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A Theorem on the Zeros of an Entire Function

by

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1. Our aim in this note is to prove the following theorem.

THEOREM: If $P(z)$ is a canonical product of genus p and order $\rho(\rho > p)$ defined by:

$$P(z) = \prod_{n=1}^{\infty} \left(1-\frac{z}{z_n}\right) \exp\left\{z/z_n+\tfrac{1}{2}(z/z_n)^2+ \dots + \frac{1}{p}(z/z_n)^p\right\},$$

where $z_1, z_2, \dots$ etc. are the zeros of $P(z)$ whose modulii $r_1, r_2, \dots$ etc. form a non-decreasing sequence such that $r_n > 1$ for all n and where $r_n \to \infty$ as $n \to \infty$, then for z in a domain exterior to the circles of radius $r_n^{-h}(h > \rho)$ described about the zeros z_n as centres, we have

$$\left|\frac{P'(z)}{P(z)}\right| < K\int_0^{\infty} \frac{n(x)r^{\rho}}{x^p(x+r)^2} dx,$$

where K is a constant independent of p and $P'(z)$ is the first derivative of $P(z)$ and $n(x)$ denotes the number of zeros whithin and on the circle $|z| = x$.

PROOF: It is sufficient to differentiate log $P(z)$ in a region in which it is regular. Such a region can always be found out: and before we tackle this problem, we would, however, like to arrange the zeros in the following way.

Let $\kappa(> 1)$ and $\kappa'(> 1)$ be two numbers so suitably chosen that the zeros of modulii $r_{N+1}, r_{N+2}, \dots$ etc. lie outside the circle with centre origin and radius κr and the zeros of modulii $r_1, r_2, \dots, r_N$ lie inside the annular region of outer radius κr and inner radius $r(\kappa')^{-1}$ respectively (these later zeros may also lie on the outer circumference of the annulus).

Now we indent all the zeros by small circles of radii $r_n^{-h}(h > \rho;$ $n = 1, 2, \dots)$. But $\sum_{n=1}^{\infty} r_n^{-h}$ is convergent since $h > \rho$ and hence after exclusion of these circles we are still left with a domain which does not include these so drawn circles. This means that if we take a point z in this excluded region, then $|r-r_n| > r_n^{-h}$.

Now we return to the mathematical formulation of the problem.

We write

$$P(z) = P_N(z)Q(z), \tag{A}$$

where

$$P_N(z) = \prod_{n=1}^{N} E(z/z_n, p),$$

and

$$Q(z) = \prod_{n=N+1}^{\infty} E(z/z_n, p),$$

$E(z/z_n, p)$ being Weierstrass's primary factor. Now from the expression for $P(z)$ we have

$$\log P(z) = \sum_{n=1}^{\infty} \left\{ \log\left(1-\frac{z}{z_n}\right) + \left(z/z_n + \tfrac{1}{2}(z/z_n)^2 + \ldots + \frac{1}{p}(z/z_n)^p\right)\right\}.$$

We can differentiate the above expression in the excluded region, for the right-hand side is regular, and uniformly and absolutely convergent. We have then

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty} \left\{ \frac{-1}{z_n(1-z/z_n)} + \frac{1}{z_n}\left(1+\frac{z}{z_n}+\ldots+\left(\frac{z}{z_n}\right)^{p-1}\right)\right\}$$

$$= \sum_{r/\kappa' < r_n \leqq \kappa r} + \sum_{r_n \geqq \kappa r} = \sum\nolimits_1 + \sum\nolimits_2 \tag{1}$$

Estimation of $\sum_1$: Let us write $r/r_n = u_n$. Then, since $|1-z/z_n| \geqq |1-r/r_n|$ we have

$$\left|\sum\nolimits_1\right| \leqq \sum_{n=1}^{N} \left\{ \frac{1}{r_n|1-u_n|} + \frac{1}{r_n}(1+u_n+\ldots+u_n^{p-1})\right\},$$

where $N = n(\kappa r)$. Again in $\sum_1$, $\kappa' > u_n \geqq 1/\kappa$ and so

$$\left|\sum\nolimits_1\right| \leqq \sum_{n=1}^{N} \frac{1}{r_n|1-u_n|} + \sum_{n=1}^{N} \frac{u_n^{p-1}}{r_n}\left(1+\frac{1}{u_n}+\ldots+\frac{1}{u_n^{p-1}}\right)$$

$$\leqq \sum_{n=1}^{N} \frac{1}{r_n|1-u_n|} + \sum_{n=1}^{N} \frac{u_n^{p-1}}{r_n}(1+\kappa+\ldots+\kappa^{p-1})$$

$$\leqq \sum_{n=1}^{N} \frac{1}{r_n|1-u_n|} + K_1 \sum_{n=1}^{N} \frac{u_n^{p-1}}{r_n}$$

But $|1-u_n| > r_n^{-h-1}$.

Hence

$$|\Sigma_1| < \sum_{n=1}^{N} r_n^h + K_1 \sum_{n=1}^{N} \frac{u_n^{p-1}}{r_n} \quad < K_2 + K_1 \sum_{n=1}^{N} \frac{u_n^{p-1}}{r_n}$$

$$< K_3 \sum_{n=1}^{N} \frac{u_n^{p-1}}{r_n} \quad < K_4 \sum_{n=1}^{N} \frac{u_n^{p}}{r_n(1+u_n)^2} \tag{2}$$

where K_4 depends on κ and κ'.

Estimation of $\sum_2$: We have

$$\sum\nolimits_2 = \sum_{n=N+1}^{\infty} \left\{ \frac{-1}{z_n(1-z/z_n)} + \frac{1}{z_n}\left(1 + \frac{z}{z_n} + \ldots + \frac{z^{p-1}}{z_n^{p-1}}\right)\right\}.$$

But in $\sum_2$, $|z/z_n| < 1/\kappa < 1$. Hence

$$\sum\nolimits_2 = -\sum_{n=N+1}^{\infty} \left\{ (z/z_n)^p + (z/z_n)^{p+1} + \ldots \right\} \frac{1}{z_n}.$$

Therefore,

$$|\Sigma_2| \leqq \sum_{n=N+1}^{\infty} \frac{1}{r_n} (u_n^p + u_n^{p+1} + \ldots)$$

$$< \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n}\left(1 + \frac{1}{\kappa} + \frac{1}{\kappa^2} + \ldots\right)$$

$$= \frac{\kappa}{\kappa - 1} \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n}.$$

But $(1+u_n)^2 < (1+1/\kappa)^2$. So we get:

$$|\Sigma_2| < \frac{\kappa}{\kappa-1}(1+1/\kappa)^2 \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n(1+u_n)^2}$$

$$= K_5 \sum_{n=N+1}^{\infty} \frac{u_n^p}{r_n(1+u_n)^2}. \tag{3}$$

Hence from (1), (2) and (3) we get:

$$\left|\frac{P'(z)}{P(z)}\right| < K_6 \sum_{n=1}^{\infty} \frac{u_n^p}{r_n(1+u_n)^2}, \quad K_6 = K_6(\kappa, \kappa')$$

$$= K_6 \sum_{n=1}^{\infty} n \left\{ \frac{u_n^p}{r_n(1+u_n)^2} - \frac{u_{n+1}^p}{r_{n+1}(1+u_{n+1})^2} \right\}$$

$$= K_6 \sum_{n=1}^{\infty} n \int_{r_n}^{r_{n+1}} d\left(\frac{-(r/x)^p}{x(1+r/x)^2}\right)$$

$$= K_6 \int_0^\infty \frac{n(x)r^p}{x^p(x+r)^2} \left\{ \frac{x(p+1)+r(p-1)}{x+r} \right\} dx.$$

Now the expression written within the curly bracket inside the integral sign is bounded in $(0, \infty)$ and monotonic increasing. Hence, we have finally

$$\left| \frac{P'(z)}{P(z)} \right| < K \int_0^\infty \frac{n(x)r^p}{x^p(x+r)^2} dx.$$

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