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# Location of Zeros of Infrapolynomials.<sup>1</sup>

by

T. S. Motzkin and J. L. Walsh

## § 1. Introduction

If  $E$  is an arbitrary point set of the  $z$ -plane, the polynomial  $q(z) \equiv z^n + \dots$  is an *underpolynomial* of  $p(z) \equiv z^n + \dots$  provided we have

$$(1.1) \quad |q(z)| < |p(z)| \text{ on } E \text{ where } p(z) \neq 0,$$

$q(z) = p(z)$  on  $E$  where  $p(z) = 0$ . If  $p(z)$  has no underpolynomial on  $E$  it is called an *infrapolynomial* on  $E$ .

The importance of infrapolynomials arises from the fact that the polynomials  $p(z) \equiv z^n + \dots$  of given degree  $n$  of minimum *norm*  $\|p(z)\|$ ,

$$\|p(z)\| = \max [w(z) |p(z)|, z \text{ on } E], \quad w(z) \geq 0,$$

$$\|p(z)\|^p = \int_E w(z) |p(z)|^p |dz|, \quad E \text{ rectifiable},$$

$$\|p(z)\|^p = \iint_E w(z) |p(z)|^p dS, \quad \iint_E dS \neq 0,$$

are all infrapolynomials on  $E$ . In particular the orthogonal polynomials  $p(z) \equiv z^n + \dots$  are infrapolynomials on  $E$ . The asymptotic properties of polynomials of minimum norm are closely associated with the logarithmic capacity of  $E$  and with Green's function for the infinite component of the complement of  $E$ . Best approximation to an arbitrary function on a set  $E$  of  $n+2$  points by a polynomial of degree  $n$  is equivalent to the problem of determining the polynomial  $z^{n+1} + \dots$  of minimum norm on  $E$ , where approximation is measured in an appropriate manner and norm is defined in the corresponding manner.

In a recent paper [8] the present writers have studied the totality of infrapolynomials of given degree on a bounded set  $E$ , with especial reference to such properties as closure, connectedness, finite generation (i.e. preservation of the property of being an infrapolynomial on a finite subset of  $E$ ), convexity, and for a real

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set  $E$  oscillation or separation properties. In the present paper we investigate the geometric location of zeros (and of the centers of gravity of zeros) of infrapolynomials on both bounded and unbounded sets. The contrast between these two kinds of sets is great and our methods involve the detailed study of the behavior of unbounded sets in the neighborhood of infinity, with especial reference to the level loci of the modulus of a rational function.

Properties of convexity of the set  $E$  play a major role. Fejér has remarked that all zeros of an infrapolynomial  $z^n + \dots$  on a set  $E$  (containing at least  $n$  points) lie in the convex hull of  $E$ , a remark to which we frequently refer. It is readily shown [8, §6] that if  $E$  is a closed bounded set (of at least  $n+1$  points), the locus of zeros of all infrapolynomials of degree  $n$  on  $E$  is precisely the convex hull of  $E$ . Thus it is not to be expected that one can go beyond Fejér's remark in determining regions of the plane that are entirely free of zeros of infrapolynomials. It is for this reason that we emphasize the behavior of the center of gravity of such zeros, or (for instance) the determination of regions which can contain at most one zero.

For a closed bounded set there is identity between weak and strong infrapolynomials, but that is not the case for unbounded sets; we shall discuss this question in more detail on another occasion. But here we consider the distinction as an important one. By a *weak* underpolynomial  $q(z)$  of  $p(z)$  we mean one for which (1.1) becomes a weak inequality; by a *strong* infrapolynomial we mean an infrapolynomial which has even no weak underpolynomial.

Let us indicate in more detail the contents of the present paper. In §2 we discuss bounded generation of infrapolynomials and also finite generation on an unbounded set. In §3 we study the behavior near infinity of various point sets, especially with regard to the existence on them of noninfrapolynomials. Sets containing the boundaries of their convex hulls turn out to be significant and are considered in §4. Sets consisting of a straight line and a point plus a straight line are treated in §§5 and 6, and applications in §7. Subsets of circular discs conclude the paper with §8. We have full information on the totality of infrapolynomials for some sets: 1) collinear sets [8, §§8, 9]; 2) certain sets containing the boundaries of their convex hulls; 3) an infinite strip plus a point. Sufficient conditions for infrapolynomials on other sets are given in §3 and necessary conditions in §§7 and 8.

The writers have further results on the location of zeros of infrapolynomials on a finite set, which are reserved for a later paper.

## § 2. Sets and Subsets

Here we consider bounded generation of infrapolynomials and also generation on the boundary of a given set. We say that an infrapolynomial on an unbounded set  $E$  is *boundedly generated* if it is also an infrapolynomial on some bounded subset of  $E$ .

**THEOREM 2.1.** *Suppose  $E_k$  is a bounded set containing at least  $n+1$  points and that the sequence  $E_k$  is monotonic increasing with limit  $E$ . If  $p(z) \equiv z^n + \dots$  is a polynomial, there exists a sequence of polynomials  $q_k(z) \equiv z^n + \dots$  respectively infrapolynomials on  $E_k$  whose limit is a weak underpolynomial  $p_0(z)$  of  $p(z)$  on  $E$ .*

The sets  $E_k$  need not be closed. If  $E$  is arbitrary, containing at least  $n+1$  points, such a sequence  $E_k$  exists.

By [8, Theorem 8] there exists a weak underpolynomial  $p_k(z)$  of  $p(z)$  on  $E_k$  which is an infrapolynomial on  $E_k$ ; we have  $|p_k(z)| \leq |p(z)|$  on  $E_k$ ; here  $p_k(z) \equiv p(z)$  is not excluded. The  $p_k(z)$  are uniformly bounded on  $E_1$  and hence on any bounded set of the plane. There exists at least one limit polynomial  $p_0(z) \equiv z^n + \dots$ , the limit of some subsequence  $p_{k_j}(z)$ , where  $k_1 = 1$ . We now define  $q_k(z)$  as  $p_{k_j}(z)$ , for  $k_j \leq k < k_{j+1}$ , and the sequence  $q_k(z)$  satisfies the required conditions.

There follows at once the

**COROLLARY.** *If  $p(z)$  in Theorem 2.1 is a strong infrapolynomial on  $E$ , it is the limit of infrapolynomials  $q_k(z)$  on the respective sets  $E_k$ .*

**THEOREM 2.2.** *Let  $E$  be a closed set and let  $F$  be the locus of centers of gravity of the zeros of all proper boundedly generated infrapolynomials of fixed degree  $n$  on  $E$ . If the center of gravity  $Z$  of the zeros of such a polynomial  $p(z)$  lies on the boundary  $C$  of  $F$ , then  $p(z)$  is a finitely generated infrapolynomial on the boundary  $B$  of  $E$ .*

If  $F$  or its boundary is empty, there is nothing to prove.

The polynomial  $p(z)$  is [5] finitely generated and a factor of an infrapolynomial

$$(2.1) \quad q(z) \equiv \sum_{k=1}^{r+1} \frac{\lambda_k \omega(z)}{z - z_k}, \quad \lambda_k > 0, \quad \omega(z) \equiv \prod_{k=1}^{r+1} (z - z_k),$$

on the set  $\{z_1, z_2, \dots, z_{r+1}\}$ ,  $n \leq r \leq 2n$ .

Let the zeros of  $q(z)$  be  $\zeta_1, \zeta_2, \dots, \zeta_r$ , and those of  $p(z)$  be  $\zeta_1, \zeta_2, \dots, \zeta_n$ . For the given  $\lambda_1, \lambda_2, \dots, \lambda_{r+1}, z_2, z_3, \dots, z_{r+1}$ , and continuously varying  $z_1$ , we define  $q(z)$  by (2.1) and still denote the zeros of  $q(z)$  by  $\zeta_1, \zeta_2, \dots, \zeta_r$ ; for  $z_1$  near its original position these zeros are determined by continuity from their original positions. We show that if  $Z = (\zeta_1 + \zeta_2 + \dots + \zeta_n)/n$  lies on  $C$  then  $z_1$  lies on  $B$ .

The function  $Z = Z(z_1)$  is analytic (except perhaps for branch points) for all finite values of  $z_1$ , for the  $\zeta_k$  lie in the convex hull of the  $z_j$ , and this function (which we shall show to be not identically constant) maps a complete neighborhood of a value  $z_1$  onto a complete neighborhood of the corresponding value  $Z$ . If  $z_1$  does not lie on  $B$ ,  $Z$  does not lie on  $C$ , contrary to our assumption.

It remains to show  $Z(z_1) \neq \text{const}$ . Since the numbers  $\lambda_k$  are all positive and (as we now assume)  $z_1$  does not coincide with any other  $z_k$ ,  $z$  in (2.1) is different from the  $z_k$ , and since  $\omega(z) \neq 0$  the value  $z_1 = z_1(z)$  is uniquely determined by  $q(z)/\omega(z) = 0$ . Thus  $q(z) = 0$  can be written

$$(2.2) \quad z_1 s(z) + t(z) = 0,$$

where  $s(z)$  and  $t(z)$  are polynomials of respective degrees  $r-1$  and  $r$ , and where  $s(z_0) = t(z_0) = 0$  is impossible.

Such an equation in  $z$  as (2.2) can have no multiple zero  $z = z^{(k)}$ , except for a finite number of values  $z_1^{(k)}$  of  $z_1$ . For a multiple zero  $z$  of (2.2) implies  $z_1 s'(z) + t'(z) = 0$ , that is to say

$$(2.3) \quad s'(z)t(z) - s(z)t'(z) = 0.$$

If equation (2.3) were an identity we should have  $s(z)/t(z) \equiv \text{const}$ , an obvious impossibility. To each of the finitely many roots of (2.3) corresponds but a single value of  $z_1$ .

We choose a value  $z_1 = z_1^*$  different from  $z_2, z_3, \dots, z_{r+1}$  and from all  $z_1^{(k)}$ . For this value of  $z_1$  we denote the zeros of  $q(z)$  by  $\zeta_1^*, \zeta_2^*, \dots, \zeta_r^*$ . When  $z_1$  traces any closed path commencing at  $z_1^*$ , avoiding  $z_2, z_3, \dots, z_{r+1}$  and the  $z_1^{(k)}$ , and returning to  $z_1^*$ , the zeros  $\zeta_1, \zeta_2, \dots, \zeta_r$  commence at  $\zeta_1^*, \zeta_2^*, \dots, \zeta_r^*$  and return to (say)  $\zeta_{p_1}^*, \zeta_{p_2}^*, \dots, \zeta_{p_r}^*$ , where  $(p_1, p_2, \dots, p_r)$  is a permutation  $p$  of  $(1, 2, \dots, r)$ . The  $g$  permutations  $p$  for all admissible paths form a group  $G$ . A closed path for  $z_1$  commencing at  $z_1^*$  can be defined by requiring that  $\zeta_1$  commence at  $\zeta_1^*$  and describe a suitable path terminating in  $\zeta_j^*$ , since (2.2) determines  $z_1 = z_1(z)$  uniquely. Thus  $G$  is transitive.

If such a sum as  $\zeta_1 + \zeta_2 + \dots + \zeta_n$  were constant, then also every sum  $\zeta_{p_1} + \zeta_{p_2} + \dots + \zeta_{p_n}$  would be constant, as would be the sum  $\sum_p (\zeta_{p_1} + \zeta_{p_2} + \dots + \zeta_{p_n})$  over all  $g$  permutations of such sums. But  $\sum_p \zeta_{p_k}$  is independent of  $k$ , namely

$$(g/r)(\zeta_1 + \zeta_2 + \dots + \zeta_r).$$

The sum  $\zeta_1 + \zeta_2 + \dots + \zeta_r$  is the negative of the coefficient of  $z^{r-1}$  in (2.2), which contains  $z_1$  and is not identically constant. This completes the proof.

**THEOREM 2.3.** *If  $E$  is a closed set, then every (weak or strong) infrapolynomial  $p(z)$  on  $E$  with no zero on  $E$  is also a (weak or strong) infrapolynomial on the boundary  $B$  of  $E$ .*

If  $q(z)$  is a weak or strong underpolynomial of  $p(z)$  on  $B$ , the quotient  $q(z)/p(z)$  is not greater than unity or less than unity on  $B$  respectively. The quotient is analytic on  $E$ , hence by the maximum principle not greater than unity or less than unity also on  $E$  respectively.

### § 3. Sets bounded or restricted at infinity

The behavior in the neighborhood of infinity of an unbounded set  $E$  may be sufficient to ensure that certain polynomials are strong infrapolynomials on  $E$ .

*If  $E$  is a closed set and if there exists a nonconstant rational function*

$$(3.1) \quad f(z) \equiv \frac{z^n + a_1 z^{n-1} + \dots}{z^n + b_1 z^{n-1} + \dots} \equiv \frac{q(z)}{p(z)}$$

*such that  $|f(z)| \leq 1$  on  $E$  then  $E$  is called a substar of degree  $n$ .*<sup>2)</sup> As the reader may verify, in the neighborhood of infinity the locus  $|f(z)| = 1$ ,  $f(z) \equiv 1 + A_0 z^{-m} + \dots$ ,  $A_0 \neq 0$ ,  $1 \leq m \leq n$ , has  $m$  concurrent asymptotes which make successively equal angles with each other, and the locus  $|f(z)| \leq 1$  consists of  $m$  alternate curvilinear sectors bounded by the branches of  $|f(z)| = 1$ ; this locus  $|f(z)| \leq 1$  is roughly starlike in shape, and such a locus is to contain  $E$  if  $E$  is a substar.

This definition of substar is useful in considering  $q(z)$  as an underpolynomial of  $p(z)$  on a given set  $E$ . But if  $q(z)$  is an underpolynomial of  $p(z)$ , we know merely  $q(z) = p(z) = 0$  at an isolated point of  $E$  where  $p(z) = 0$ ; we cannot deduce  $|f(z)| \leq 1$ . Nevertheless it is possible to adjoin such isolated points (finite in number) to a known substar.

**THEOREM 3.1.** *If  $E$  is a substar of degree  $n$  and  $z_1, z_2, \dots, z_\nu$  are a finite number of points forming a set  $E_1$ , then  $E + E_1$  is a substar of degree  $n$ .*

In the plane of  $w = f(z)$ , the image of  $E$  lies in the disc  $D_1$ :  $|w| \leq 1$ . There exists an infinite disc  $D_2$ :  $|w - (1 + \varepsilon)| \geq \varepsilon (> 0)$  which contains the images of both  $E$  and  $E_1$ . A suitable linear

<sup>2)</sup> If this definition is modified so that  $f(\infty)$  is not necessarily unity, a wholly different circle of ideas appears. This new situation has been considered by S. Bernstein [1, p. 56; 2] and de Bruijn [3].

transformation maps  $D_2$  onto  $D_1$ , leaving  $w = 1$  invariant, <sup>3)</sup> and this linear transformation of  $f(z)$  defines a rational function  $g(z)$  of form (3.1) such that  $|g(z)| \leq 1$  on  $E + E_1$ .

In Theorem 3.1 it is not essential to assume  $E_1$  finite, but it is sufficient to assume for some  $\epsilon (> 0)$  the inequality  $|f(z) - (1 + \epsilon)| \geq \epsilon$  on  $E_1$ . In particular if the part of a closed set  $E$  in the neighborhood of infinity is a substar of degree  $n$ , then with the deletion from  $E$  of a finite number of arbitrarily small discs the residual subset of  $E$  is also a substar of degree  $n$ .

If there exists a sequence of points  $z_k$  in an arbitrary closed set  $E$  such that  $\lim_{k \rightarrow \infty} (z_k / |z_k|) = e^{i\theta}$ ,  $|z_k| \rightarrow \infty$ , we say that  $e^{i\theta}$  is a *direction point* of  $E$ . Such direction points form a closed subset  $\sigma$  of the unit circumference and their behavior may be sufficient to imply that  $E$  is not a substar.

**THEOREM 3.2.** *A closed unbounded set  $E$  is not a substar under either of the following conditions:*

- 1) *the measure of  $\sigma$  is greater than  $\pi$ ;*
- 2)  *$\sigma_1$  cannot be a set of direction points of a substar, where  $\sigma$  contains an arc of length greater than  $\pi/(k+1)$  and  $\sigma_1$  is the union of  $\sigma$  and an arc of length less than  $\pi/k$  whose endpoints belong to  $\sigma$ .*

Of course the procedure indicated in 2) can be continued by iteration. If  $E$  is a substar,  $\sigma$  must lie on a set  $A$  of alternate arcs of the unit circumference, which is divided into  $2m$  equal arcs. Under the conditions of 2) we have  $m < k+1$  if  $E$  is a substar, and an arc of length less than  $\pi/k$  whose endpoints belong to  $\sigma$  must lie wholly in  $A$ .

Theorem 3.2 gives sufficient but not necessary conditions that  $\sigma$  not belong to a substar; this is shown by the example that  $\sigma$  consists of arcs  $77^\circ$ ,  $5^\circ$ ,  $5^\circ$  separated by three arcs of  $91^\circ$  each. A second example, where  $\sigma$  is countably infinite and hence of measure zero, is that  $\sigma$  consists of the points  $e^{i\theta}$ ,  $\theta = 0, \pi, \pm\pi/2^k$  ( $k = 1, 2, \dots$ ).

<sup>3)</sup> Also a suitable linear transformation maps  $D_1$  onto the halfplane  $u \leq 1$  ( $w = u + iv$ ), leaving  $w = 1$  invariant. This linear transformation defines a rational function  $h(z)$  of form (3.1) such that  $\Re[h(z)] \leq 1$ ,  $h(\infty) = 1$  on  $E$ . Then the function

$$1 - h(z) \equiv \frac{0 \cdot z^n + \dots}{z^n + \dots}$$

has the property  $\Re[1 - h(z)] \geq 0$  on  $E$ ,  $1 - h(\infty) = 0$ ; the existence of such a function  $h(z)$  is necessary and sufficient that  $E$  be a substar.

If  $m = n$  for  $f(z)$  defined by (3.1), the locus  $|f(z)| = 1$  is a *stelloid*, a type studied by F. Lucas.

We remark that *every finite set  $\sigma$  belongs to a substar*. Let the direction points of  $E$  be the exponentials of  $2\pi i\varphi_j$ . By Dirichlet's theorem [6, p. 169] there exists an integer  $m$  such that all  $m\varphi_j$  differ from integers by less than  $1/4$ . Then the numbers  $e^{2\pi i m\varphi_j}$  lies on the right half of the unit circumference and the numbers  $e^{2\pi i\varphi_j}$  lie on a set  $A$  of alternate arcs.

If, for given  $E$ ,  $\sigma$  lies interior to a set  $A$  of alternate arcs of the unit circumference, which is divided into  $2m$  equal arcs, then the part of  $E$  in a suitable neighborhood of infinity is a substar. However, if  $\sigma$  merely lies in a set  $A$  of such closed arcs, supplementary conditions are here necessary.

**THEOREM 3.3.** *Let the set  $\sigma$  of direction points of a closed set  $E$  be a set  $A$  of alternate closed arcs of the unit circumference, which is divided into  $m$  equal arcs. Then a necessary condition that the part of  $E$  in some neighborhood of infinity be a substar is that the totality of distances be bounded, from points of  $E$  to the set  $B$  consisting of all halflines from 0 through points of  $A$ .*

*If  $\sigma$  contains  $A$  as a proper subset, then  $E$  is not a substar in a neighborhood of infinity.*

If  $\sigma$  contains  $A$ , the number  $m$  pertaining to  $A$  is the only possible number characteristic of the locus  $|f(z)| = 1$ ,  $f(z) \equiv 1 + A_0 z^{-m} + \dots$  used in defining „substar”; the endpoints of  $A$  give the only possible directions of asymptotes. Given  $\epsilon (> 0)$ , for  $|z|$  sufficiently large the distance from a possible locus  $|f(z)| = 1$  to one of its asymptotes is less than  $\epsilon$ ; the distance from an asymptote to  $B$  is bounded, so the distance from the set  $|f(z)| \leq 1$  and hence from  $E$  to  $B$  is bounded. In particular  $\sigma$  can contain no point not in  $A$ .

A relation of substars to infrapolynomials is given by

**THEOREM 3.4.** *Let  $E$  be a closed set and let  $F$  be a union of components of the complement of  $E$  such that  $E + F$  is not a substar of degree  $n$ . Then every polynomial  $p(z) \equiv z^n + \dots$  with no zero on  $F$  or at isolated points of the boundary  $B$  of  $F$  is a strong infrapolynomial on  $E$ .*

If  $q(z) \equiv z^n + \dots \neq p(z)$  is a weak underpolynomial of  $p(z)$  on  $E$ , consider  $f(z) \equiv q(z)/p(z)$ . On  $E$  and hence on  $B$  (contained in  $E$ ) we have  $|f(z)| \leq 1$  except perhaps at the zeros of  $p(z)$ ; in the zeros of  $p(z)$  on  $B$  (necessarily at non-isolated points of  $B$ ) we have also  $|f(z)| \leq 1$ . Thus in  $F$  we have by the principle of maximum modulus  $|f(z)| < 1$ . On  $E + F$  except perhaps at zeros of  $p(z)$  we have  $|f(z)| \leq 1$ . Hence  $E + F$  with the possible deletion of some zeros of  $p(z)$  is a substar and by Theorem 3.1 the set  $E + F$  is also a substar of degree  $n$  contrary to hypothesis.



Illustrations of Theorem 3.4 are that  $E$  is a parabola or a hyperbola of eccentricity less than  $2^{1/2}$ ; we choose  $F$  as the exterior of the curve.

A special case of Theorem 3.4 deserves explicit statement:

**THEOREM 3.5.** *Let  $E$  be a closed bounded set and let  $F$  be the unbounded component of the complement of  $E$ . Then every polynomial  $z^n + \dots$  with no zeros on  $F$  or at isolated points of the boundary of  $F$  is an infrapolynomial on  $E$ .*

#### § 4. Sets containing the boundaries of their convex hulls

If a closed set  $E$  is convex the situation is relatively simple:

**THEOREM 4.1.** *If  $E$  is a closed convex set containing more than one point, the set of infrapolynomials of degree  $n$  on  $E$  is precisely the set of polynomials  $p(z) \equiv z^n + \dots$  having all their zeros on  $E$ .*

Every infrapolynomial on  $E$  has all its zeros on  $E$  (Fejér). Conversely if  $p(z)$  has all its zeros on  $E$ , each zero is a zero of the third kind, namely a zero at a limit point of  $E$ , so  $p(z)$  is an infrapolynomial on  $E$ , by [8, Theorem 2].

If all points of a closed set  $E$  (containing more than one point) are collinear and if  $E$  contains the boundary of its convex hull,  $E$  must itself be convex, namely a finite closed interval, a closed half-line or a line; the infrapolynomials are detailed in Theorem 4.1.

If the points of  $E$  are not collinear and if  $E$  contains the boundary of its convex hull, there are the following cases: (i)  $E$  is bounded; (ii)  $E$  is unbounded and contained in a sector of angle less than  $\pi$ ; (iii) the convex hull of  $E$  is a parallel strip; (iv) the convex hull of  $E$  is a halfplane; (v) the convex hull of  $E$  is the entire plane.

**THEOREM 4.2.** *If  $E$  is a bounded closed noncollinear set which contains the boundary of its convex hull  $H$ , then the set of infrapolynomials of degree  $n$  on  $E$  is precisely the set of infrapolynomials of degree  $n$  on  $H$ .*

Every infrapolynomial on  $E$  or  $H$  has its zeros on  $H$  (Fejér). Conversely, every polynomial  $z^n + \dots$  with all its zeros on  $H$  is (Theorem 3.5) an infrapolynomial on both  $E$  and  $H$ .

**THEOREM 4.3.** *If  $E$  is a noncollinear closed set which contains the boundary of its convex hull  $H$ , and if  $H$  is a parallel strip or halfplane or is contained in a sector of angle less than  $\pi$ , then the conclusion of Theorem 4.2 holds.*

Proof as for Theorem 4.2, using Theorem 3.4 in conjunction with Theorems 3.2 and 3.3.

We have thus established the conclusion of Theorem 4.2 in the

cases (i)–(iv) enumerated before. This conclusion is not valid in the case (v). However, we have

**THEOREM 4.4.** *If  $E$  is a closed set which is not a substar of degree  $n$  then every polynomial  $z^n + \dots$  is a strong infrapolynomial on  $E$ .*

Theorem 4.4 follows from the definition of substar in conjunction with Theorem 3.1 and also from Theorem 3.4 if  $F$  is chosen the null set. Every set  $E$  of Theorem 4.4 has the entire plane as its convex hull, for if the convex hull of a set  $E$  is not the entire plane  $E$  lies in some halfplane, so  $E$  is a substar of degree unity and of every higher degree.

The converse of Theorem 4.4 is clearly true, that if every polynomial  $z^n + \dots$  is a strong infrapolynomial on a closed set  $E$ , then  $E$  is not a substar of degree  $n$ .

If  $E$  is a closed set having the entire plane as its convex hull, and is a substar of degree  $n$  ( $> 1$ ), the totality of infrapolynomials  $z^n + \dots$  on  $E$  may be quite complicated in character; indeed the set of infrapolynomials need not be closed, although [8, Theorem 3] this set is closed if  $E$  is an arbitrary bounded set.

**THEOREM 4.5.** *If  $E$  is the set  $xy = 0$ , the set of (strong or weak) infrapolynomials on  $E$  of degree two is not closed; this set consists of all polynomials  $z^2 + az + b$ ,  $a \neq 0$  and of the polynomials  $z^2 + b$ ,  $b$  real.*

The polynomial  $z^2 + b$ ,  $b$  real, has only zeros of the third kind, hence is a strong infrapolynomial on  $E$ .

Suppose the polynomial  $p(z) \equiv z^2 + az + b$  to have the underpolynomial  $q(z) \equiv z^2 + a_1z + b_1 \not\equiv p(z)$  on  $E$ . Then we write

$$(4.1) \quad f(z) \equiv \frac{q(z)}{p(z)} \equiv 1 + \frac{(a_1 - a)z + (b_1 - b)}{z^2 + az + b} \equiv 1 + \frac{a_1 - a}{z} + \dots$$

If  $a_1 \neq a$  the locus  $|f(z)| = 1$  can have but one asymptote, so near infinity one or the other of the lines  $x = 0$ ,  $y = 0$  lies partly in  $|f(z)| < 1$  and partly in  $|f(z)| > 1$ , which is impossible. If  $a_1 = a$ , equation (4.1) gives us  $f(z) \equiv 1 + (b_1 - b)/(z^2 + az + b) + \dots$ ,  $b_1 \neq b$ , and the directions of the two asymptotes of the locus  $|f(z)| = 1$ , a hyperbola with center  $-a/2$ , are not the directions of the lines  $0x$  and  $0y$  unless we have

$$(4.2) \quad \Re(b_1 - b) = 0;$$

thus (4.2) and  $a = 0$  are necessary. The polynomial  $z^2 + az + b$ ,  $a \neq 0$  has no underpolynomial.

If the conditions (4.2) and  $a = 0$  are fulfilled, the locus  $|f(z)| = 1$  is a hyperbola with the axes as asymptotes. Consequently either the locus  $|f(z)| \leq 1$  or the locus  $|f(z)| \geq 1$  must contain  $E$ , and the one

or the other occurs according to which locus contains  $0$ , namely according as  $|b_1/b| \leq 1$  or  $|b_1/b| \geq 1$ . Thus the polynomial  $z^2+b$  has the underpolynomial  $z^2+b_1$  provided merely  $|b_1/b| \leq 1$  and  $\Re(b_1-b) = 0$ . Theorem 4.5 is established.

It is striking that each of the polynomials  $z^2+i$  and  $z^2-i$  is an underpolynomial of the other.

### § 5. The straight line: underpolynomials

**THEOREM 5.1.** *Let  $E$  be the axis of reals  $0x$ . Let  $q(z) \equiv z^n + b_1 z^{n-1} + \dots$  be a weak underpolynomial of  $p(z) \equiv z^n + a_1 z^{n-1} + \dots$  on  $E$ . Then  $\Re(b_1 - a_1) = 0$ . Moreover if all zeros of  $p(z)$  lie in  $y > 0$ , and if  $q(z) \not\equiv p(z)$ , then  $\Im(b_1 - a_1) > 0$ .*

Delete the zeros of  $p(z)$  and the corresponding zeros of  $q(z)$  on  $E$  (i.e. zeros of the third kind). It is sufficient to prove the result for the remaining factors, which we continue to denote by  $p(z)$  and  $q(z)$  respectively, both of degree  $n$ . We set  $Q(z) \equiv q(z) - p(z) \equiv Az^{n-1} + \dots$ ,  $A = b_1 - a_1$ , whence on  $E$ :  $|q(z)/p(z)| \leq 1$ ,

$$\left| 1 + \frac{Q(z)}{p(z)} \right| \leq 1.$$

Then at every  $z$  on  $E$  we have either  $Q(z) = 0$  or

$$(5.1) \quad \frac{\pi}{2} < \arg [Q(z)/p(z)] < \frac{3\pi}{2}.$$

We have if  $A \neq 0$  approximately  $\arg [Q(z)/p(z)] = \arg [A/z]$  when  $z$  is numerically large, which if  $\Re(A) \neq 0$ , for  $z \rightarrow -\infty$  and  $z \rightarrow +\infty$  gives a contradiction.

We remark too that if we have  $Q(z) \equiv Az^{n-m} + \dots$ ,  $A \neq 0$ ,  $m$  odd, then  $\Re(A) = 0$ , namely  $\Re(b_m - a_m) = 0$ . If  $m$  here is even, we deduce merely  $\Re(A) \leq 0$ ,  $\Re(b_m - a_m) \leq 0$ .

If all zeros of  $p(z)$  lie in  $y > 0$ , with  $Q(z) \not\equiv 0$ , let  $r$  denote the number of distinct real zeros of  $Q(z)$  of odd order,  $r \leq n-1$ . Since  $p(z) \neq 0$  on  $E$ ,  $\arg [Q(z)/p(z)]$  is continuous on the set  $E'$ , namely  $E$  minus the zeros of  $Q(z)$  of odd order; as  $z$  traces each of the  $r+1$  components of  $E'$  from left to right, by (5.1) the net change in the argument is at most  $\pi$ , so the total net increase in the argument is numerically not greater than  $(r+1)\pi$ . However,  $\arg [p(z)]$  increases monotonically in  $-\infty < x < +\infty$  from  $-\pi$  to zero. Let  $r_0$  denote the number of nonreal zeros of  $Q(z)$ ,  $r_0 \leq n-1-r$ ; the total numerical increase in  $\arg [Q(z)]$  on  $E'$  is not greater than  $r_0\pi$  ( $\leq (n-1-r)\pi$ ). Thus the total algebraic increase on  $E'$  of  $\arg [Q(z)/p(z)]$  is not greater than  $-(1+r)\pi$ , hence must be pre-

cisely  $-(1+r)\pi$ . All real zeros of  $Q(z)$  must be of order one and all other zeros must be in the open upper halfplane. Also all net changes in  $\arg [Q(z)/p(z)]$  in any component of  $E'$  must be precisely  $-\pi$ . Consideration of the component bounded by  $+\infty$  then shows  $\Im(A) > 0$  if  $A \neq 0$ .

If  $Q_0(z) \equiv z^{r_0} + \dots$  has precisely the zeros of  $Q(z)$  in the upper halfplane, the real values of  $z$  at which  $\arg [Q(z)/p(z)]$  jumps from  $3\pi/2$  to  $\pi/2$ , namely the real zeros of  $Q(z)$  are precisely the real points where  $\arg [Q_0(z)/p(z)]$  decreases as it passes through the values  $\pi/2$  and  $-\pi/2$  in alternation (not including conceivable points where  $\pm\pi/2$  is a maximum or minimum) as  $z$  traces  $E$  from left to right. Thus the real zeros of  $Q(z)$  are uniquely determined by the zeros of  $Q_0(z)$ .

It remains to show that  $A = 0$  is impossible. If  $A = 0$  our previous discussion shows that  $r \leq n-2$ ,  $r_0 \leq n-2-r$ . The total algebraic increase in  $\arg [Q(z)/p(z)]$  is not greater than  $-(r+2)\pi$ , which contradicts the fact that the total net increase is numerically not greater than  $(r+1)\pi$ .

The proof as given establishes also the

**COROLLARY.** *If under the conditions of the first part of Theorem 5.1 all zeros of  $p(z)$  lie in  $y \geq 0$ , then we have  $\Im(b_1 - a_1) \geq 0$ . Moreover, if the zeros of  $p(z)$  ( $\neq q(z)$ ) are not all real, we have  $\Im(b_1 - a_1) > 0$ .*

## § 6. Point plus straight line

**THEOREM 6.1.** *Let  $E$  be  $0x$  plus  $i$ . If  $p(z) \equiv z^n + a_1 z^{n-1} + \dots$  has its zeros  $\zeta_i$  in  $y > 0$  and  $\Im(-a_1) < 1$ , then  $p(z)$  is an intrapolynomial on a finite subset of  $E$  consisting of  $n+1$  points.*

For real  $z$  we consider  $\arg [Q(z)]$ , where  $Q(z) \equiv p(z)/(z-i)$ . For large  $|z|$ ,  $\arg (z-i)$  behaves like  $-1/z$ , whereas  $\arg [p(z)]$  behaves like  $-\sum \Im(\zeta_i)/z = \Im(a_1)/z$ . As  $z$  traces the entire  $x$ -axis from left to right,  $\arg (z-\zeta_i)$  increases continuously from a limit of  $-\pi$  to a limit of zero, so  $\arg [Q(z)]$  varies continuously from a limit of  $-(n-1)\pi$  to a limit of zero; moreover  $\arg [Q(z)]$  approaches these two limits from below and above respectively. Then for certain values  $z_1, z_2, \dots, z_n$  we have  $\arg [Q(z_k)] = -(n-k)\pi$ . The polynomials  $\omega_k(z) \equiv \omega(z)/(z-z_k)$ , where  $\omega(z) \equiv \prod_0^n (z-z_k)$ ,  $z_0 = i$ , are linearly independent and any polynomial  $p(z) \equiv z^n + \dots$  can be expressed as

$$\sum_{k=0}^n \frac{p(z_k)\omega(z)}{\omega'(z_k)(z-z_k)} \equiv \sum_{k=0}^n \lambda_k \omega_k(z),$$

$\sum \lambda_k = 1$ . We have  $\lambda_k = p(z_k)/\omega_k(z_k)$ ,  $\arg \lambda_k = \arg [Q(z_k)] = 0$

(mod  $\pi$ ),  $k \neq 0$ ; consequently  $(\sum \lambda_k = 1) \arg \lambda_0 = \arg [p(z_0)] - \arg [\omega_0(z_0)] = 0 \pmod{\pi}$ . From this last equation it follows that the  $z_1, z_2, \dots, z_n$  are unique; otherwise there exist at least  $n+1$  such points, say two sets of such points differing only in  $z_1$ , which contradicts the unique determination of  $\arg [\omega_0(z_0)] \pmod{\pi}$ .

From the uniqueness of the  $z_k$  follows the order  $z_1 < z_2 < \dots < z_n$ . We have also  $(k > 0) \arg [\omega_k(z_k)] = \arg (z_k - i) + (n - k)\pi$ ,  $\arg \lambda_k = \arg [p(z_k)/(z_k - i)] - (n - k)\pi = 0 \pmod{2\pi}$ , and hence  $\lambda_k > 0$ . The number  $\lambda_0$  varies continuously with  $p(z)$  and since  $p(z_k) \neq 0$ , is never zero. When the zeros of  $p(z)$  are symmetric in  $0y$ , the numbers  $z_k$  will also be symmetric, whence  $\arg \lambda_0 = \arg [p(i)/\omega_0(i)] = 0 \pmod{2\pi}$ ,  $\lambda_0 > 0$ , and thus we have  $\lambda_0 > 0$  for every  $p(z)$ .

Since all the  $\lambda_k$  are positive,  $p(z)$  is [7] an infrapolynomial on the set  $z_0, z_1, \dots, z_n$ .

We shall prove also the

**COROLLARY.** *If in Theorem 6.1 we replace  $y > 0$  by  $y \geq 0$ , it follows that  $p(z)$  is an infrapolynomial on a closed bounded subset of  $E$ .*

The factors of  $p(z)$  corresponding to zeros of the third kind are clearly infrapolynomials on a bounded subset of  $E$ , namely on a suitable closed interval of  $0x$ . The remaining factor of  $p(z)$  is by Theorem 6.1 an infrapolynomial on a finite subset of  $E$ , so  $p(z)$  is [8, Theorem 2] an infrapolynomial on the set consisting of this interval of  $0x$  plus the finite subset of  $E$ .

**THEOREM 6.2.** *Under the conditions of the Corollary to Theorem 6.1 except that  $\Im(-a_1) \leq 1$ ,  $p(z)$  is a strong infrapolynomial on  $E$ .*

Suppress [8, Theorem 2] the real zeros (if any) of  $p(z)$ .

We suppose  $\Im(-a_1) = 1$ , as is allowable by Theorem 6.1. If the conclusion of Theorem 6.2 is false, there exists a weak underpolynomial  $q(z) \equiv z^n + b_1 z^{n-1} + \dots \neq p(z)$  of  $p(z)$  on  $E$ . By Theorem 5.1 we have  $\Im(-b_1) < 1$ .

Consider  $P(z) \equiv (1 - \varepsilon)p(z) + \varepsilon q(z)$  and  $Q(z) \equiv (1 - 2\varepsilon)p(z) + 2\varepsilon q(z)$  for  $\varepsilon$  ( $0 < \varepsilon \leq 1/3$ ) so small that the zeros of  $P(z)$  lie in  $y > 0$ . Then  $Q(z)$  is a weak underpolynomial of  $P(z)$  on  $E$ ; for  $w = 0$  lies in the closed half of the  $w$ -plane containing  $w = q(z)$  bounded by the perpendicular bisector of the segment from  $w = p(z)$  to  $w = q(z)$ , hence  $w = 0$  also lies in the closed half of the  $w$ -plane containing  $w = Q(z)$  bounded by the perpendicular bisector of the segment from  $w = P(z)$  to  $w = Q(z)$ . But this contradicts Theorem 6.1.

**THEOREM 6.3.** *Let  $E$  be  $0x$  plus the point  $i$ . If  $p(z) \equiv z^n + a_1 z^{n-1} + \dots$  is a boundedly generated proper infrapolynomial on  $E$ , then its roots lie in  $0 < y < 1$  and  $\mathfrak{J}(-a_1) < 1$ .*

A fundamental polynomial  $z^r + \dots$  on a set has only simple zeros, and those on the given set. A proper polynomial has no zeros on the given set.

The polynomial  $p(z)$  is [5] a factor of some infrapolynomial of degree  $r$  on a subset of  $r+1$  points of  $E$ , hence a weighted sum of fundamental polynomials with positive weights whose sum is unity. For each fundamental polynomial but one the sum of the pure imaginary parts of its zeros is unity, and for that one the sum is zero, whence  $\mathfrak{J}(-a_1) < 1$ . Since the convex hull of  $E$  is  $0 \leq y \leq 1$  and  $p(z)$  is proper, it follows that the zeros of  $p(z)$  lie in  $0 < y < 1$ .

**COROLLARY.** *If the word proper is omitted in Theorem 6.3., the zeros of  $p(z)$  lie in  $0 \leq y \leq 1$  and  $\mathfrak{J}(-a_1) \leq 1$ .*

If  $p(i) = 0$ , we write  $p(z) \equiv q(z)(z-i)$ , where [8, Lemma 5]  $q(z)$  is an infrapolynomial on  $0x$ , so the conclusion of the Corollary follows.

If  $p(i) \neq 0$ , let  $p(z)$  be an infrapolynomial on a bounded subset  $E_1$  of  $E$ . We can suppress the zeros of  $p(z)$  of the third kind on  $E_1$ ; the remaining factor is a proper infrapolynomial on  $E_1$  and therefore on  $E$ , and the conclusion of the Corollary follows from Theorem 6.3.

We note that in the Corollary the zeros of  $p(z)$  lie in  $0 \leq y < 1$  and we have  $\mathfrak{J}(-a_1) < 1$  unless  $p(i) = 0$ .

**THEOREM 6.4.** *Let  $E$  be  $0x$  plus the point  $i$ . If  $p(z) \equiv z^n + a_1 z^{n-1} + \dots$  is a (weak or strong) infrapolynomial on  $E$ , then its zeros lie in  $0 \leq y \leq 1$  and  $\mathfrak{J}(-a_1) \leq 1$ .*

Delete the zeros of  $p(z)$  on  $0x$ , without change of notation. Suppose  $\mathfrak{J}(-a_1) > 1$ , whence  $n > 1$ . Let the set  $E'$  consist of the line  $L: y = \varepsilon (> 0)$  plus the point  $i$ , where  $\varepsilon$  is chosen so small that no zeros of  $p(z)$  lie on or below  $L$ , and the sum of the distances of their zeros from  $L$  is greater than  $1-\varepsilon$ . By Theorem 2.1 and by the Corollary to Theorem 6.3 applied to  $E'$ , there exists a weak underpolynomial  $p_0(z) \equiv z^n + b_1 z^{n-1} + \dots$  of  $p(z)$  on  $E'$  for which the sum of distances of its zeros from  $L$  is not greater than  $1-\varepsilon$ . Hence we have  $p_0(z) \neq p(z)$  and  $\mathfrak{J}(b_1 - a_1) > 0$ . Consider  $f(z) \equiv p_0(z)/p(z)$  in the halfplane  $\pi: y \leq \varepsilon$ . On  $\pi$  the function  $f(z)$  is analytic and we have  $|f(z)| \leq 1$ . In  $\pi$ ,  $\max |f(z)|$  is attained only on the boundary, whence  $|f(z)| < 1$  interior to  $\pi$ , in particular on  $y = 0$ .

If  $|p_0(i)| < |p(i)|$  or if  $p(i) = 0$ , then  $p_0(z)$  is a strong underpolynomial on  $E$ , contrary to hypothesis.

If  $|p_0(i)| = |p(i)| \neq 0$ , we write

$$f(z) \equiv \frac{z^n + b_1 z^{n-1} + \dots}{z^n + a_1 z^{n-1} + \dots} \equiv 1 + \frac{b_1 - a_1}{z} + \dots,$$

so the locus  $|f(z)| = 1$  has but one asymptote. This asymptote is parallel to  $0x$  since it has no point interior to  $\pi$ . The asymptote depends continuously on  $f(z)$  and remains parallel to  $0x$  and disjoint from the halfplane  $y \leq 0$  if we modify  $p_0(z)$  slightly without changing  $\Re(a_1 - b_1) = 0$ . This change can be made so that  $|p_0(i)|$  is decreased, and also so that the entire locus  $|f(z)| = 1$  (which is originally disjoint from  $y \leq 0$ ) remains disjoint from  $y \leq 0$ . Theorem 6.4 follows.

The proof just given contains also the proof of the

**COROLLARY.** *Let  $E$  be the halfplane  $y \leq 0$  plus the point  $i$ , and let the zeros of the polynomial  $p(z) \equiv z^n + a_1 z^{n-1} + \dots$ ,  $\Im(-a_1) > 1$ , lie in the halfplane  $y \geq 0$ . Then there exists a strong underpolynomial of  $p(z)$  on  $E$ .*

## § 7. Point plus strip, and applications

**THEOREM 7.1.** *Let  $E$  denote the point set  $0 \geq y \geq A$  (where  $0 \geq A \geq -\infty$ ) plus the point  $i$ . Then  $p(z) \equiv z^n + \dots$  is a boundedly generated infrapolynomial on  $E$  if and only if all zeros of  $p(z)$  lie in  $1 \geq y \geq A$  and those zeros in  $1 \geq y > 0$  either have the sum of ordinates less than unity or consist of the single point  $i$ , a simple zero.*

If  $p(z)$  satisfies the given conditions, the bounded generation follows by the method of proof of the Corollary to Theorem 6.1.

Conversely, if  $p(z)$  is a boundedly generated infrapolynomial on  $E$ , namely an infrapolynomial on a bounded subset  $E_1$  of  $E$ , its zeros lie in the convex hull of  $E_1$ , hence in  $1 \geq y \geq A$ . We assume, as is allowable without loss of generality,  $A = -\infty$ . If  $p(i) = 0$ , the polynomial  $p(z)/(z-i)$  is an infrapolynomial on  $E_1-i$  and its zeros lie in the convex hull of  $E_1-i$ , hence in  $0 \geq y \geq A$ . If  $p(i) \neq 0$ , we suppress (without change of notation) the factors of  $p(z)$  corresponding to zeros of  $p(z)$  on or below  $0x$ . The sum of ordinates of the zeros of  $p(z)$  is now not greater than unity, by the Corollary to Theorem 6.4; we may assume this sum equal to unity. By Theorem 2.2 the polynomial  $p(z)$  is finitely generated for the boundary of  $E$ , which contradicts Theorem 6.3.

**THEOREM 7.2.** *If the words „boundedly generated” in Theorem 7.1*

are replaced by „weak” or by „strong” the condition is valid if modified to read „all the zeros of  $p(z)$  lie in  $1 \geq y \geq A$  and those zeros in  $1 \geq y > 0$  have the sum of ordinates not greater than unity.”

If  $p(z)$  satisfies the given conditions, it is a strong infrapolynomial on  $E$ , as follows from Theorem 6.2 after there are deleted from  $p(z)$  the zeros of the third kind in  $y < 0$ .

Conversely, if  $p(z)$  is a weak infrapolynomial on  $E$ , it remains such a polynomial after deletion of zeros of the third kind in  $y < 0$ . The conclusion now follows from the Corollary to Theorem 6.4.

If  $E$  is the set of Theorem 7.1 and if the points  $z_1, z_2, \dots, z_n$  lie on  $E$ , then any polynomial

$$(7.1) \quad p(z) \equiv \sum_{k=1}^n \frac{\lambda_k \omega(z)}{z-z_k}, \quad \lambda_k > 0, \quad \sum \lambda_k = 1,$$

$$\omega(z) \equiv \prod_1^n (z-z_k),$$

or any factor of  $p(z)$  is [8, Theorem 13, Lemmas 3 and 6] an infrapolynomial on  $E$ . Thus we have by the method of proof of Theorem 6.3 the following consequence of Theorem 7.1:

**THEOREM 7.3.** *If the points  $z_1, z_2, \dots, z_n$  lie on the set  $E$  of Theorem 7.1, then all zeros of  $p(z)$  defined by (7.1) or of any factor of  $p(z)$  lie in the strip  $1 \geq y \geq A$ , and those zeros in  $1 \geq y \geq 0$  have the sum of ordinates less than unity. If all zeros of  $p(z)$  lie in  $y \geq 0$ , the sum of their ordinates is not greater than  $1-\lambda_1$ , where  $z_1 = i$ .*

Theorem 7.3 is of particular interest because the zeros of  $p(z)$  are precisely the zeros of the derivative of the function  $\prod_1^n (z-z_k)^{\lambda_k}$ , and the zeros of  $p(z)$  may be precisely the zeros of the derivative of a polynomial with the exception of the zeros of the original polynomial [cf. 7].

Still another application of Theorem 7.1 is

**THEOREM 7.4.** *Let  $E$  be the union of a point 0 and a convex set  $E_1$  to which 0 is exterior. If  $p(z) \equiv z^n + \dots$  is a boundedly generated infrapolynomial on  $E$ , the center of gravity of the zeros of  $p(z)$  lies in 0 or in the convex hull  $H$  of  $E_1$  plus the set  $E_2$  found by shrinking  $E_1$  toward 0 in the ratio  $n: n-1$ .*

A variable line  $L$  separating 0 from  $E_1$  separates the zeros of  $p(z)$  (other than perhaps 0 itself) from 0 and the center of gravity of all zeros of  $p(z)$  lies in the halfplane containing  $E_1$  bounded by the line found by shrinking  $L$  toward 0 in the ratio  $n: n-1$ . The center of gravity of all zeros of  $p(z)$  also lies in the convex hull of  $E$ , from which the conclusion follows.

We also have the



**COROLLARY.** *If  $E$  is the set of Theorem 7.4, and if the points  $z_1, z_2, \dots, z_n$  lie on  $E$  with  $z_1 = 0$ , then the center of gravity of the zeros of  $p(z)$  defined by (7.1) lies in the set  $E_3$  found by shrinking  $E_1$  toward 0 in the ratio  $n - 1 : n - 2 + \lambda_1$ .*

We add the related remark that if  $E$  contains at least  $n + 1$  points and consists of a set  $E_1$  plus a point  $P: z = z_0$  not in the convex hull  $H$  of  $E_1$ , and if an infrapolynomial  $p(z) \equiv z^n + \dots$  on  $E_1 + P$  vanishes in  $P$ , then all other zeros of  $p(z)$  lie in  $H$ ; indeed [8, Lemma 5], the polynomial  $p(z)/(z - z_0)$  is an infrapolynomial on  $E_1$  and its zeros lie in  $H$ . In particular  $P$  is not a double zero of  $p(z)$ ; also a suitably chosen neighborhood of  $P$  cannot contain a double zero of  $p(z)$  (Theorem 7.2; for bounded  $E$  also Theorem 8.2,  $r_1 = 0$ ).

In the present study of zeros of infrapolynomials, the following theorem due to De Bruijn and Springer [4] is significant:

**THEOREM 7.5.** *For  $n > 1$  and  $\lambda_k > 0$ , if  $\beta_1, \beta_2, \dots, \beta_{n-1}$  are the zeros of*

$$p^*(z) \equiv \sum_1^n \frac{\lambda_k}{z - \alpha_k}, \quad \sum_1^n \lambda_k = 1,$$

then we have

$$(7.2) \quad \sum_1^{n-1} |\Im(\beta_k)| \leq \sum_1^n |\Im(\alpha_k)| - \sum_1^n \lambda_k |\Im(\alpha_k)|.$$

Theorem 6.3 follows at once from Theorem 7.5.

A corollary of Theorem 7.5 is that the sum of the distances of the  $\beta_k$  to  $0x$  (or to any other line) is not greater than the sum of the distances of the  $\alpha_k$  to  $0x$  (or to the other line).

From the definition of  $p^*(z)$  in Theorem 7.5 we may write

$$\sum_1^{n-1} \beta_k = \sum_1^n \alpha_k - \sum_1^n \lambda_k \alpha_k,$$

and by taking pure imaginary parts

$$(7.3) \quad \sum_1^{n-1} \Im(\beta_k) = \sum_1^n \Im(\alpha_k) - \sum_1^n \lambda_k \Im(\alpha_k).$$

We introduce the notation  $2\Im^+(z) \equiv |\Im(z)| + \Im(z)$ , so  $\Im^+(z) \equiv \Im(z)$  if  $\Im(z) \geq 0$ ,  $\Im^+(z) = 0$  if  $\Im(z) \leq 0$ . Addition of (7.2) and (7.3) yields

$$(7.4) \quad \sum_1^{n-1} \Im^+(\beta_k) \leq \sum_1^n \Im^+(\alpha_k) - \sum_1^n \lambda_k \Im^+(\alpha_k),$$

whence we have (this result is implicit with De Bruijn and Springer)

**THEOREM 7.6.** *Under the conditions of Theorem 7.5, the sum of the positive ordinates of the  $\beta_k$  is not greater than the sum of the positive ordinates of the  $\alpha_k$  and is less than the latter sum unless the latter sum is zero.*

An easily formulated corollary applies to the sums of the distances of the  $\alpha_k$  and  $\beta_k$  lying on one side of any line.

An application of Theorems 7.5 and 7.6 is

**THEOREM 7.7.** *Let  $E$  be a point set consisting of a non-empty subset  $E_0$  in the halfplane  $y > 0$  plus the subset  $E_1 = E - E_0$  in the halfplane  $y \leq 0$ . If  $p(z)$  is a finitely generated proper infrapolynomial on  $E$ , the sum of the absolute values of the ordinates of the zeros of  $p(z)$  is less than the sum of the absolute values of the ordinates of the points of  $E$ ; the sum of the positive ordinates of the zeros of  $p(z)$  is less than the sum of the ordinates of the points of  $E_0$ .*

Theorem 7.7 can be sharpened if the structure (7.1) of  $p(z)$  is known, that is, if the  $\lambda_k$  are known; compare (7.2) and (7.4).

If  $E_0$  consists of the single point  $z = i$ , we deduce from Theorem 7.7 the second part of Theorem 7.1, and with (7.4) deduce also Theorem 7.3. Moreover, in the first part of Theorem 7.3 it is true that the sum of the ordinates of the zeros of  $p(z)$  in the strip  $1 \geq y \geq 0$  is not greater than  $1 - \lambda_1$ , where  $z_1 = i$ ; that the ordinate of each zero of  $p(z)$  is not greater than  $1 - \lambda_1$  follows also by a limiting case of Theorem 8.1; compare [9, § 1.5].

## § 8. Subsets of several circular discs

The frequently used result of Fejér can be expressed in the following form, so far as concerns bounded sets: *If a set  $E$  consists of at least  $n$  points and lies in the (circular) disc  $C$ , then all zeros of an infrapolynomial  $z^n + \dots$  on  $E$  lie in  $C$ .* More general situations of interest, where  $E$  may lie in several circular discs, are now to be considered. These new results are analogous to, and derived from, known results on the location of zeros of the derivative of a polynomial.

We state first a theorem due to Walsh [9, §1.5], in a form suited to our present discussion:

**THEOREM 8.1.** *If the points  $z'_1, z'_2, \dots, z'_m$  lie in the disc  $C_1$ :  $|z - \alpha'| \leq r'$  and the points  $z''_1, z''_2, \dots, z''_n$  lie in the disc  $C_2$ :  $|z - \alpha''| \leq r''$ , then all zeros of the function*

$$(8.1) \quad f(z) \equiv \sum_{k=1}^m \frac{\lambda'_k}{z - z'_k} + \sum_{k=1}^n \frac{\lambda''_k}{z - z''_k}, \quad \lambda'_k > 0, \lambda''_k > 0, \quad \sum \lambda'_k + \sum \lambda''_k = 1,$$

lie in  $C_1, C_2$ , and the disc

$$(8.2) \quad C: |z - \sum \lambda'_k \alpha' - \sum \lambda''_k \alpha''| \leq \sum \lambda'_k r' + \sum \lambda''_k r'.$$

If any of the discs  $C_1, C_2, C$  is disjoint from the other two discs, it contains precisely  $m-1, n-1$ , or 1 zero of  $f(z)$  respectively.

If  $C_1$  and  $C_2$  have two external tangents,  $C$  also is tangent to them; indeed for fixed  $\alpha'$  and  $\alpha''$  but variable  $\lambda'_k$  and  $\lambda''_k$ , the class of discs  $C$  is precisely the totality of discs with centers on the segment  $\alpha'\alpha''$  tangent to those external tangents.

We may state at once the

**COROLLARY.** *Under the conditions of Theorem 8.1, if a zero  $z_0$  of  $f(z)$  lies exterior to  $C_1$  and  $C_2$ , some disc of the class  $C$  containing  $z_0$  contains all zeros of  $f(z)$  exterior to  $C_1$  and  $C_2$ ; in particular, if no disc of the class containing  $z_0$  intersects  $C_1$  or  $C_2$ , no other zero of  $f(z)$  lies exterior to  $C_1$  and  $C_2$ .*

Under the conditions of the Corollary, if  $r'$  and  $r''$  are small relative to  $|\alpha' - \alpha''|$ , there exists a region  $D$  bounded by a segment of each of the external tangents to  $C_1$  and  $C_2$  and by the arcs  $A$  (assumed not to intersect, not tangent to  $C_1$  or  $C_2$ ) intercepted by them on the two circles of the class  $C$  which are tangent respectively to  $C_1$  and  $C_2$ ; if the two arcs  $A$  intersect, the region  $D$  becomes the sum of two disjoint regions; if  $z_0$  lies in  $D$ , no disc  $C$  containing  $z_0$  can intersect  $C_1$  or  $C_2$ , so no zero of  $f(z)$  other than  $z_0$  lies exterior to  $C_1$  and  $C_2$ . Otherwise expressed, the convex hull of  $C_1$  and  $C_2$  minus  $D$  contains all zeros of  $f(z)$  except perhaps one.

Although the Corollary is less precise than Theorem 8.1, it has the present advantage of not involving the  $\lambda'_k$  and the  $\lambda''_k$ , and thus can be applied to the study of arbitrary infrapolynomials.

**THEOREM 8.2.** *Let  $C_1: |z - \alpha'| \leq r'$  and  $C_2: |z - \alpha''| \leq r''$  be discs having two common external tangents, and let  $C$  denote generically an arbitrary disc having with  $C_1$  and  $C_2$  these same external tangents and whose center lies on the segment  $\alpha'\alpha''$ . If a closed set  $E$  containing at least  $n+1$  points lies in  $C_1$  and  $C_2$ , then all zeros of an arbitrary infrapolynomial  $p(z) \equiv z^n + \dots$  on  $E$  lie in  $C_1, C_2$ , and some  $C$  depending on  $p(z)$ . Thus if a zero  $z_0$  of  $p(z)$  lies exterior to  $C_1$  and  $C_2$ , some disc  $C$  containing  $z_0$  contains all zeros of  $p(z)$  exterior to  $C_1$  and  $C_2$ ; moreover, if no disc  $C$  containing  $z_0$  intersects  $C_1$  or  $C_2$ , no other zero of  $f(z)$  lies exterior to  $C_1$  and  $C_2$ .*

The zeros of  $p(z)$ , which we may assume proper, are [5] among those of some function  $f(z)$  defined by (8.1), where the  $z'_k$  and  $z''_k$  lie on  $E$ , so Theorem 8.2 is a consequence of Theorem 8.1 and its Corollary. The degenerate case  $r' = 0$  is not excluded in Theorem

8.2; in this case a suitably chosen neighborhood of  $\alpha'$  cannot contain two zeros of  $p(z)$ .

If the set  $E$  possesses various kinds of symmetry, Theorem 8.1 and therefore also Theorem 8.2 may be capable of improvement [cf. 9, §3.7].

**THEOREM 8.3.** *Under the conditions of Theorem 8.2 suppose  $\alpha'' = \bar{\alpha}'$ ,  $r' = r''$ ,  $|\alpha' - \alpha''| > 4r'$ . Then  $p(z)$  cannot have more than one real zero. An infrapolynomial  $p(z)$  cannot have two conjugate nonreal zeros exterior to  $C_1$  and  $C_2$ . A real infrapolynomial has at most one zero, necessarily real, exterior to  $C_1$  and  $C_2$ .*

If  $p(z)$  has two real zeros, necessarily on the interval  $|x - (\alpha' + \alpha'')/2| \leq r'$ , they are zeros of an infrapolynomial of degree two, hence [5] zeros of some  $f(z)$  defined by (8.1), where the points  $z'_k$  and  $z''_k$  lie on  $E$ . Then these zeros are zeros also of the function

$$f_1(z) \equiv \sum_{k=1}^m \frac{\lambda'_k}{z - z'_k} + \sum_{k=1}^n \frac{\lambda''_k}{z - z''_k},$$

and are zeros of the function  $f(z) + f_1(z)$ . The notation can be chosen so that the points  $z'_k$  and  $\bar{z}'_k$  lie in  $C_1$  and the points  $z''_k$  and  $\bar{z}''_k$  in  $C_2$ . For the function  $f(z) + f_1(z)$  the center of the disc  $C$  of Theorem 8.1 is  $(\alpha' + \alpha'')/2$  and  $C$  is disjoint from  $C_1$  and  $C_2$ , so  $C$  contains precisely one zero of  $f(z) + f_1(z)$ . Thus  $C$  cannot contain two zeros of  $p(z)$ . This proof shows also that  $p(z)$  cannot have a real zero of multiplicity greater than unity.

If  $p(z)$  has two conjugate nonreal zeros exterior to  $C_1$  and  $C_2$ , these zeros are zeros also of some  $f(z)$  defined by (8.1), and zeros of  $f(z) + f_1(z)$  defined as before. These zeros must lie in  $C$ , yet  $C$  cannot contain two zeros of  $f(z) + f_1(z)$ .

Theorem 8.3 is stronger than Theorem 8.2 with the present hypothesis, for a disc  $C$  (of Theorem 8.2) may intersect  $C_1$  and still contain a segment of  $0x$ . In Theorem 8.3 the set  $E$  need not be symmetric in  $0x$ .

Another case of symmetry is [cf. 9, §3.7]

**THEOREM 8.4.** *Under the conditions of Theorem 8.2 suppose  $\alpha'' = -\alpha'$ ,  $r' = r''$ ,  $|\alpha| > 2r'$ . Then any pair  $(\zeta, -\zeta)$  of distinct opposite zeros of  $p(z)$  lies in  $C_1$  and  $C_2$ .*

If  $\zeta$  and  $-\zeta$  are zeros of  $p(z)$ , they are zeros of an infrapolynomial of degree two, hence zeros of  $f(z)$  defined by (8.1), where the points  $z'_k$  and  $z''_k$  lie on  $E$ . Then  $\zeta$  and  $-\zeta$  are zeros also of

$$f_1(z) \equiv \sum_{k=1}^m \frac{\lambda'_k}{z + z'_k} + \sum_{k=1}^n \frac{\lambda''_k}{z + z''_k},$$

and are zeros of  $f(z)+f_1(z)$ . We may assume that the points  $z'_k$  and  $-z''_k$  lie in  $C_1$ , the points  $z''_k$  and  $-z'_k$  in  $C_2$ . The function  $f(z)+f_1(z)$  obviously admits the zero  $z = 0$ , and has (by Theorem 8.1) no other zero exterior to  $C_1$  and  $C_2$ , so  $\zeta$  and  $-\zeta$  lie in  $C_1$  and  $C_2$ . If  $z = 0$  is a zero of  $p(z)$  it is necessarily a simple zero. The set  $E$  need not be symmetric in 0. Fekete [5] proved that for  $E$  symmetric in 0, under the conditions of Theorem 8.4, the zeros other than 0 of all odd infrapolynomials lie in  $C_1$  and  $C_2$ , a conclusion which is contained in Theorem 8.4.

Theorem 8.1 admits an extension, which we formulate in a special case [cf. 9, § 3.3]:

**THEOREM 8.5.** *If the points  $z_1^{(j)}, z_2^{(j)}, \dots, z_{n_j}^{(j)}$  lie in the discs  $C_j$ ;  $|z-\alpha^{(j)}| \leq r$ , where  $\alpha^{(j)}$  is real,  $j = 1, 2, \dots, m$ , then all zeros of the function*

$$(8.3) \quad f(z) \equiv \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\lambda_k^{(j)}}{z-z_k^{(j)}}, \quad \lambda_k^{(j)} > 0, \quad \sum_{j=1, k=1}^{j=m, k=n_j} \lambda_k^{(j)} = 1$$

lie in the discs  $C_j$  and in the discs  $C'_j$  of common radius  $r$  whose centers are the zeros  $\alpha_1, \alpha_2, \dots, \alpha_{m-1}$  of

$$(8.4) \quad \varphi(z) \equiv \sum_{j=1}^n \sum_{k=1}^{n_j} \frac{\lambda_k^{(j)}}{z-\alpha^{(j)}}.$$

If any of the discs  $C_j$  or  $C'_j$  is disjoint from all the other discs, it contains precisely  $n_j-1$  zeros or 1 zero of  $f(z)$  respectively.

Let us suppose the notation  $\alpha^{(1)} < \alpha_1 < \alpha^{(2)} < \dots < \alpha_{m-1} < \alpha^{(m)}$ . A corollary to Theorem 8.5 is readily formulated analogous to that of Theorem 8.1; we proceed directly to the analogue of Theorem 8.2:

**THEOREM 8.6.** *Let the discs  $C_j$ ;  $|z-\alpha^{(j)}| \leq r$  be given, where  $\alpha^{(j)}$  is real,  $j = 1, 2, \dots, m$ ,  $\alpha^{(j+1)} > \alpha^{(j)}$ , and let  $C'_j$  denote generically an arbitrary disc of radius  $r$  whose center lies on the segment  $\alpha^{(j)} < x < \alpha^{(j+1)}$ ,  $j = 1, 2, \dots, m-1$ . If a closed set  $E$  containing at least  $n+1$  points lies in the discs  $C_j$ , then all zeros of an arbitrary infrapolynomial  $p(z) \equiv z^n + \dots$  on  $E$  lie in the  $C_j$  and some fixed set  $C'_j$  ( $j = 1, 2, \dots, m-1$ ). If a zero  $z_0$  of  $p(z)$  exterior to  $C_j$  and  $C_{j+1}$  lies in the infinite strip  $\alpha^{(j)} < x < \alpha^{(j+1)}$ , some disc  $C'_j$  contains  $z_0$  and all the other zeros of  $p(z)$  in that strip exterior to  $C_j$  and  $C_{j+1}$ ; in particular, if no disc  $C'_j$  containing  $z_0$  intersects  $C_j$  or  $C_{j+1}$ , no other zero of  $p(z)$  lies in the strip mentioned.*

Of course all zeros of  $p(z)$  lie in the strip  $|y| \leq r$ . Theorem 8.6 follows from Theorem 8.5; details are left to the reader. Both

theorems extend to discs  $C_j$  which are not equal but have two common external tangents. Theorem 8.6 is analogous to, and indeed a generalization of, the separation properties [8, §§8, 9] for real infrapolynomials.

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