfrak{Y})$, *and consists of those* ϕ *in* $\Phi(\mathfrak{X})$ *for which*

$$|\hat{x}(\phi)| \leqq ||x||_A \quad \textit{for all} \quad x \in \mathfrak{X} \tag{6.2}$$

(*and so for all* $x \in \mathfrak{X}_A^+$).

Similarly $\hat{A}\Phi(\mathfrak{Y}_{A^{-1}}^-) = \Phi(\mathfrak{X})$, *and* $\Phi(\mathfrak{Y})$ *consists of those* ψ *in* $\Phi(\mathfrak{Y}_{A^{-1}}^-)$ *for which*

$$|\hat{y}(\psi)| \leqq ||y||_{A^{-1}} \quad \textit{for all} \quad y \in \mathfrak{Y}_{A^{-1}}^-.$$

PROOF. If $\phi = \hat{A}\psi \in \hat{A}\Phi(\mathfrak{Y})$ then (6.2) certainly holds, for

$$|\hat{x}(\hat{A}\psi)| = |(Ax)^\wedge(\psi)| \leqq ||Ax||_{\mathfrak{Y}} = ||x||_A.$$

On the other hand, if (6.2) holds then clearly ϕ has a well-defined

[8]) $\Phi(\mathfrak{X})$ is $\Phi_{\mathfrak{X}}$, in the notation of [10], Chapter 3.

extension to $\mathfrak{X}_A^+$, and $(A^{-1}\cdot)^\wedge(\phi)$ determines a homomorphism on $\mathfrak{Y}$ onto the complex numbers, ψ say. Then $\phi = \hat{A}\psi \in \hat{A}\Phi(\mathfrak{Y})$. The proof of the second part is similar.

Example 2° (*continued*). Let us determine the carrier spaces for $\mathfrak{X} = L^1(0, \infty)$ and its inflation, bearing in mind that $\mathfrak{X}$ is not the usual group algebra for the group of positive reals, but is to be regarded as a closed subalgebra of $L^1(-\infty, \infty)$, with the form of product (4.1). [9])

The homomorphisms ϕ of $\Phi(\mathfrak{X})$ are in one-to-one correspondence with the maximal modular ideals of $\mathfrak{X}$. Let $\mathfrak{R}$ and R stand for the algebras obtained from $L^1(-\infty, \infty)$ and $L^1(0, \infty)$ by adjunction of the (common) identity. Every maximal ideal $\mathfrak{M}(\neq L^1(0, \infty))$ in $\mathfrak{R}$ intersects R in a maximal ideal of R; moreover, if M_1 is a maximal ideal in R contained in some maximal ideal $\mathfrak{M}_1$ in $\mathfrak{R}$, then $M_1 = \mathfrak{M}_1 \cap R$. On the other hand, the maximal modular ideals in $L^1(0, \infty)$ are precisely the intersections with $L^1(0, \infty)$ of the maximal ideals in R other than $L^1(0, \infty)$ itself. Thus to every $\mathfrak{M}$ will correspond a maximal modular ideal in $L^1(0, \infty)$, namely $\mathfrak{M} \cap L^1(0, \infty)$. But the $\mathfrak{M}$ are in one-to-one correspondence with the elements χ of the character group of $(-\infty, \infty)$, and from this it can be deduced that the maximal modular ideals M induced in $L^1(0,\infty)$ as a subalgebra of $L^1(-\infty, \infty)$ are those given by relations of the form

$$\hat{x}(\phi_{i\chi}) = \hat{x}(M) = \int_0^\infty x(t)e^{-i\chi t}dt \quad (\text{all } x \in L^1(0, \infty))$$

Here $\phi_{i\chi}$, M and χ are corresponding elements, and χ ranges over $(-\infty, \infty)$.

The $\phi_{i\chi}$ so determined do not exhaust $\Phi(\mathfrak{X})$. We can obtain the whole space as follows. Every ϕ in the space, being a bounded linear functional on $L^1(0, \infty)$, determines by

$$\hat{x}(\phi) = \int_0^\infty x(t)g_\phi(t)dt \quad (\text{all } x \in L^1(0, \infty))$$

an essentially bounded measurable function $g_\phi = g$. Then, for any x, y in $L^1(0, \infty)$,

$$\begin{aligned}(x \cdot y)^\wedge(\phi) &= \int_0^\infty g(t)dt \int_0^t x(u)y(t-u)du \\ &= \int_0^\infty x(u)du \int_u^\infty y(t-u)g(t)dt, \quad \text{by Fubini's theorem,} \\ &= \int_0^\infty x(u)du \int_0^\infty y(t)g(t+u)\,dt,\end{aligned}$$

[9]) For the following remarks cf. [9], § 7,2, VI, and § 31, 1.

while

$$\hat{x}(\phi)\hat{y}(\phi) = \int_0^\infty x(u)du \int_0^\infty y(t)g(t)g(u)dt.$$

These imply

$$g(t+u) = g(t)g(u), \quad \text{all} \quad t, u > 0,$$

and this identity, with the measurability of g, implies that [10])

$$g_\phi(t) = e^{-(\omega+i\chi)t}$$

for some complex number $z = \omega + i\chi$ depending upon ϕ, with $\omega \geqq 0$. We have $\hat{x}(\phi_z) = x^\vee(z)$ in the notation of (1.3), and the elements of $\Phi(\mathfrak{X})$ can be identified with the points of the half-plane $\Re(z) \geqq 0$, the region of convergence of the Laplace transform.

If $\mathfrak{X}_A^+$ is the inflation of $L^1(0, \infty)$ by $A = A_F$, of (1.3), Theorem 9 shows that $\Phi(\mathfrak{X}_A^+)$ can be identified with the compact set F. $\Phi(C(F))$ is also F.

7. Inflations using *C(F)*

We end with a discussion of the case $\mathfrak{Y} = C(F)$, and obtain a generalization suggested by Example 2°. We consider the following situation.

Let I stand for any one of the real number sets $I_1 = (0, \infty)$, ${}_1I = (-\infty, 0)$, or $I_2 = (-\infty, \infty)$. Let $\mathfrak{A} = \mathfrak{A}(I)$ be a complex commutative Banach algebra without identity, whose elements are functions on I to some general linear space $\mathfrak{R}$, so that $x(t) \in \mathfrak{R}$ whenever $x(\cdot) \in \mathfrak{A}$, $t \in I$. Let $||\cdot||$ be the norm of $\mathfrak{A}$, and suppose that the linear operations of addition and scalar multiplication in $\mathfrak{A}$ are those induced from $\mathfrak{R}$.

Let $\Phi = \Phi(\mathfrak{A})$ be the carrier space of $\mathfrak{A}$. We know that $x \to \hat{x}$ is a homomorphic mapping of $\mathfrak{A}$ into $C_0(\Phi)$, the space of all continuous functions on the locally compact space Φ which vanish at ∞, with the uniform norm.

Let F be a compact subset of Φ, which does not contain the zero homomorphism. Let A_F be the induced mapping $x \to \hat{x}$ of $\mathfrak{A}$ into $C(F)$, the space of all continuous functions on F, and write

$$|x|_F = ||x||_{A_F} = \max_{\phi \in F} |\hat{x}(\phi)|. \tag{7.1}$$

In what circumstances does $A_F \in \mathfrak{I}(\mathfrak{A}, C(F))$? Since $|x|_F \leqq ||x||$, $\Omega(1)$ certainly holds. $\Omega(2)$, will not hold in general; it is valid in

[10]) [1], Corollary to Theorem 4.17.3.

the case of Example 2°, but not for Example 4°. Conditions for $\Omega(3)$ are given by the Stone-Weierstrass theorem: $A_F \mathfrak{A}$ certainly separates the points of F, and does so strongly since F does not contain the zero homomorphism; if also $A_F \mathfrak{A}$ is self-adjoint on F, (contains the complex conjugate $\overline{f(\phi)}$ whenever it contains $f(\phi)$, for $\phi \in F$) it follows that $A_F \mathfrak{A}$ is dense in $C(F)$. Assume that $\Omega(2)$ does hold: if δ_F is such that $\hat{\delta}_F(\phi) = 1$ for $\phi \in F$ then $x \cdot \delta_F = x$ for all $x \in \mathfrak{A}$, contradicting the assumption that $\mathfrak{A}$ has no identity; therefore $A_F \mathfrak{A}$ is a proper subset of $C(F)$. Thus $\Omega(2)$ and self-adjointness together imply $\Omega(3)$. $\Omega(4)$ obviously holds.

To reproduce the characteristics of Example 2° we need some assumptions concerning the translation operation

$$\begin{gathered} \tau^h x(\cdot) = x(\cdot + h) \quad (h \in J), \\ \tau^0 x(\cdot) = x(\cdot) \end{gathered} \tag{7.2}$$

We shall assume:

(i) that $\mathfrak{A}$ is closed under translations τ^h for every $h \in J$, J being one of the sets I_1, ${}_1I$, or I_2, not necessarily distinct from I.

(ii) that τ^h is a strongly continuous operator function of h at 0, in the sense

$$||\tau^h x - x|| \to 0 \quad \text{as} \quad h \to 0 \quad \text{in} \quad J, \quad \text{for every } x \in \mathfrak{A};$$

and

(iii) that the product in $\mathfrak{A}$ is so defined that

$$\tau^h(x \cdot y) = (\tau^h x) \cdot y = x \cdot (\tau^h y) \quad (\text{all } h \in J, \text{all } x, y \in \mathfrak{A}).$$

Notice that, if $\mathfrak{K}$ is an algebra, (iii) holds only if the product in $\mathfrak{A}$ is not that induced from $\mathfrak{K}$ (except in the trivial case when all elements of $\mathfrak{A}$ are constant on I); the given property is characteristic of convolution-type products.

(Example 2° can be considered as a case where $I = I_1$, $J = {}_1I$. We have to remember in defining τ^h that $x(t) = 0$ for $t < 0$).

We now prove

THEOREM 10. *Let $\mathfrak{A}(I)$, J and F be defined as above, and suppose that $A_F \mathfrak{A}$ is self-adjoint on F and that $\Omega_{A_F}(2)$ is valid. Then $A_F \in \mathfrak{I}_p(\mathfrak{A}, C(F))$, and $\mathfrak{A}^+_{A_F}$ contains for each of its elements x a derivative*

$$x^d = \lim\nolimits_{A_F} h^{-1}(\tau^h x - x) \quad (h \to 0 \quad \text{in} \quad J).$$

PROOF. The assumptions make A_F a proper inflator. Now if $\phi \in \Phi(\mathfrak{A})$, and $x, y \in \mathfrak{A}$,

$$(\tau^h(x\cdot y))^\wedge(\phi) = (\tau^h x)^\wedge(\phi)\cdot \hat{y}(\phi) = \hat{x}(\phi)\cdot(\tau^h y)^\wedge(\phi).$$

It follows that

$$(\tau^h x)^\wedge(\phi) = \hat{x}(\phi) f(h),$$

where f is some numerical function depending upon ϕ. From $\tau^{h+k} = \tau^h\tau^k$ and the definition of τ^0 we deduce

$$f(h+k) = f(h)f(k), \quad f(0) = 1. \tag{7.3}$$

Also, if $h \in J$, for each $\phi \in F$ we can find an x such that $\hat{x}(\phi) \neq 0$, and then

$$\begin{aligned} |\hat{x}(\phi)|\cdot|f(h+k)-f(h)| &= |(\tau^{h+k}x-\tau^h x)^\wedge(\phi)| \\ &\leqq \begin{cases} \|\tau^k(\tau^h x)-(\tau^h x)\| & \text{if } \ k\in J \\ \|(\tau^{h+k}x)-\tau^{-k}(\tau^{h+k}x)\| & \text{if } \ -k \in J \end{cases} \\ &\to 0 \quad \text{as} \quad k\to 0, \end{aligned}$$

by assumption (i); hence $f(h)$ is continuous on both sides at every $k \in J$. It is therefore a measurable function, and we conclude as before that

$$f(h) = e^{sh}, \quad (\tau^h x)^\wedge(\phi) = \hat{x}(\phi)e^{sh},$$

for some complex number s which will in general depend upon ϕ.

Let S be the image of F under the mapping $\phi \to s(\phi) = s$. It is clear that the mapping is continuous, for since $A_F\mathfrak{A}$ is dense in $C(F)$ an x can be found for which $\hat{x}(\phi) \neq 0$ for $\phi \in F$. S is therefore a compact set in the complex plane. The proof proceeds now as in § 4, 4°. For $h \in J$, $|h| \leqq h_0$, we have

$$\begin{aligned} |\tau^h x|_F &= \max_{\phi\in F} |\hat{x}(\phi)e^{sh}| \\ &\leqq \max_{\phi\in F} |\hat{x}(\phi)|\cdot\max_{s\in S}|e^{sh}| = \beta(h_0)\cdot|x|_F, \quad \text{say.} \end{aligned}$$

so that τ^h is extendible to a bounded linear operator mapping $\mathfrak{A}^+_{AF}$ into itself. It should be emphasized that our assumptions do not make the elements of the inflation functions on I to $\mathfrak{R}$, so that for $x \in \mathfrak{A}^+_{AF}$, $x(\cdot+h)$ must be taken to be defined as $\tau^h x$, rather than the converse.

Writing $\alpha^h = h^{-1}(\tau^h-1)$, we get

$$\begin{aligned} |\alpha^h x-\alpha^k x|_F &= \max_{\phi\in F}\left|\left(\frac{e^{sh}-1}{h}-\frac{e^{sk}-1}{k}\right)\hat{x}(\phi)\right| \\ &\leqq \max_{s\in S}\left|s+O(|s^2h|e^{|sh|})-s+O(|s^2k|e^{|sk|})\right|\times\max_{\phi\in F}|\hat{x}(\phi)| \\ &= \{O(|h|)+O(|k|)\}\cdot|x|_F, \end{aligned}$$

valid for every $x \in \mathfrak{A}^+_{A_F}$; and all Cauchy subsequences of $(\alpha^h x)$, $h \to 0$, define the derivative x^d as an element of $\mathfrak{A}^+_{A_F}$. This concludes the proof.

We note that if x has a derivative already defined as a strong limit in $\mathfrak{A}$, then this coincides with x^d. The operation ^{d} is linear, and has the product law

$$(x \cdot y)^d = x^d \cdot y = x \cdot y^d$$

Clearly $(x^d)^\wedge(\phi) = \hat{x}(\phi) \cdot s(\phi)$.

We notice also that $\mathfrak{A}^+_{A_F}$ is an algebra with identity, $\delta = \delta_F$ say, whose defining property is $\hat{\delta}(\phi) = 1$. Now $\mathfrak{A}$ is without identity; let $\mathfrak{A}_1$ be the algebra obtained by adjoining an identity, e say. Since δ and e have the same algebraic properties, we may embed $\mathfrak{A}_1$ in $\mathfrak{A}^+_{A_F}$ by making $x+\lambda e$ correspond to $x+\lambda\delta$ for every $x \in \mathfrak{A}$ and scalar λ. Then $\mathfrak{A} \subset \mathfrak{A}_1 \subset \mathfrak{A}^+_{A_F}$. The norm of $\mathfrak{A}_1$, given by $||x+\lambda\delta|| = ||x||+|\lambda|$, dominates $|\cdot|_F$.

Since s as a function of ϕ belongs to $C(F)$, there is some element in $\mathfrak{A}^+_{A_F}$ to which it corresponds: it is δ^d, for $(\delta^d)^\wedge(\phi) = \delta(\phi)s(\phi) = s(\phi)$. We see that differentiation can be written as multiplication by δ^d: $x^d = (x \cdot \delta)^d = x \cdot \delta^d$.

Integration in $\mathfrak{A}^+_{A_F}$ can be defined as follows. Suppose F so chosen that $s(\phi)$ does not vanish in F: then $[s(\phi)]^{-1}$ belongs to $C(F)$ and so determines an element, q say. We define integration to be the operation of multiplying by q, and write $x^i = q \cdot x$. Then

$$(x^d)^i = (x^i)^d = x;$$

q may be identified with $(\delta^d)^{-1}$. These formal calculations suggest that a Mikusinski-like calculus exists in $\mathfrak{A}^+_{A_F}$; but the space must always possess divisors of zero, and the generalized functions envisaged here are essentially different from Mikusinski's.

The adjoint operator A^*_F maps the space $rca(F)$ into $\mathfrak{A}^*$, and thus the adjoint deflation $\mathfrak{A}^{*-}$ consists of those bounded linear functionals x^* on $\mathfrak{A}$ which can be written in the form

$$x^*(z) = \int_F \hat{z}(\phi)d\mu(\phi) \quad (\text{all} \quad z \in \mathfrak{A}^+_{A_F})$$

for some $\mu \in rca(F)$. Here $A^*_F\,\mu = x^*$, and $||x^*||_{A_F^{*-1}} = \int_F|d\mu(\phi)|$.

The A_F-weak completion of $\mathfrak{A}$ consists of the limits of sequences (x_n), $x_n \in \mathfrak{A}$, for which $\int_F(\hat{x}_n(\phi)-\hat{x}_m(\phi))d\mu(\phi) \to 0$, for all $\mu \in rca(F)$. Since $C(F)$ is not weakly complete in general, $\mathfrak{A}^+_{A_F}$ is properly contained in the A_F-weak completion of $\mathfrak{A}$.

We observe that Example 2° is a case covered by the theorem, since F there is supposed real (contained in the real axis) and has non-empty interior; for the first of these conditions is necessary and sufficient for $A_F \mathfrak{A}$ to be self-adjoint on F, and the second implies $\Omega_{A_F}(2)$. For complex F, the example falls outside the scope of the theorem.

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