COMPOSITIO MATHEMATICA

A. M. OSTROWSKI

Three theorems on products of power series

Compositio Mathematica, tome 14 (1959-1960), p. 41-49
<http://www.numdam.org/item?id=CM_1959-1960__14__41_0>

Three Theorems on Products of Power Series.

by

A. M. Ostrowski [1])

1. All numbers considered in the following are real. By a well-known theorem of Cauchy, from the relations

$$\frac{s_n}{S_n} \to \alpha \; (n \to \infty), \quad S_\nu > 0, \quad \sum_{\nu=0}^{\infty} S_\nu = \infty$$

follows

$$\frac{\sum\limits_{\nu=0}^{n} s_\nu}{\sum\limits_{\nu} S_\nu} \to \alpha.$$

More generally we have

$$\underline{\lim}_{n\to\infty} \frac{s_n}{S_n} \leqq \underline{\overline{\lim}}_{n\to\infty} \frac{\sum\limits_{\nu=0}^{n} s_\nu}{\sum\limits_{\nu=0}^{n} S_\nu} \leqq \overline{\lim}_{n\to\infty} \frac{s_n}{S_n}.$$

This can be interpreted in the following way. Put

$$\varphi(z) = \sum_{\nu=0}^{\infty} s_\nu z^\nu, \qquad \Phi(z) = \sum_{\nu=0}^{\infty} S_\nu z^\nu \tag{1}$$

and

$$\frac{\varphi(z)}{1-z} = \sum_{\nu=0}^{\infty} r_\nu z^\nu, \qquad \frac{\Phi(z)}{1-z} = \sum_{\nu=0}^{\infty} R_\nu z^\nu;$$

then we have

$$\overline{\lim}_{n\to\infty} \frac{s_n}{S_n} \geqq \underline{\overline{\lim}}_{n\to\infty} \frac{r_n}{R_n} \geqq \underline{\lim}_{n\to\infty} \frac{s_n}{S_n}. \tag{2}$$

We are first going to prove that this result remains true if the series $\frac{1}{1-z}$ is replaced by

$$\Psi(z) = \sum_{\nu=0}^{\infty} T_\nu z^\nu, \qquad T_\nu > 0 \;\; (\nu = 0, 1, \ldots), \tag{3}$$

[1]) The preparation of this paper was sponsored in part by the Office of Naval Research.

provided that we have

$$\overline{\lim_{\nu\to\infty}} \frac{T_{\nu+1}}{T_\nu} \leqq 1. \tag{4}$$

We then obtain the following theorem which appears to be quite useful although very easy to prove.

2. THEOREM 1. *Consider the power series* (1) *and* (3) *with positive* S_ν, T_ν *and assume that*

$$\sum_{\nu=0}^{\infty} S_\nu = \infty \tag{5}$$

and (4) *holds. If we then put*

$$\varphi(z)\Psi(z) = f(z) = \sum_{\nu=0}^{\infty} r_\nu z^\nu, \qquad \Phi(z)\Psi(z) = F(z) = \sum_{\nu=0}^{\infty} R_\nu z^\nu, \tag{6}$$

we have the relation (2).

3. We prove first the

LEMMA. *Under the hypothesis of the theorem* 1 *we have for any fixed integer* m

$$\frac{T_{n-m}}{R_n} \to 0 \quad (n \to \infty). \tag{7}$$

PROOF. We have by (4) for any fixed $k > 0$

$$\overline{\lim_{\nu\to\infty}} \frac{T_{\nu+k}}{T_\nu} \leqq 1, \tag{8}$$

and therefore

$$\overline{\lim_{n\to\infty}} \frac{T_{n-m}}{R_n} \leqq \overline{\lim_{n\to\infty}} \frac{T_{n-|m|}}{R_n}.$$

It is therefore sufficient to prove that for $m \geqq 0$

$$\sum_{\nu=0}^{n} \frac{S_\nu T_{n-\nu}}{T_{n-m}} \to \infty \ (n \to \infty).$$

Choose an integer $k > m$, then we have by (8)

$$\underline{\lim_{n\to\infty}} \sum_{\nu=0}^{n} S_\nu \frac{T_{n-\nu}}{T_{n-m}} \geqq \underline{\lim} \sum_{\nu=m}^{k} S_\nu \frac{T_{n-m-(\nu-m)}}{T_{n-m}} \geqq \sum_{\nu=m}^{k} S_\nu,$$

and our assertion follows from (5).

4. PROOF OF THE THEOREM 1. It is sufficient to prove the right hand inequality in (2), since we can replace s_ν by $-s_\nu$; thus we have to prove

$$(9) \qquad \overline{\lim_{n\to\infty}} \frac{\sum_{\nu=0}^{n} s_\nu T_{n-\nu}}{\sum_{\nu=0}^{n} S_\nu T_{n-\nu}} \leqq \overline{\lim_{\nu\to\infty}} \frac{s_\nu}{S_\nu}.$$

We can assume that the right hand side of (9) is $< \infty$. But then (9) follows at once if we prove:

If for a constant c and an integer m we have

$$(10) \qquad \frac{s_\nu}{S_\nu} \leqq c \qquad (\nu \geqq m),$$

then we have also

$$(11) \qquad \overline{\lim_{n\to\infty}} \frac{\sum_{\nu=0}^{n} s_\nu T_{n-\nu}}{\sum_{\nu=0}^{n} S_\nu T_{n-\nu}} \leqq c.$$

To prove (11) under the condition (10), put

$$\operatorname*{Max}_{\nu<m} |s_\nu - cS_\nu| = d$$

and consider

$$\frac{1}{R_n}\sum_{\nu=0}^{n} s_\nu T_{n-\nu} - c = \frac{1}{R_n}\sum_{\nu=0}^{n} (s_\nu - cS_\nu) T_{n-\nu} \leqq d \sum_{\mu=0}^{m} \frac{T_{n-\mu}}{R_n}.$$

But by our Lemma each of the $m+1$ terms $\frac{T_{n-\mu}}{R_n}$ on the right tends to 0 and (11) follows.

5. If one of the conditions (4) or (5) of the theorem 1 is not satisfied sometimes the following corollary can be used.

COROLLARY 1. *The relation* (2) *of the theorem* 1 *remains true if the conditions* (4) *and* (5) *are replaced by*

$$(12) \qquad \overline{\lim_{\nu\to\infty}} \frac{T_{\nu+1}}{T_\nu} < \underline{\lim_{\nu\to\infty}} \sqrt[\nu]{S_\nu}.$$

To prove this choose a positive γ such that

$$(13) \qquad \overline{\lim_{\nu\to\infty}} \frac{T_{\nu+1}}{T_\nu} < \frac{1}{\gamma} < \underline{\lim_{\nu\to\infty}} \sqrt[\nu]{S_\nu}$$

and put

$$(14) \qquad s'_\nu = s_\nu \gamma^\nu, \qquad S'_\nu = S_\nu \gamma^\nu, \qquad T'_\nu = T_\nu \gamma^\nu.$$

Then in multiplying (13) by γ we obtain from (14)

$$\overline{\lim_{\nu\to\infty}} \frac{T'_{\nu+1}}{T'_\nu} < 1 < \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{S'_\nu}. \tag{15}$$

But then it follows that the radius of convergence of $\sum_{\nu=0}^{\infty} S'_\nu z^\nu$ is < 1 and therefore $\sum_{\nu=0}^{\infty} S'_\nu = \infty$. The theorem 1 can be applied to the sequences (14) and we obtain, putting

$$r'_n = \sum_{\nu=0}^{n} S'_\nu T'_{n-\nu} = \gamma^n r_n, \qquad R'_n = \sum_{\nu=0}^{n} S'_\nu T'_{n-\nu} = \gamma^n R_n,$$

from

$$\overline{\lim_{n\to\infty}} \frac{s'_n}{S'_n} \geq \overline{\lim_{n\to\infty}} \frac{r'_n}{R'_n} \geq \underline{\lim_{n\to\infty}} \frac{s'_n}{S'_n}$$

again (2).

6. If we take in the theorem 1, $S_\nu \equiv 1$, then our result contains a „summability" statement; the corresponding transformation is then in Hardy's terminology [2]) a *regular and positive transformation* (Hardy p. 52). If the theorem 1 is applied twice starting with $S_\nu \equiv 1$ we obtain an "inclusion theorem" (Hardy p. 66) with a statement more special and more general than that given by Hardy (Hardy p. 67, Theorem 19), since this inclusion theorem does not assume the convergence of the S_ν but refers to the more general situation (2).

7. We obtain another corollary from theorem 1 if we put there $s_\nu = S_{\nu+1}$. Then we have

$$z\varphi(z) = \Phi(z) - S_0, \qquad z\varphi(z)\Psi(z) = \Phi(z)\Psi(z) - S_0\Psi(z),$$

$$r_\nu = R_{\nu+1} - S_0 T_{\nu+1},$$

$$\frac{r_\nu}{R_\nu} = \frac{R_{\nu+1}}{R_\nu} - S_0 \frac{T_{\nu+1}}{R_\nu}.$$

But here we have by the lemma of No. 3, $\frac{T_{\nu+1}}{R_\nu} \to 0$ $(\nu \to \infty)$, therefore

$$\overline{\lim_{\nu\to\infty}} \frac{r_\nu}{R_\nu} = \overline{\lim_{\nu\to\infty}} \frac{R_{\nu+1}}{R_\nu}, \qquad \underline{\lim_{\nu\to\infty}} \frac{r_\nu}{R_\nu} = \underline{\lim_{\nu\to\infty}} \frac{R_{\nu+1}}{R_\nu},$$

and obtain from (2) the

Corollary 2. *Consider the power series*

$$\Phi(z) = \sum_{\nu=0}^{\infty} S_\nu z^\nu, \quad \Psi(z) = \sum_{\nu=0}^{\infty} T_\nu z^\nu, \quad \Phi(z)\Psi(z) = \sum_{\nu=0}^{\infty} R_\nu z^\nu \tag{16}$$

[2]) G. H. Hardy, Divergent Series, Oxford (1949).

and assume that S_ν and T_ν are positive and the relations (4) *and* (5) *hold. Then we have*

$$\overline{\lim_{n\to\infty}} \frac{S_{n+1}}{S_n} \geqq \overline{\lim_{n\to\infty}} \frac{R_{n+1}}{R_n} \geqq \underline{\lim_{n\to\infty}} \frac{S_{n+1}}{S_n}. \tag{17}$$

8. If we drop now the assumption (5) we can still prove a result in the direction of the Corollary 2 though considerably weaker.

THEOREM 2. *Consider two power series with positive coefficients*

$$\sum_{\nu=0}^{\infty} S_\nu z^\nu, \qquad \sum_{\nu=0}^{\infty} T_\nu z^\nu \tag{18}$$

and assume that we have

$$\frac{S_{\nu+1}}{S_\nu} \to 1, \qquad \frac{T_{\nu+1}}{T_\nu} \to 1 \qquad (\nu \to \infty); \tag{19}$$

then putting

$$\left(\sum_{\nu=0}^{\infty} S_\nu z^\nu\right)\left(\sum_{\nu=0}^{\infty} T_\nu z^\nu\right) = \sum_{\nu=0}^{\infty} R_\nu z^\nu \tag{20}$$

we have

$$\frac{R_{\nu+1}}{R_\nu} \to 1 \qquad (\nu \to \infty). \tag{21}$$

9. PROOF. Assume ε arbitrary with $0 < \varepsilon < 1$; then there exists an $n_0 = n_0(\varepsilon)$ such that for $\nu \geqq n_0$

$$(1-\varepsilon)S_\nu \leqq S_{\nu+1} \leqq (1+\varepsilon)S_\nu, \tag{22}$$

$$(1-\varepsilon)T_\nu \leqq T_{\nu+1} \leqq (1+\varepsilon)T_\nu. \tag{23}$$

For an integer $m > n_0$ and for $n > m+n_0$ we use

$$R_{n+1} = (S_{n+1}T_0 + \dots + S_{m+1}T_{n-m}) + (S_m T_{n-m+1} + \dots + S_0 T_{n+1}).$$

Apply the right hand inequalities of (22) and (23) in the first resp. the second parenthesis; then we have

$$R_{n+1} \leqq (1+\varepsilon)(S_n T_0 + \dots + S_m T_{n-m}) + (1+\varepsilon)(S_m T_{n-m} + \dots + S_0 T_n)$$

which gives

$$R_{n+1} \leqq (1+\varepsilon)(R_n + S_m T_{n-m}).$$

Applying the left hand sides of inequalities (22) and (23) in the same way, we get

$$R_{n+1} \geqq (1-\varepsilon)(R_n + S_m T_{n-m}) > (1-\varepsilon)R_n,$$

$$1-\varepsilon \leqq \frac{R_{n+1}}{R_n} \leqq (1+\varepsilon)\left(1 + \frac{S_m T_{n-m}}{R_n}\right). \tag{24}$$

Take an arbitrarily great but fixed integer K and assume ε and m such that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^K > \frac{1}{2}, \qquad m-K > n_0(\varepsilon).$$

Then for any $n > m+K+n_0$ we have

$$\begin{aligned} R_n &\geqq S_m T_{n-m} + \sum_{k=1}^{K} S_{m-k} T_{n-m+k} \\ &\geqq S_m T_{n-m} + \sum_{k=1}^{K} \frac{S_m}{(1+\varepsilon)^k} (1-\varepsilon) T_{n-m} \\ &> (K+1)\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^K S_m T_{n-m} > \frac{K+1}{2} S_m T_{n-m}, \end{aligned}$$

hence from (24)

$$(1-\varepsilon) \leqq \frac{R_{n+1}}{R_n} \leqq (1+\varepsilon)\left(1+\frac{2}{K+1}\right)$$

and (21) is proved.

10. We give finally a third theorem on products of power series which follows easily from the theorem 1.

THEOREM 3. *Consider the four series*

$$(25) \quad \varphi(z)=\sum_{\nu=0}^{\infty} s_\nu z^\nu, \quad \Phi(z)=\sum_{\nu=0}^{\infty} S_\nu z^\nu, \quad \psi(z)=\sum_{\nu=0}^{\infty} t_\nu z^\nu, \quad \Psi(z)=\sum_{\nu=0}^{\infty} T_\nu z^\nu,$$

put

$$\varphi(z)\psi(z) = \sum_{\nu=0}^{\infty} r_\nu z^\nu, \qquad \Phi(z)\Psi(z) = \sum_{\nu=0}^{\infty} R_\nu z^\nu$$

and assume that S_ν and T_ν are positive and

$$(26) \qquad \sum_{\nu=0}^{\infty} S_\nu = \infty, \qquad \sum_{\nu=0}^{\infty} T_\nu = \infty,$$

$$(27) \qquad \overline{\lim_{\nu\to\infty}} \frac{T_{\nu+1}}{T_\nu} \leqq 1, \qquad \overline{\lim_{\nu\to\infty}} \frac{S_{\nu+1}}{S_\nu} \leqq 1.$$

Assume further that we have as $\nu \to \infty$

$$(28) \qquad \frac{s_\nu}{S_\nu} \to \alpha, \qquad \frac{t_\nu}{T_\nu} \to \beta,$$

with finite α and β. Then we have

$$(29) \qquad \frac{r_\nu}{R_\nu} \to \alpha\beta \quad (\nu \to \infty).$$

11. Proof. Since $\frac{r_\nu}{R_\nu}$ is homogeneous both in s_ν, S_ν and in t_ν, T_ν of dimension 0, each of the constants α, β, which is $\neq 0$, can be assumed as 1. We have therefore only three cases to consider

$$\text{a) } \alpha = \beta = 1; \qquad \text{b) } \alpha = 1,\ \beta = 0; \quad \text{c) } \alpha = \beta = 0.$$

(The case b′) $\alpha = 0$, $\beta = 1$ is by symmetry equivalent to the case b)).

Consider first the case a),

$$\frac{s_\nu}{S_\nu} \to 1, \qquad \frac{t_\nu}{T_\nu} \to 1. \tag{30}$$

In applying then the theorem 1 we have

$$\frac{\sum_{\nu=0}^{n} s_\nu T_{n-\nu}}{\sum_{\nu=0}^{n} S_\nu T_{n-\nu}} \to 1 \qquad (n \to \infty). \tag{31}$$

On the other hand it follows from (28) that

$$\overline{\lim_{\nu\to\infty}} \frac{s_{\nu+1}}{s_\nu} \leqq 1 \text{ and } \sum_{\nu=0}^{\infty} s_\nu = \infty.$$

We can therefore, *assuming that all s_ν are* > 0, apply the theorem 1 in replacing there $\varphi(z)$ by $\psi(z)$, $\Phi(z)$ by $\Psi(z)$ and $\Psi(z)$ by $\varphi(z)$. We obtain then

$$\frac{\sum_{\nu=0}^{n} t_\nu s_{n-\nu}}{\sum_{\nu=0}^{n} T_\nu s_{n-\nu}} = \frac{\sum_{\nu=0}^{n} s_\nu t_{n-\nu}}{\sum_{\nu=0}^{n} s_\nu T_{n-\nu}} \to 1 \quad (n \to \infty). \tag{32}$$

In multiplying (31) and (32) we obtain $\frac{r_n}{R_n} \to 1$, that is (29), proved now in the case (30) *under the additional assumption that all s_ν are* > 0.

12. It is now easy to prove the case a) *without* the assumption that *all* s_ν are positive. Indeed we can find in any case a positive number A, such that under the assumption (30) the sums $s'_\nu = \frac{s_\nu + AS_\nu}{1+A}$ are positive for all $\nu = 0, 1, \ldots.$ But then we have, since $\frac{s'_\nu}{S_\nu} \to 1$,

$$\frac{\sum_{\nu=0}^{n} (s_\nu + A S_\nu) t_{n-\nu}}{(1+A) R_n} \to 1,$$

$$\frac{r_n}{R_n} + A \frac{\sum_{\nu=0}^{n} S_\nu t_{n-\nu}}{R_n} \to 1 + A.$$

Here the second term on the left tends to A by the theorem 1 (applied in interchanging the S_ν and the T_ν) and we obtain again $\frac{r_n}{R_n} \to 1$. The case a) is now completely proved.

13. Assume now the case b), that is $\frac{s_\nu}{S_\nu} \to 1$, $\frac{t_\nu}{T_\nu} \to 0$. If we then put $t'_\nu = t_\nu + T_\nu$ we have $\frac{t_\nu}{T_\nu} \to 1$ and it follows by the first part of our theorem, already proven,

$$\frac{\sum_{\nu=0}^{n} s_\nu (t_{n-\nu} + T_{n-\nu})}{R_n} \to 1$$

$$\frac{r_n}{R_n} + \frac{\sum_{\nu=0}^{n} s_\nu T_{n-\nu}}{\sum_{\nu=0}^{n} S_\nu T_{n-\nu}} \to 1.$$

Here the second term on the left tends to 1 by the theorem 1 and we obtain $\frac{r_n}{R_n} \to 0$ $(n \to \infty)$. Thus our theorem is proved in the case b).

The case c) $\frac{s_\nu}{S_\nu} \to 0$, $\frac{t_\nu}{T_\nu} \to 0$ is reduced to the case b) by exactly the same argument. The theorem 3 is proved.

14. If we take in the theorem 3 e.g.

$$\Phi(z) = (1-z)^{-r-1}, \quad \Psi(z) = (1-z)^{-s-1} \quad (r, s > -1)$$

we obtain

$$S_\nu = \binom{\nu+r}{\nu}, \qquad T_\nu = \binom{\nu+s}{\nu}, \qquad R_\nu = \binom{\nu+s+r+1}{\nu}.$$

It follows therefore, that if we have

$$(33) \quad s_\nu \sim \binom{\nu+r}{\nu}, \qquad t_\nu \sim \binom{\nu+s}{\nu} \qquad (\nu \to \infty,\ r, s > -1),$$

then

$$(34) \qquad s_0 t_\nu + s_1 t_{\nu-1} + \ldots + s_\nu t_0 \sim \binom{\nu+r+s+1}{\nu} \quad (\nu \to \infty).$$

This is the theorem 41 in Hardy (l.c. p. 98).

15. The above results are probably valid under much more general conditions. Here we have gone into them only as far as they arose naturally in the course of other investigations.

(Oblatum 3-3-58).

Numerical Analysis Research,
University of California in Los Angeles, U.S.A.
and
Mathematische Anstalt der Universität Basel,
Switzerland.