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# QUANTIFICATION OF NUMBER-THEORETIC FUNCTIONS

by

S. C. Kleene

A class $C$ of one-place number-theoretic functions has been called a *basis* for a class $D$ of predicates of a function variable $\alpha$, [1]) if, for each predicate $B(\alpha)$ of $D$,

$$(1a) \qquad (E\alpha)B(\alpha) \rightarrow (E\alpha)[\alpha \in C \;\&\; B(\alpha)],$$

whence, since the converse implication is immediate,

$$(1b) \qquad (E\alpha)B(\alpha) \equiv (E\alpha)[\alpha \in C \;\&\; B(\alpha)].$$

We showed in [7, XXVI] that the hyperarithmetical functions are not a basis for the predicates $(x)R(\alpha, x)$ with $R$ recursive. [2])

In this paper we give some more information about this situation. Extending the use of the term "basis", say $C$ is a *basis* for a predicate $P(a)$ expressed in the form $(E\alpha)B(a, \alpha)$ with a certain $B$ (or for the quantifier in this expression), if $C$ is a basis in the above sense for the class $\{B(0, \alpha), B(1, \alpha), B(2, \alpha), \ldots\}$; thus when $B(a, \alpha) \equiv (x)R(a, \alpha, x)$, if

$$(2b) \qquad (a)\{(E\alpha)(x)R(a, \alpha, x) \equiv (E\alpha)[\alpha \in C \;\&\; (x)R(a, \alpha, x)]\}.$$

Dually, $C$ is a basis for $P(a)$ expressed in the form $(\alpha)(Ex)R(a, \alpha, x)$ with a certain $R$ (or for the quantifier $(\alpha)$ in this expression), if

$$(3b) \qquad (a)\{(\alpha)(Ex)R(a, \alpha, x) \equiv (\alpha)[\alpha \in C \rightarrow (Ex)R(a, \alpha, x)]\},$$

which is equivalent to saying it is a basis for $\bar{P}(a)$ expressed as $(E\alpha)(x)\bar{R}(a, \alpha, x)$.

By [7, XXVI], the hyperarithmetical functions are not a basis for $(E\alpha)(x)\bar{T}_1^{\alpha}(a, a, x)$, with $\bar{T}_1^{\alpha}(a, a, x)$ itself as the $R(a, \alpha, x)$, and

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[1]) The term "basis" was suggested to us by G. Kreisel in correspondence in 1952, but investigations of bases for various classes $D$ of predicates $B(\alpha)$ were begun late in the 1940's by Kreisel and us independently of each other. (The results numbered (1)—(3) in [5, 5.5] were known to us in 1950 before we learned of Kreisel's work; and [5, the first part of Corollary Theorem 7] was communicated to him in 1952.)

[2]) A *hyperarithmetical function* can be defined as a function general recursive in a hyperarithmetical predicate, or equivalently as a function whose representing predicate [4, p. 199] is hyperarithmetical [7, p. 210]. We presuppose acquaintance with our [4], [5], [6], [7].

a particular value $f$ of $a$ that refutes (2) is constructed; but the question remains whether for some other recursive $R(a, \alpha, x)$ they might be a basis. However a slight recasting (in § 1 below) of the argument of [7] shows that they are not; indeed, whenever a predicate $P(a)$ can be expressed in the form $(E\alpha)(x)R(a, \alpha, x)$ (or in the dual form) with a recursive $R$ and the hyperarithmetical functions as basis, it is hyperarithmetical, and then (using § 2) it is so expressible with the hyperarithmetical functions located lower than itself in the hyperarithmetical hierarchy as basis (except if it is already at the bottom). For this remark, a hyperarithmetical function or predicate is to be located in the hierarchy by the least $|y|$ (where $y \in O$) for which it is recursive in $H_y$. By Spector's [9, Theorem 5] here only $|y|$ matters and not $y$ itself. [3]) Thus the hyperarithmetical predicates are characterizable as the least class of predicates from which one cannot escape by a definition of the form $P(a) \equiv (E\alpha)(x)R(a, \alpha, x)$ with $R$ recursive and the functions recursive in the predicates of the class as basis. [4])

A definition of the form $P(a) \equiv (E\alpha)(x)R(a, \alpha, x)$ with a class $C$ of functions as basis is of course not the same as a definition of the form

$$P(a) \equiv (E\alpha)_{\alpha \in C}(x)R(a, \alpha, x), \text{ i.e. } P(a) \equiv (E\alpha)[\alpha \in C \,\&\, (x)R(a, \alpha, x)],$$

with $C$ as the range of the variable $\alpha$, since (2) is required to hold for the former. In the definition with $C$ as basis rather than merely as range, the class $C$ enters only as a lower bound; the definition means the same to persons with various universes of functions, so long as each person's universe includes at least $C$ (of which he may have no exact conception). In the definition with $C$ merely as range, the class $C$ enters exactly; to prove $\bar{P}(a)$, it is then insufficient to derive a contradiction from the supposition that $(x)R(a, \alpha, x)$ for some function $\alpha$, but one must show rather that any such $\alpha \notin C$.

The definition $P(f, a) \equiv (E\alpha)_{\alpha \in \mathrm{HA}}(x)\bar{T}_1^{\alpha}(f, a, x)$, with the (one-place) hyperarithmetical functions as the range of $\alpha$, leads outside the class of the hyperarithmetical predicates. For by § 2, each hyperarithmetical predicate $P(a) \equiv P(f_0, a)$ for some $f_0$; so $\bar{P}(a, a)$, and hence $P(f, a)$ and $P(a, a)$ ($\equiv (E\alpha)_{\alpha \in \mathrm{HA}}(x)\bar{T}_1^{\alpha}(a, a, x)$) are not hyperarithmetical.

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[3]) The theorems below are stated and proved without using Spector's [9, Theorem 5], which enters only in some of the discussion.

[4]) A predicate or function is *recursive in* (the predicates or functions of) a class $C$, if it is recursive in some finite list $\Psi$ of members of $C$. In this paper "recursive" means general recursive except where otherwise indicated.

By Lemma 1 below with [5, Lemma 1], $(E\alpha)_{\alpha\epsilon\mathrm{HA}}(x)\overline{T}_1^{\alpha}(a, a, x) \equiv (\alpha)(Ex)T_1^{\alpha}(\varphi(a), \varphi(a), x)$ with a recursive $\varphi$; so $(E\alpha)_{\alpha\epsilon\mathrm{HA}}(x)\overline{T}_1^{\alpha}(a, a, x)$ is recursive in $(E\alpha)(x)\overline{T}_1^{\alpha}(a, a, x)$. We do not know whether $(E\alpha)(x)\overline{T}_1^{\alpha}(a, a, x)$ is recursive in $(E\alpha)_{\alpha\epsilon\mathrm{HA}}(x)\overline{T}_1^{\alpha}(a, a, x)$. Another open problem is whether $(E\alpha)(x)\overline{T}_1^{\alpha}(a, a, x)$ is expressible as $(E\alpha)(x)R(a, \alpha, x)$ with a recursive $R$ and a basis (closed under relative recursiveness) less than the functions general recursive in that predicate itself (cf. [5, 5.5 (5)]).

Quantification of function variables ranging over segments of the hyperarithmetical hierarchy is considered in § 3. This we relate to the ramified analytic hierarchy, in which to the 0 level belong the arithmetical predicates, to the 1 level the predicates expressible using besides quantification of number variables also quantification of variables ranging over the arithmetical functions (i.e. expressible as analytic predicates under [5, 2.1] except restricting the range of the function variables to be the arithmetical functions), to the 2 level those expressible using also quantification of variables ranging over the functions of the 1 level, etc. [5]) Using in one direction the result of § 2, we find that each level corresponds precisely to $\omega$ levels of the hyperarithmetical hierarchy. Thus in the union of all the finite levels are exactly the hyperarithmetical predicates located below $\omega^2$; but when all transfinite levels, indexed by members $y$ of $O$ (or via Spector's [9, Theorem 5] by ordinals $|y| < \omega_1$), are included also, we get precisely all the hyperarithmetical predicates. Here quantification of predicate variables can replace quantification of function variables; thus exactly the hyperarithmetical predicates located below $\omega^2$ are expressible in Church's ramified second-order arithmetic $\mathrm{A}^{2/\omega}$ of level $\omega$ [2, p. 353] under the classical interpretation of the symbolism (rather than that suggested in his Footnote 577). In expressing a predicate in these ramified hierarchies, it suffices to use (besides number quantifiers) a single higher-type quantifier (either existential or universal as we choose), with the one-place functions or predicates of a lower level not merely as range but also as basis.

## 1. Predicates expressible with the hyperarithmetical functions as basis

**1.1** THEOREM 1. *If* $P(a) \equiv (E\alpha)(x)R(a, \alpha, x)$ *with recursive* $R$

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5) R. O. Gandy asked what one obtains thus (without specifically mentioning transfinite levels) in a letter to us dated November 14, 1955, and the present answer was given in our reply dated December 24, 1955.

*and the hyperarithmetical functions as basis, then $P(a)$ is hyperarithmetical.*

Proof. Applying (2b) with $C = \mathrm{HA}$ and the following lemma, the conclusion follows under the second definition of "hyperarithmetical predicate" [7, p. 210].

1.2 Lemma 1. *For any recursive $R$,*

$$(E\alpha)_{\alpha\in \mathrm{HA}}(x)R(a, \alpha, x) \equiv (\alpha)(Ex)S(a, \alpha, x) \tag{4}$$

*with a recursive $S$.*

Proof. $(E\alpha)_{\alpha\in \mathrm{HA}}(x)R(a, \alpha, x)$

$\equiv (Ey)(E\alpha)[y \in O \ \&\ \{\alpha \text{ is recursive in } H_y\} \ \&\ (x)R(a, \alpha, x)]$ (by the first definition of "hyperarithmetical predicate" [7, p. 210]) [2])

$\equiv (Ey)(Ee)[y \in O \ \&\ \{e \text{ is a Gödel number from } H_y \text{ of a total function } \alpha_e\} \ \&\ (x)R(a, \alpha_e, x)]$

$\equiv (Ey)(Ee)[y \in O \ \&\ (i)(Et)T_1^{H_y}(e, i, t) \ \&\ (\beta)\{(i)(Et)T_1^{H_y}(e, i, \beta(i)) \rightarrow (x)R(a, \lambda i\, U(\beta(i)), x)\}]$

$\equiv (Ey)(Ee)[y \in O \ \&\ (i)H_{y^*}(\varphi(e, i)) \ \&\ (\beta)\{(i)H_{y^*}(\psi(e, i, \beta(i))) \rightarrow (x)R(a, \lambda i\, U(\beta(i)), x)\}]$ (for some primitive recursive $\varphi$ and $\psi$, by [5, Lemma 1]) [6])

$\equiv (Ey)(Ee)[(\alpha)(Ex)R_1(y, \alpha, x) \ \&\ (i)(\alpha)(Ex)T_1^{\alpha}((\tau(y^*))_0, \varphi(e, i), x) \ \&\ (\beta)\{(i)(E\alpha)(x)\bar{T}_1^{\alpha}((\tau(y^*))_1, \psi(e, i, \beta(i)), x) \rightarrow (x)R(a, \lambda i\, U(\beta(i)), x)\}]$ (with $R_1$ recursive, by [6, Theorem II] and [5, Theorem 9])

$\equiv (\alpha)(Ex)S(a, \alpha, x)$ (with recursive $S$, by advancing quantifiers and applying [5, p. 316, Steps 1—4]).

Remark 1. This proof is essentially an improved version of the proof of [7, (2) p. 209]. Substituting this (with $\bar{T}_1^1(\bar{\alpha}(x), a, a)$, $R(a, \alpha, x)$ as the $R(a, \alpha, x)$, $S(a, \alpha, x)$) and continuing as before gives an improved proof of [7, XXVI].

## 2. Bases for hyperarithmetical predicates

2.1 According to [5, Theorem 9], for each $y \in O$, $H_y(a)$ and $\bar{H}_y(a)$ are each expressible in the form $(E\alpha)(x)R(a, \alpha, x)$ with a recursive $R$.

In Part 1 of Theorem 2 we give as a function of $y \in O$ (with $|y|$ as in Row 1 of the table) a basis $C$ for the predicate $H_y(a)$, expressed in the form $(E\alpha)(x)R(a, \alpha, x)$ with a recursive $R$, which consists of the functions each recursive in $H_{w_0}$ for some $w_0 \leq_O y$

[6]) In connection with a $y \in O$ or $3 \cdot 5^z \in O$, we use the abbreviations $y^*$ for $2^y$ and $z_n$ for $\{z\}(n_O)$, as in [5] and [7]. Continuing from [6, (I)—(XXIII)]: (XXIV) *If $a <_O b$, then $a^* \leq_O b <_O b^*$.*

with $|w_0| =$, or in other cases $|w_0| <$, a specified ordinal (Row 2). To do this, we determine an $f_0$ such that $H_y(a) \equiv (E\alpha)(x) \bar{T}_1^\alpha(f_0, a, x)$, and, for each $a$ for which $H_y(a)$ is true, find such a $w_0$ and an $\alpha$ recursive in $H_{w_0}$ such that $(x)\bar{T}_1^\alpha(f_0, a, x)$. The $w_0$, and the Gödel number $d_0$ from $H_{w_0}$ of the $\alpha$, as functions of $a$, will be partial recursive in $H_{u_0}$ for some $u_0 \leqq_O y$ (with $|u_0|$ as in Row 3). Part 2 (with Rows 4, 5) provides similar information about $\bar{H}_y(a)$ expressed in the same form. We obtain these results by defining, by recursion on $y$ over $O$, simultaneously for Parts 1 and 2, the $f_j$ and the Gödel numbers $h_j$ and $g_j$ of the aforesaid partial recursive functions $w_j$ and $d_j$. The ordinals $|w|$ which we obtain are always less than the ordinal $|y|$ which locates the predicate itself in the hierarchy, except trivially for $|y| = 0$; and likewise in the corollary.

In Parts 3—5, these basis results are stated to be the best describable in terms of the hyperarithmetical hierarchy. (We have not considered whether they could be improved in terms of the finer structure of Kleene-Post [8].)

TABLE. Bases for $H_y(a)$ and $\bar{H}_y(a)$.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0. | Case | 1 | 1 | 4 | 5 | 5 | ... | 2 | 3 | 4 | 5 | 5 | ... | 5 |
| 1. | $\|y\|$ | 0 | 1 | 2 | 3 | 4 | ... | $\xi$ | $\xi+1$ | $\xi+2$ | $\xi+3$ | $\xi+4$ | ... | $\eta+3$ |
| 2. | $\|w_0\|$ | 0 | 0 | 0 | 0 | 1 | ... | $<\xi$ | $<\xi$ | $\xi$ | $\xi$ | $\xi+1$ | ... | $\eta$ |
| 3. | $\|u_0\|$ | 0 | 0 | 1 | 2 | 3 | ... | 0 | $\xi$ | $\xi+1$ | $\xi+2$ | $\xi+3$ | ... | $\eta+2$ |
| 4. | $\|w_1\|$ | 0 | 0 | 0 | 1 | 2 | ... | $<\xi$ | $\xi$ | $\xi$ | $\xi+1$ | $\xi+2$ | ... | $\eta+1$ |
| 5. | $\|u_1\|$ | 0 | 0 | 0 | 0 | 0 | ... | 0 | 0 | 0 | 0 | 0 | ... | 0 |

$\xi$ is a limit ordinal, $\eta$ an arbitrary ordinal, $< \omega_1$.

THEOREM 2. PARTS 1 AND 2. *Let* $u_0(y) = \max((y)_0, 1), u_1(y) = 1$. *Thus if* $y \in O$, *then* $u_j(y) \leqq_O y$, *and for* $|y|$ *as in Row* 1 *of the table* $|u_j(y)|$ *is as in Row* $3+2j$ $(j = 0, 1)$.

*There are primitive recursive functions* $f(y)$ *and* $k(y)$ *with the following properties. For* $j = 0, 1$, *let* [7])

$$f_j(y) = (f(y))_j, \qquad g_j(y) = (k(y))_{0,j}, \qquad h_j(y) = (k(y))_{1,j},$$

$$d_j(y, a) \simeq \{g_j(y)\}(H_{u_j(y)}, a), \qquad w_j(y, a) \simeq \{h_j(y)\}(H_{u_j(y)}, a).$$

*Also for* $y \in O$, *let*

$$H_{y,0}(a) \equiv H_y(a), \qquad H_{y,1}(a) \equiv \bar{H}_y(a).$$

*Now for* $j = 0, 1$ *and* $y \in O$,

(5) $H_{y,j}(a) \equiv (E\alpha)(x)\bar{T}_1^\alpha(f_j(y), a, x)$,

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[7]) We write $\{z\}(\Psi, a)$ for the $\{z\}^\Psi(a)$ of [4, p. 341], $A\Psi$ for $A^{m_1,\dots,m_l}$ [4, p.344]. and (with complicated $\Psi$) $T_1(\Psi, z, a, y)$ for $T_1^\Psi(z, a, y)$ [4, p. 292].

(6) $H_{y,j}(a) \to [d_j(y, a)$ *and* $w_j(y, a)$ *are defined,* $w_j(y, a) \leqq_O y$, *for* $|y|$ *as in Row* 1 $|w_j(y, a)|$ *is as in Row* $2+2j$, $\lambda l\,\{d_j(y, a)\}(H_{w_j(y,a)}, l)$ *is completely defined, and* $(x)\bar{T}_1^{\alpha}(f_j(y), a, x)$ *is true for* $\alpha = \lambda l\,\{d_j(y, a)\}(H_{w_j(y,a)}, l)]$.

PARTS 3—5. *The table is the best possible (of its type) for the results stated in Parts* 1 *and* 2. *Thus* (PART 3) *for each* $y \in O$, $H_y(a)$ *is not expressible in the form* $(E\alpha)(x)R(a, \alpha, x)$ ($R$ *recursive*) *with a basis consisting of the functions each recursive in* $H_{w_0}$ *for some* $w_0 \leqq_O y$ *with* $|w_0|$ *restricted to be a smaller ordinal, or to belong to a smaller segment of the ordinals, than in Row* 2. *Similarly* (PART 4) *for* $\bar{H}_y(a)$, $|w_1|$ *and Row* 4. *Also* (PART 5) *with the given restriction on* $|w_0|$ (*Row* 2), *no smaller* $|u_0|$ *will suffice than that given in Row* 3.

**2.2** REMARK 2. In Parts 1 and 2, the $f_0(y)$, $f_1(y)$ play the role of the $(\tau(y))_1$, $(\tau(y))_0$ of [5, Theorem 9], which is reproved in the process of obtaining the more detailed results stated now. Parts 3—5 are related to the proof of [5, Theorem 7].

**2.3** PROOF OF THEOREM 2, PARTS 1 AND 2. (Parts 3—5 will be proved in 2.5.) The demonstration that $f(y)$ and $k(y)$ have the stated properties is to be by induction on $y$ over $O$. We give a treatment by cases on $y$ (beginning with the more complicated cases), in which we work out case definitions of $f(y)$ and $k(y)$ that suffice for establishing their properties in the cases. In the cases, we write $p$, $q$ for Gödel numbers of $f(y)$, $k(y)$. The case treatments should be followed by definitions (left to the reader) of $f(y)$ and $k(y)$, combining the case definitions with the help of the recursion theorem (as e.g. in the proofs of [5, Lemmas 3—5]). Of course the combined definitions logically precede the proof of the properties by induction and cases. [6])

CASE 5: $y = b^* = c^{**} = d^{***}$ where $d \in O$. Then $u_0(y) = (y)_0 = b$, and we want $w_0(y, a) = d = (y)_{0,0,0}$ and $w_1(y, a) = c = (y)_{0,0}$ (cf. Rows 2,4). So we shall take $h_0(y) = \Lambda H_b a\ (y)_{0,0,0}$, $h_1(y) = \Lambda H_1 a\ (y)_{0,0}$.

PART 1. We reduce $H_y(a)$ to the desired form, thus: $H_{y,0}(a) \equiv H_y(a)$

(a) $\equiv (En)(t)(Es)T_3^{H_d}(m, a, n, t, s)$ (for some $m$, by [4, Theorem XI* with (17), and IV*, pp. 285, 292, 295] or [7, XII* and V*, pp. 197, 198])

(b) $\equiv (En)(t)(Es)(Ev)\{v = \Pi_{i<s}p_i^{(v)_i}\ \&\ (i)_{i<s}[\{H_d(i)\ \&\ (v)_i{=}0\} \vee \{\bar{H}_d(i)\ \&\ (v)_i = 1\}]\ \&\ T_3^1(v, m, a, n, t, s)\}$ (cf. [5, 2.3 and 2.5])

(c) $\equiv (En)(t)(Er)\{(r)_1{=}\Pi_{i<(r)_0}p_i^{(r)_{1,i}}\ \&(i)_{i<(r)_0}[\{H_d(i)\&(r)_{1,i}{=}0\}\vee \{\bar{H}_d(i)\ \&\ (r)_{1,i} = 1\}]\ \&\ T_3^1((r)_1, m, a, n, t, (r)_0)\}$

(d) $\equiv (En)(t)(Er)\{(r)_1 = \Pi_{i<(r)_0} p_i^{(r)_{1,i}} \;\&\; (i)_{i<(r)_0}(Ej)_{j<2}[H_{d,j}(i) \;\&\; (r)_{1,i} = j] \;\&\; T_3^1((r)_1, m, a, n, t, (r)_0)\}$

(e) $\equiv (En)(t)(Er)\{(r)_1 = \Pi_{i<(r)_0} p_i^{(r)_{1,i}} \;\&\; (i)_{i<(r)_0}(Ej)_{j<2}[(E\alpha_1)(x)\bar{T}_1^{\alpha_1}(f_j(d), i, x) \;\&\; (r)_{1,i} = j] \;\&\; T_3^1((r)_1, m, a, n, t, (r)_0)\}$ (by hyp. ind., since $d <_O y$)

(f) $\equiv (En)(t)(Er)(E\alpha_2)\{(r)_1 = \Pi_{i<(r)_0} p_i^{(r)_{1,i}} \;\&\; (i)_{i<(r)_0}(Ej)_{j<2}[(x)\bar{T}_1^{\lambda s(\alpha_2(s))_i}(f_j(d), i, x) \;\&\; (r)_{1,i} = j] \;\&\; T_3^1((r)_1, m, a, n, t, (r)_0)\}$ (using $(i)_{i<a}(E\alpha)A(i, \alpha) \equiv (E\alpha)(i)_{i<a}A(i, \lambda s\,(\alpha(s))_i)$, which is analogous to [4, (19) p. 285])

(g) $\equiv (En)(t)(Er)(E\alpha_2)(x)\{(r)_1 = \Pi_{i<(r)_0} p_i^{(r)_{1,i}} \;\&\; (i)_{i<(r)_0}(Ej)_{j<2}[\bar{T}_1^{\lambda s(\alpha_2(s))_i}(f_j(d), i, (x)_j) \;\&\; (r)_{1,i} = j] \;\&\; T_3^1((r)_1, m, a, n, t, (r)_0)\}$ (using [4, (20) p. 285])

(h) $\equiv (E\alpha)(t)(x)\{$same scope with $n$, $r$, $\alpha_2$ replaced by $(\alpha(0))_0$, $(\alpha(t))_1$, $\lambda s\,(\alpha(2^t \cdot 3^s))_2\}$ (cf. [5, Footnote 10])

(i) $\equiv (E\alpha)(x)\{$same scope with $t$, $x$, $f_j(d)$ replaced by $(x)_0$, $(x)_1$, $(\{p\}((y)_{0,0,0}))_j\}$

(j) $\equiv (E\alpha)(x)\bar{T}_1^{\alpha}(\varphi_0^5(p, y), a, x)$ (for some primitive recursive $\varphi_0^5$, by [5, Lemma 12]).

Accordingly in this case we shall take $f_0(y) = \varphi_0^5(p, y)$.

To get the basis result for $H_y(a)$, we shall evaluate the existentially-bound variables successively, starting with the $n$ of (a).

Consider any $a$ such that $H_y(a)$. Then (using [5, (11)] twice),

$$(7) \quad n = n^{H_b}(a) = \mu n(t)(Es)T_3^{H_d}(m, a, n, t, s) = \mu n(t)H_c(\psi(a, n, t)) = \mu n\bar{H}_b(\chi(a, n)) \quad \text{(for some recursive } \psi, \chi)$$

is an $n$ for (a), and hence for (b)—(g); i.e. the scope of $(En)$ in (a) is true when $n$ has this value, and hence the scopes of $(En)$ in (b)—(g) are also true, since under the method of the reduction (a)—(g) each of these scopes is equivalent to the preceding.

Now, for this $a$ and $n$, consider any $t$. Then

$$(8) \qquad r = r(t) = \mu r\{(r)_1 = \Pi_{i<(r)_0} p_i^{(r)_{1,i}} \;\&\; (i)_{i<(r)_0}[\{H_d(i) \;\&\; (r)_{1,i} = 0\} \vee \{\bar{H}_d(i) \;\&\; (r)_{1,i} = 1\}] \;\&\; T_3^1((r)_1, m, a, n, t, (r)_0)\}$$

is an $r$ for (c) and hence for (d)—(g).

For this $a$, $n$, $t$ and $r$, consider any $i < (r)_0$. Then

$$(9) \qquad j = j(i) = \mu j[\{H_d(i) \;\&\; j = 0\} \vee \{\bar{H}_d(i) \;\&\; j = 1\}]$$

is a $j$ for (d), and hence for (e).

Then also, using the hypothesis of the induction (since $d <_O y$),

$$(10) \qquad \alpha_1 = \lambda s\,\alpha_1(i, s) = \lambda s\,\{d_{j(i)}(d, i)\}(H_{w_{j(i)}(d,i)}, s)$$

is an $\alpha_1$ for (e). Here

(11) $d_{j(i)}(d, i) = \{g_{j(i)}(d)\}(H_{u_{j(i)}(d)}, i) = \{(\{q\}(d))_{0,j(i)}\}(H_{u_{j(i)}(d)}, i),$

(12) $H_{w_{j(i)}(d,i)} = \lambda l\, H_d(\rho(w_{j(i)}(d, i), d, l))$

by [5, Lemma 3] since $w_{j(i)}(d, i) \leqq_O d$ (by the hyp. ind.),

(13) $w_{j(i)}(d, i) = \{h_{j(i)}(d)\}(H_{u_{j(i)}(d)}, i) = \{(\{q\}(d))_{1,j(i)}\}(H_{u_{j(i)}(d)}, i),$

(14) $H_{u_{j(i)}(d)} = \lambda l\, H_d(\rho(u_{j(i)}(d), d, l))$

by ([5, Lemma 3] since $u_{j(i)}(d) \leqq_O d$.

Next

(15) $\alpha_2 = \lambda s\, \alpha_2(t, s) = \lambda s\, \tilde{\alpha}_1((r(t))_0; s)$

is an $\alpha_2$ for (f) (where $i$ is no longer free in the scope) and hence for (g).

Finally

(16) $\alpha = \lambda l\, 2^n \cdot 3^{r(l)} \cdot 5^{\alpha_2((l)_0, (l)_1)}$

is an $\alpha$ for (h), and hence for (i) and (j).

Combining (7)—(16) (using (12) in (10) before using (13), and noting that $\lambda jy\, u_j(y)$ is recursive), we can write $\alpha(l) = \varphi^{H_d}(n^{H_b}(a), q, d, a, l)$ with $\varphi^{H_d}(n, q, d, a, l)$ partial recursive uniformly in $H_d$. So if we put $\beta_0^5(n, q, d, a) = \Lambda H_d l\, \varphi^{H_d}(n, q, d, a, l)$, $d_0(y, a) = \beta_0^5(n^{H_b}(a), q, d, a)$ where $d = (y)_{0,0,0}$, and $g_0(y) = \Lambda H_b a\, \beta_0^5(n^{H_b}(a), q, d, a)$, we will have what we need for this case and part.

PART 2. The reduction begins with $\bar{H}_y(a) \equiv (t)(Es)T_2^{H_c}(m, a, t, s)$, and the rest of the treatment is similar to Part 1, but simpler as there is no $(En)$.

CASE 4: $y = b^* = c^{**}$ where $c \in O$ and $|c|$ is 0 or a limit ordinal. Then $u_0(y) = (y)_0 = b$, and we take $w_j(y, a) = c = (y)_{0,0}$.

PART 1. $H_{y,0}(a) \equiv H_y(a)$

(a) $\equiv (En)(t)\bar{T}_2^{H_c}(m, a, n, t)$

(b) $\equiv (En)(t)(v)\{v = \Pi_{i<t}\, p_i^{(v)_i} \;\&\; (i)_{i<t}[\{H_c(i) \;\&\; (v)_i = 0\} \vee \{\bar{H}_c(i) \;\&\; (v)_i = 1\}] \rightarrow \bar{T}_2^1(v, m, a, n, t)\}$

(c) $\equiv (En)(t)(v)\{v \neq \Pi_{i<t}\, p_i^{(v)_i} \vee (Ei)_{i<t}[\{\bar{H}_c(i) \vee (v)_i \neq 0\} \;\&\; \{H_c(i) \vee (v)_i \neq 1\}] \vee \bar{T}_2^1(v, m, a, n, t)\}$

(d) $\equiv (En)(t)(v)\{v \neq \Pi_{i<t}\, p_i^{(v)_i} \vee (Ei)_{i<t}[\{(E\alpha_1)(x)\bar{T}_1^{\alpha_1}(f_1(c), i, x) \vee (v)_i \neq 0\} \;\&\; \{(E\alpha_0)(x)\bar{T}_1^{\alpha_0}(f_0(c), i, x) \vee (v)_i \neq 1\}] \vee \bar{T}_2^1(v, m, a, n, t)\}$

(e) $\equiv (En)(t)(v)(E\alpha_0)(E\alpha_1)\{v \neq \Pi_{i<t} p_i^{(v)_i} \vee (Ei)_{i<t}[\{(x)\bar{T}_1^{\alpha_1}(f_1(c), i, x) \vee (v)_i \neq 0\} \;\&\; \{(x)\bar{T}_1^{\alpha_0}(f_0(c), i, x) \vee (v)_i \neq 1\}] \vee \bar{T}_2^1(v, m, a, n, t)\}$.

The reduction is completed substantially as before. For the evaluation, consider any $a$ such that $H_y(a)$. Then

(17) $n = n^{H_b}(a) = \mu n(t) T_2^{H_c}(m, a, n, t) = \mu n \overline{H}_b(\psi(a, n))$

is an $n$ for (a)—(f). For this $a$ and $n$, consider any $t$, $v$ such that not $v \neq \Pi_{i<t}\, p_i^{(v)_i} \vee \overline{T}_2^1(v, m, a, n, t)$. Then

(18) $i = i(t, v) = \begin{cases} 0 \text{ if } v \neq \Pi_{i<t}\, p_i^{(v)_i} \vee \overline{T}_2^1(v, m, a, n, t), \\ \mu i[\{\overline{H}_c(i) \vee (v)_i \neq 0\} \& \{H_c(i) \vee (v)_i \neq 1\}] \text{ otherwise} \end{cases}$

(cf. [4, Theorem XX (c) p. 337]) is an $i$ for (c)—(d). For this $a, n, t, v$ and $i$, if not $(v)_i \neq \overline{\mathrm{sg}}(j)$, then

(19) $\alpha_j = \lambda s\, \alpha_j(t, v, s) = \lambda s \begin{cases} 0 \text{ if } v \neq \Pi_{i<t}\, p_i^{(v)_i} \vee \\ \qquad \overline{T}_2^1(v, m, a, n, t) \vee (v)_i \neq \overline{\mathrm{sg}}(j), \\ \{d_j(c, i)\}(H_{w_j(c, i)}, s) \text{ otherwise} \end{cases}$

is an $\alpha_j$ for (d) ($j = 0, 1$). Then for any $t$, $v$ (not necessarily such that not $v \neq \Pi_{i<t}\, p_i^{(v)_i} \vee \overline{T}_2^1(v, m, a, n, t)$), the $\alpha_j$ of (19) for the $i$ of (18) (whether or not $(v)_i \neq \overline{\mathrm{sg}}(j)$) is an $\alpha_j$ for (e)—(f). The evaluation is completed much as before.

PART 2. Identical with Case 5 Part 2.

CASE 3: $y = b^*$ where $b = 3 \cdot 5^z$ and $b \in O$. Then $u_0(y) = (y)_0 = b$.

PART 1. The reduction of $H_y(a)$ is further simplified from Case 5 Part 1 and Cases 5 and 4 Part 2, as there is also no $(t)$. After evaluating $r = r^{H_b}(a)$ (for $a$ such that $H_y(a)$) and $j = j(i)$ (for $i < (r)_0$), we note that, by the hyp. ind. (since $b <_O y$) with Rows 2 and 4 of Case 2, the numbers $w_{j(i)}(b, i)$ for $i < (r)_0$ are $<_O b$. Hence by [6, p. 409], they are linearly ordered by $<_O$, and we choose the highest among them for $w_0(y, a)$. Using [6, (XV) p. 410], then

(20) $w_0(y, a) = w_{j((s)_0)}(b, (s)_0)$ where $s = \mu s\{(s)_0 < (r)_0 \;\&\; (i)_{i<(r)_0} [\mathrm{enm}([w_{j((s)_0)}(b, (s)_0)]^*, (s)_{i+1}) = w_{j(i)}(b, i)]\}$.

Corresponding to (11)—(13) (using the formulas for $r$ and $j$), for $i < (r)_0$

(21) $d_{j(i)}(b, i) = \{g_{j(i)}(b)\}(H_1, i) = \{((\{q\}(b))_{0, j(i)}\}(H_1, i)$
$= \{(\chi_0^{H_b}(q, b, a))_i\}(H_1, i)$ where $\chi_0^{H_b}(q, b, a) = \Pi_{i<(r)_0}\, p_i^{(\{q\}(b))_{0, j(i)}}$,

(22) $H_{w_{j(i)}(b, i)} = \lambda l\, H_{w_0(y, a)}(\rho(w_{j(i)}(b, i), w_0(y, a), l))$,

(23) $w_{j(i)}(b, i) = \{h_{j(i)}(b)\}(H_1, i) = \{((\{q\}(b))_{1, j(i)}\}(H_1, i)$
$= \{(\chi_1^{H_b}(q, b, a))_i\}(H_1, i)$ where $\chi_1^{H_b}(q, b, a) = \Pi_{i<(r)_0}\, p_i^{(\{q\}(b))_{1, j(i)}}$.

Combining (20) and (23) and the formulas for $r$ and $j$,

(24) $w_0(y, a) = \psi^{H_b}(q, b, a)$

with a $\psi^{H_b}$ partial recursive uniformly in $H_b$; so we take $h_0(y) = \Lambda H_b a\, \psi^{H_b}(q, b, a)$ where $b = (y)_0$. Combining (21)—(24) and the formulas for $r$, $\alpha_1$, $\alpha_2$ and $\alpha$,

(25) $\alpha = \lambda l\, \varphi^{H_w}(r^{H_b}(a), \psi^{H_b}(q, b, a), \chi_0^{H_b}(q, b, a), \chi_1^{H_b}(q, b, a), l)$

for $w = w_0(y, a)$, where $\varphi^{H_w}(r, s, t_0, t_1, l)$ is partial recursive uniformly in $H_w$. So we shall put $\beta_0^3(r,s,t_0,t_1) = \Lambda H_w l\ \varphi^{H_w}(r,s,t_0,t_1,l)$, and $g_0(y) = \Lambda H_b a\ \beta_0^3(r^{H_b}(a), \psi^{H_b}(q, b, a), \chi_0^{H_b}(q, b, a), \chi_1^{H_b}(q, b, a))$ where $b = (y)_0$.

PART 2. Simplify from Case 4 Part 1.

CASE 2: $y = 3 \cdot 5^z$ and $y \in O$. PART 1. [6]) $H_y(a) \equiv H_{z_{(a)_1}}((a)_0) \equiv (E\alpha)(x)\bar{T}_1^{\alpha}(f_0(z_{(a)_1}), (a)_0, x)$, etc. Taking $w_0(y, a) = z_{(a)_1} = [(y)_2]_{(a)_1}$, the evaluation is straightforward (cf. (10)—(14)). PART 2 is similar.

CASE 1: $y = 1$ or $y = 1^*$. PART 1. $H_y(a) \equiv (Et)R(a, t)$ (with a recursive $R) \equiv (E\alpha)R(a, \alpha(0)) \equiv (E\alpha)(x)R(a, \alpha(0)) \equiv (E\alpha)(x)\bar{T}_1^{\alpha}(m_0^1, a, x)$. When $H_y(a)$, then $\lambda l\ \mu t R(a, t)$ is an $\alpha$. PART 2. $\bar{H}_y(a) \equiv (x)R(a, x) \equiv (E\alpha)(x)R(a, x) \equiv (E\alpha)(x)\ \bar{T}_1^{\alpha}(m_1^1, a, x)$. When $\bar{H}_y(a)$, $\lambda l\ 0$ is an $\alpha$.

2.4 REMARK 3. In Cases 5, 4, 1, a simpler treatment can be given when $|d|$, $|c|$, $|y|$, respectively, $= 0$. If $|w_0|$ be increased to the present $|u_0|$ in Cases 3, 4 and 5, then the $|u_0|$ can be made 0.

2.5 PROOF OF THEOREM 2, PARTS 3—5. CASE 5. PART 3. SUBCASE 1: $|d|$ is a limit ordinal, i.e. $d = 3 \cdot 5^z$. The next stronger restriction on $|w_0|$ would make the functions each recursive in $H_e$ for some $e <_O d$ a basis for $H_y(a)$ expressed in the form $(E\alpha)(x)R(a, \alpha, x)$ with a recursive $R$; here of course $R(a, \alpha, x)$ can be $R(\tilde{\alpha}(x), a, x)$ with this $R$ also recursive, by [4, Theorem IV* p. 292 with uniformity]. Each $e <_O d$ is $<_O z_n$ for some $n$ (by [6, (VI) p. 408]), so $H_e$ is recursive in $H_{z_n}$ (by [7, XIV]); and for any $n$, $z_n <_O d$. Now, for any predicate $P(a)$ so expressed with such a basis, $P(a) \equiv (En)(E\alpha)[\{\alpha$ is recursive in $H_{z_n}\}$ & $(x)R(\tilde{\alpha}(x), a, x)] \equiv (En)(Em)[\{m$ is a Gödel number from $H_{z_n}$ of a total function $\alpha_m\}$ & $(x)R(\tilde{\alpha}_m(x), a, x)] \equiv (En)(Em)(x)(Ev)[(i)_{i<x}T_1(H_{z_n}, m, i, (v)_i)$ & $R(\Pi_{i<x}\ p_i^{U((v)_i)}, a, x)] \equiv (En)(Em)\ \bar{H}_{z_n^{**}}(\psi(m, a))$ (with a recursive $\psi$, by using [5, Lemma 1] twice) $\equiv (En)(Em)\bar{H}_{z_{n+2}}(\rho(z_n^{**}, z_{n+2}, \psi(m, a)))$ (by [5, Lemma 3] with (XXIV) [6])) $\equiv (En)(Em)\bar{H}_d(2^{\rho(z_n^{**}, z_{n+2}, \psi(m, a))} \cdot 3^{n+2}) \equiv H_c(\varphi(a))$ (with a recursive $\varphi$). This is absurd when $P(a) \equiv H_y(a)$ (where $y = c^{**}$), since $H_y$ is of higher degree than $H_c$ (by [8, (11)] or [7, XIV]). SUBCASE 2: $|d|$ is a successor ordinal, i.e. $d = e^* \neq 1$. A stronger restriction on $|w_0|$ would make the functions recursive in $H_e$ a basis for $H_y(a)$ expressed in the form $(E\alpha)(x)R(\tilde{\alpha}(x), a, x)$ with a recursive $R$. For any $P(a)$ so expressed with such a basis, proceeding similarly to Subcase 1 but without the $(En)$, $P(a) \equiv (Em)(x)(Ev)[(i)_{i<x}T_1^{H_e}(m, i, (v)_i)$ & $R(\Pi_{i<x}\ p_i^{U((v)_i)}, a, x)] \equiv H_b(\varphi(a))$ (with a recursive $\varphi$), which is absurd when $P(a) \equiv$

$H_y(a)$ $(y = b^*)$. SUBCASE 3: $|d|$ is 0, i.e. $d = 1$. Immediate, as no further restriction on $|w_0|$ is possible.

PART 4. Were the functions recursive in $H_d$ a basis for $\bar{H}_y(a)$ expressed in the form $(E\alpha)(x)R(\tilde{\alpha}(x), a, x)$ with recursive $R$, we would have as in Part 3 Subcase 2 (with $d, y$ instead of $e, b$), $\bar{H}_y(a) \equiv H_y(\varphi(a))$. This is absurd; for then $H_y(a)$ and $\bar{H}_y(a)$ would each be of the form $(Ex)R^{H_b}(a, x)$ with an $R^{H_b}(a, x)$ recursive in $H_b$, which by [4, Theorem VI* (c) p. 292] would make $H_y$ recursive in $H_b$.

PART 5. Were $u_0 = c$ instead of $b$, we would have $H_y(a) \equiv (Et)T_1^{H_c}(g_0(y),\ a,\ t)\ \&\ (t)\{T_1^{H_c}(g_0(y),\ a,\ t) \rightarrow (x)(Ev)[(i)_{i<x} T_1^{H_d}(U(t), i, (v)_i)\ \&\ \bar{T}_1^1(\Pi_{i<x}p_i^{U((v)_i)}, f_0(y), a, x)]\} \equiv \bar{H}_y(\varphi(a))$ (with a recursive $\varphi$), whence $\bar{H}_y(a) \equiv H_y(\varphi(a))$, which is absurd (as in Part 4).

CASE 4. PARTS 3 AND 4. SUBCASE 1: $|c|$ is a limit ordinal. Like Case 5 Part 3 Subcase 1 with $c, b$ in place of $d, c$ (and for Part 4 $\bar{H}_y$ in place of the $H_y$). SUBCASE 2: $|c| = 0$. Immediate. PART 5. Like Case 5 Part 5 with $c, c$ in place of $d, c$.

CASE 3. PART 3. Increasing the restriction on $|w_0|$ would mean the functions recursive in $H_e$ for some fixed $e <_O b$ are a basis for $H_y(a)$. We could then argue as in Case 5 Part 3 Subcase 2 with $e, e^{***}$ in place of $e, b$. PART 4. Like Case 5 Part 3 Subcase 1 with $\bar{H}_y(a), b, y$ in place of $H_y(a), d, c$; the contradiction is then as in Case 5 Part 4.

PART 5. If we had as $u_0$ an $e <_O b = 3 \cdot 5^z$, then we would have $e <_O z_n$ for some $n$, and $H_y(a) \equiv (Et)T_1^{H_e}(g_0(y), a, t)\ \&\ (Es) [T_1^{H_e}(h_0(y),\ a,\ s)\ \&\ U(s) <_O b]\ \&\ (m)(t)(s)\{T_1^{H_e}(g_0(y),\ a,\ t)\ \& T_1^{H_e}(h_0(y),\ a,\ s)\ \&\ m \geqq n\ \&\ U(s) <_O z_m \rightarrow (x)(Ev)[(i)_{i<x} T_1(\lambda l\, H_{z_m}(\rho(U(s), z_m, l)), U(t), i, (v)_i)\ \&\ \bar{T}_1^1(\Pi_{i<x}\, p_i^{U((v)_i)}, f_0(y), a, x)]\}$.[8]) Replacing the first two $H_e$'s by $\lambda l\, H_{z_n}(\rho(e, z_n, l))$ and the last two by $\lambda l\, H_{z_m}(\rho(e, z_m, l))$, and $<_O$ by $\lambda ab\, (Ex)V(a, b, x)$ with a recursive $V$ given by [6, (32), (VII) and (VIII), p. 408], this expression for $H_y(a)$ comes by [5, Lemma 1] to the form $H_{z_n*}(\psi(a))$

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[8]) The propositional connectives and quantifiers, when applied to partial predicates, are to be understood in the strong senses [4, pp. 334, 336, 337]. Substitutions using the $\lambda$-operator do not lead outside the class of functions and predicates which are partial recursive (partial recursive in $\Psi$); i.e. [5, 1.3] holds reading "partial recursive (partial recursive in $\Psi$)" in place of "general recursive", by [4, Lemma VI p. 344 with Theorem XVII (a) p. 329]. Here a function or predicate which is general recursive (general recursive in $\Psi$) for total functions as values of its function variables can always be extended to one that is partial recursive (partial recursive in $\Psi$) when partial functions are allowed as values, by choosing some particular system E for [4, p. 275] and employing it as on [4, p. 326].

& $(m)\bar{H}_{z_m^{**}}(\chi(m, a))$ with recursive $\psi$ and $\chi$, and thence, replacing $H_{z_n^*}$ by $\lambda l\ H_b(2^{\rho(z_n^*, z_{n+1}, l)} \cdot 3^{n+1})$ and similarly with $m$, to the form $\bar{H}_y(\varphi(a))$ with a recursive $\varphi$, which is absurd.

Case 2. Parts 3 and 4. Like Case 3 Part 3. Part 5. Immediate.

2.6 Remark 4. The partial recursive $\lambda a\ d_j(y, a)$ is not in general completely defined. Thus if $\lambda a\ d_0(y, a)$ were completely defined in Case 5 (or 4), the method used in Case 5 Part 5, omitting the existence condition and with $d$, $b$ (or $c$, $b$) in place of $d$, $c$, would give $H_y(a) \equiv \bar{H}_y(\varphi(a))$ with a recursive $\varphi$.

2.7 Corollary. Part 1. *For each $y \in O$ with $|y|$ as in Row 1 of the table, if $P(a)$ is recursive in $H_y$, then $P(a)$ is expressible in the form $(E\alpha)(x)R(a, \alpha, x)$ ($R$ recursive) with a basis consisting of the functions each recursive in $H_w$ for some $w \leqq_O y$ with $|w| = |w_1|$ as in Row 4.* Part 2. *This result is the best possible (of its type), $\bar{H}_y(a)$ being a $P(a)$ for which it cannot be improved* (by Part 4 of the theorem).

Proof. Part 1. By the following lemma with Theorem 1 Parts 1 and 2 (and [7, XIV]), since the restriction on $|w_0|$ in the table (Row 2) is always at least as strict as on $|w_1|$ (Row 4).

2.8 Lemma 2. *If a class $C$ closed under recursive operations is a basis for each of $Q(a)$ and $\bar{Q}(a)$ expressed in the form $(E\alpha)(x)R(a, \alpha, x)$ with a recursive $R$, then it is a basis for any predicate $P(a)$ recursive in $Q$ expressed in that form.*

Proof. Say $e$ is a Gödel number of $P(a)$ from $Q$. Then $P(a)$

(a) $\equiv U(\mu s T_1^Q(e, a, s)) = 0$

(b) $\equiv (Es)(Ev)[v = \Pi_{i<s}\, p_i^{(v)_i}\ \&\ (i)_{i<s}[\{Q(i)\ \&\ (v)_i = 0\} \vee \{\bar{Q}(i)\ \&\ (v)_i = 1\}]\ \&\ T_1^1(v, e, a, s)\ \&\ U(s) = 0]$ (since for each $Q$, $e$ and $a$, $T_1^Q(e, a, s)$ for at most one $s$)

(c) $\equiv (Er)[(r)_1 = \Pi_{i<(r)_0}\, p_i^{(r)_{1,i}}\ \&\ (i)_{i<(r)_0}(Ej)_{j<2}[Q_j(i)\ \&\ (r)_{1,i} = j]\ \&\ T_1^1((r)_1, e, a, (r)_0)\ \&\ U((r)_0) = 0]$ (where $Q_0(i) \equiv Q(i)$, $Q_1(i) \equiv \bar{Q}(i)$)

(d) $\equiv (Er)[(r)_1 = \Pi_{i<(r)_0}\, p_i^{(r)_{1,i}}\ \&\ (i)_{i<(r)_0}(Ej)_{j<2}[(E\alpha_1)(x) R_j(a, \alpha_1, x)\ \&\ (r)_{1,i} = j]\ \&\ T_1^1((r)_1, e, a, (r)_0)\ \&\ U((r)_0) = 0]$ (with a recursive $R_j$, by hypothesis),

etc. as for Theorem 1 Case 5 Part 1. For $a$ such that $P(a)$, after evaluating $r$ and $j$ (for each $i < (r)_0$), the hypothesis gives for each such $i$ an $\alpha_1 = \lambda s\ \alpha_1(i, s) \in C$ for (d), and thence we get an $\alpha = \lambda l\ 2^r \cdot 3^{\tilde{\alpha}_1((r)_0; l)} \in C$ for the final expression (g).

2.9 Remark 5. Using additional details from the theorem in the proof of Lemma 2 with $H_y(a)$ as the $Q(a)$, functions $f(e, y)$, $g(e, y)$, $h(e, y)$, $d(e, y, a)$, $w(e, y, a)$ can be obtained which for the $P(a)$ in the corollary recursive in $H_y$ with Gödel number $e$ are analogous to the $f_j(y)$, $g_j(y)$, $h_j(y)$, $d_j(y, a)$, $w_j(y, a)$ for the $H_{y,j}(a)$ in the theorem. By the present method, $u(y) = y$ simply, which we do not know to be the best result.

## 3. The ramified analytic hierarchy

**3.1** The ramified analytic hierarchy is to consist of a class $\mathrm{An}_y$ of number-theoretic predicates and functions for each $y \in O$. A function shall $\in \mathrm{An}_y$, if its representing predicate [4, p. 199] $\in \mathrm{An}_y$. A predicate shall $\in \mathrm{An}_y$, if it is expressible explicitly in terms of (general) recursive predicates of number and function variables and the operations of the predicate calculus with quantification of number variables and of function variables each ranging over (the one-place number-theoretic functions which $\in$) $\mathrm{An}_u$ for a respective $u <_O y$.

**3.2** Clearly $z <_O y \to \mathrm{An}_z \subset \mathrm{An}_y$. By the technique of [5, 2.3 and 2.5] or by Theorem 3 below, the class of the predicates of $\mathrm{An}_y$ is closed under recursive (or indeed, arithmetical) operations; so the functions of $\mathrm{An}_y$ are equivalently those recursive in predicates of $\mathrm{An}_y$. Furthermore, by Theorem 3 below with [7, XIV], $z <_O y \to \mathrm{An}_z \neq \mathrm{An}_y$. By Theorem 3 with Spector's [9, Theorem 5], $\mathrm{An}_y$ is actually determined by $|y|$, so we may write $\mathrm{An}_y$ also as $\mathrm{An}^{|y|}$. By Theorem 3 and Theorem 3 relativized with [7, p. 210 lines 6—4 from below], no more predicates become definable, if after reaching any level of the hierarchy we relativize the ordinal notations for defining higher levels to any predicate of that level; i.e. for each $u \in O$, every $\mathrm{An}^Q_z$ defined as above except using $O^Q$, $<^Q_O$ for some $Q \in \mathrm{An}_u$ instead of $O$, $<_O$ will be an $\mathrm{An}_y$ for some $y \in O$.

**3.3** By the method used to define $+_O$ in [6, § 22)] (also cf. [3, p. 18]), we find a primitive recursive function $b \cdot_O a$ such that $b \cdot_O 1 = 1$ if $b \neq 0$, $1 \cdot_O a = 1$ if $a \neq 0$, $2^z \cdot_O a = (z \cdot_O a) +_O a$ if $z \neq 0$, $(3 \cdot 5^z) \cdot_O a = 3 \cdot 5^d$ where $(n)[d_n \simeq z_n \cdot_O a]$ if $a \neq 1$, and $b \cdot_O a = 7$ otherwise. [6]) Using induction on $b$ for the case $a >_O 1$ (cf. [6, (XVI)]):

(XXV) *If* $a, b \in O$, *then* (a) $b \cdot_O a \in O$ *and* (b) $(c)[a >_O 1 \mathbin{\&} c <_O b \to c \cdot_O a <_O b \cdot_O a]$.

When $a, b \in O$, $|b \cdot_O a| = |b|\,|a|$ with $|b|$ as multiplier and $|a|$ as multiplicand.

**3.4** THEOREM 3. *Let* $w \in O \mathbin{\&} |w| = \omega$ (*so for* $y \in O$, $|w +_O y \cdot_O w| = \omega + |y|\omega = (1+|y|)\omega$). *For each* $y \in O$, *the predicates of* $\mathrm{An}_y$ *are exactly those each recursive in* $H_v$ *for some* $v <_O w +_O y \cdot_O w$.

PROOF, by induction on $y$ over $O$. CASE 1: $y = 1$. By the definition 3.1, the predicates of $\mathrm{An}_1$ are exactly the arithmetical predicates (cf. [5, 2.1] or [4, p. 239 with Theorem VII p. 285]; and by [7, IV, XI and XII], these are exactly the predicates each recursive in $\mathrm{H}_v$ for some $v <_O w = w +_O 1 \cdot_O w$. CASE 2: $y = 2^z$ where $z \in O$. Consider any expression under 3.1 for a predicate $P$ of $\mathrm{An}_y$. Using the hypothesis of the induction for $u \leqq_O z$, each of the function variables in this expression has as range the functions each recursive in $H_v$ for some $v <_O w +_O u \cdot_O w \leqq_O w +_O z \cdot_O w$ (by (XXV) and [6, (XVI)]). Hence by Lemma 4 (a) below, $P$ is recursive in $H_v$ for some $v <_O (w +_O z \cdot_O w) +_O w$. But $((w +_O z \cdot_O w) +_O w) \sim (w +_O (z \cdot_O w +_O w))$, [9]) and $w +_O (z \cdot_O w +_O w) = w +_O y \cdot_O w$. So to each $v <_O (w +_O z \cdot_O w) +_O w$, there is a $\bar{v}$ with $(\bar{v}) \sim (v)$ such that $\bar{v} <_O w +_O y \cdot_O w$; and $H_{\bar{v}}$ is $H_v$ by [5, 6.5]. Thus $P$ is a predicate recursive in $H_{\bar{v}}$ for some $\bar{v} <_O w +_O y \cdot_O w$. Conversely, any such predicate is recursive in $H_v$ ($= H_{\bar{v}}$) for a $v$ (with $(v) \sim (\bar{v})$) $<_O (w +_O z \cdot_O w) +_O w$, and so by Lemma 4 (b) is expressible explicitly using as the only function quantification one over the functions each recursive in $H_v$ for some $v <_O w +_O z \cdot_O w$, i.e. by the hypothesis of the induction, over $\mathrm{An}_z$; and so by 3.1, the predicate $\in \mathrm{An}_y$. CASE 3: $y = 3 \cdot 5^z$ and $y \in O$. By the definition 3.1 with [6, (VI)], $\mathrm{An}_y = \bigcup_{k=0,1,2,\ldots} \mathrm{An}_{z_k}$; and the set of the predicates each recursive in $H_v$ for some $v <_O w +_O y \cdot_O w$ is likewise the union for $k = 0, 1, 2, \ldots$ of the sets of the predicates each recursive in $H_v$ for some $v <_O w +_O z_k \cdot_O w$. These sets are respectively the same as the sets $\mathrm{An}_{z_k}$, by the hypothesis of the induction.

**3.5** LEMMA 3. *There is a partial recursive function $\tau(u, m, a, b)$ such that, if $u = 3 \cdot 5^z \in O$, then $H_u(\tau(u, m, a, b))$ for $m = 0, 1, 2, \ldots$ is an enumeration (with repetitions) of the predicates $P(a, b)$ each recursive in $H_v$ for some $v <_O u$.*

PROOF. Suppose $P(a, b)$ is recursive in $H_v$ for such a $v$. Then $v <_O z_k$ for some $k$, so $P$ is recursive in $H_{z_k}$ and hence $P(a, b) \equiv (Et)P(a, b) \equiv H_{z_k*}(S_1^{2,1}(e, a, b))$ (for some $e$, by [5, Lemma 1])[6]) $\equiv H_{z_{k+1}}(\rho(z_k*, z_{k+1}, S_1^{2,1}(e, a, b)))$ (using (XXIV) and [5, Lemma 3]) $\equiv H_u(\tau(u, m, a, b))$ upon putting $\tau(u, m, a, b) = [2 \exp \rho(z_{(m)_0}*, z_{(m)_0+1}, S_1^{2,1}((m)_1, a, b))] \cdot 3^{(m)_0+1}$ for $z = (u)_2$, and $m = 2^k \cdot 3^e$.

LEMMA 4. *Let $w \in O$ and $|w| = \omega$. For any given $z >_O 1$: (a) If a number-theoretic predicate $P$ is expressible explicitly*

[9]) For any $a, b, c \in O$, by induction on $c$, $((a +_O b) +_O c) \sim (a +_O (b +_O c))$, i.e. $(a +_O b) +_O c$ and $a +_O (b +_O c)$ are of the same h-type as notations of $O$ [5, p. 328], though they may not be the same number [6, Footnote 29].

*in terms of recursive predicates of number and function variables and the operations of the predicate calculus with quantification of number variables and of function variables each ranging over the (one-place) functions each recursive in $H_v$ for some $v <_O$ some $u$ specified for the variable with $1 <_O u \leqq_O z$, then $P$ is recursive in $H_v$ for some $v <_O z +_O w$, and* (b) *conversely, indeed using in the explicit expression besides number quantifiers a single function quantifier (either existential or universal as we choose) whose $u = z$, i.e. one over the functions each recursive in $H_v$ for some $v <_O z$.*

Proof. (a) Consider an expression as described for a predicate $P(a_1, \ldots, a_n)$. Bring this expression to prenex form. Consider in this prenex form a function quantifier with its scope, $(E\alpha)A(\alpha)$ or $(\alpha)A(\alpha)$, where $\alpha$ ranges over the functions each recursive in $H_v$ for some $v <_O u$. According as $u = 2^x \neq 1$ (using [6, (V)] and [7, XIV]) or $u = 3 \cdot 5^z$ (using Lemma 3), $(E\alpha)A(\alpha)$ is replaceable by

$$(Em)[(a)(Eb)T_1^{H_x}(m, a, b) \; \& \; A(\lambda a \, U(\mu b T_1^{H_x}(m, a, b)))] \text{ or}$$
$$(Em)[(a)(Eb)H_u(\tau(u, m, a, b)) \; \& \; A(\lambda a \, \mu b H_u(\tau(u, m, a, b)))]$$

and $(\alpha)A(\alpha)$ dually); [8]) by [5, Lemma 3], $H_x$ is replaceable herein by $\lambda t \, H_z(\rho(x, z, t))$ and $H_u$ similarly. After carrying out these replacements, successively for each function quantifier, the general recursive scope of the prenex form will have been transformed into a predicate $Q(b_1, \ldots, b_m)$ partial recursive in $H_z$. Writing $Q(b_1, \ldots, b_m) \equiv (Et)Q(b_1, \ldots, b_m)$, we can complete the definition of the latter by [4, Example 4 p. 337] relativized to $H_z$ to obtain $(Et)R(b_1, \ldots, b_m, t)$ with $R$ primitive recursive in $H_z$. Thus we finally obtain an expression for $P(a_1, \ldots, a_n)$ built by the predicate calculus with number quantifiers only from predicates general recursive in $H_z$, i.e. $P(a_1, \ldots, a_n)$ is arithmetical in $H_z$, and hence recursive in $H_{z+_O k_O}$ for some $k$ (e.g. by [7, IV* p. 197 with XII* p. 198]. But $z +_O k_O <_O z +_O w$ by [6, (XVI)].

(b) Conversely, suppose $P(a_1, \ldots, a_n)$ is recursive in $H_v$ for some $v <_O z +_O w$. Then for some $k$, $P(a_1, \ldots, a_n)$ is recursive in $H_{z+_O k_O}$, hence is arithmetical in $H_z$, and hence is expressible in prenex form with number quantifiers only and a scope $Q(b_1, \ldots, b_m)$ recursive in $H_z$. Let $\bar{Q}(a) \equiv Q((a)_0, \ldots, (a)_{m-1})$. By Corollary Theorem 2 (since $z >_O 1$), $\bar{Q}(a)$ is expressible in the form $(E\alpha)(x)R(a, \alpha, x)$ ($R$ recursive) with basis, and therefore range, the functions each recursive in $H_v$ for some $v <_O z$. But $Q(b_1, \ldots, b_m) \equiv \bar{Q}(p_0^{b_1} \cdot \ldots \cdot p_{m-1}^{b_m})$. Applying Corollary Theorem 2 to $\bar{Q}(a)$, a universal quantifier $(\alpha)$ can be secured instead.

3.6 For each $y \in O$, a (number-theoretic) predicate shall $\in A_y^2$

if it is expressible explicitly in terms of the predicates $a+b=c$, $a \cdot b = c$, $n$-ary predicate variables ($n = 0, 1, 2, \ldots$) with number variables as arguments, and the operations of the predicate calculus with quantification of number variables and of $n$-ary predicate variables each ranging over the $n$-ary predicates of $\mathrm{A}^2_u$ for a respective $u <_O y$.

**3.7** It is immediate for $|y| \leqq \omega$, and will follow in the general case from the corollary below with Spector's [9, Theorem 5], that $\mathrm{A}^2_y$ depends only on $|y|$, and so may be written also as $\mathrm{A}^{2/|y|}$. For $|y| = 1, 2, 3, \ldots$ or $\omega$, the predicates of $\mathrm{A}^{2/|y|}$ are exactly the number-theoretic predicates expressible in Church's ramified second-order arithmetic $\mathrm{A}^{2/1}$, $\mathrm{A}^{2/2}$, $\mathrm{A}^{2/3}, \ldots$ or $\mathrm{A}^{2/\omega}$, respectively, under the classical interpretation of the symbolism; and for $|y| = 0$, in his $\mathrm{A}^1$ [2, pp. 353, 321].

**3.8** Corollary. *Let $w \in O$ and $|w| = \omega$. For each $y \in O$, the predicates of $\mathrm{A}^2_y$ are likewise exactly those each recursive in $H_v$ for some $v <_O w +_O y \cdot_O w$.*

From the theorem, by:

Lemma 5. *For each $y \in O$:* (a) *If a predicate $\in \mathrm{A}^2_y$, it $\in \mathrm{An}_y$, and* (b) *conversely, indeed with an expression under* 3.6 *having besides number quantifiers only quantifiers with one-place predicate variables corresponding to, and of the same respective kinds existential or universal as, the function quantifiers in an expression under* 3.1).

Proof, by induction on $y$ over $O$. (a) Consider an expression under 3.6 for a predicate $R(a_1, \ldots, a_n)$ of $\mathrm{A}^2_y$. In this expression, ignoring for the moment the ranges of the predicate variables, any existential quantifier with its scope, $(EP)A(P)$, where $P$ is an $n$-ary predicate variable, is replaceable for $n = 0$, $n = 1$ or $n > 1$ by

$$A(0=0) \vee A(1=0),\ (E\alpha)A(\lambda t\,[\alpha(t)=0]) \text{ or }$$
$$(E\alpha)A(\lambda t_1 \ldots t_n\,[\alpha(p_0^{t_1} \cdot \ldots \cdot p_{n-1}^{t_n}) = 0]),$$

respectively, where $\alpha$ is a 1-ary function variable. In the $n = 1$ and $n > 1$ cases, given any predicate $P$ for the former, there is a function $\alpha$ recursive in $P$ for the latter, and vice versa. Now say the range of $P$ is the $n$-ary predicates of $A^2_u$, for a fixed $u <_O y$, i.e. by the hypothesis of the induction (a) and (b), the $n$-ary predicates of $\mathrm{An}_u$. The set of these predicates is closed under recursive operations, by 3.2. So as the range of $\alpha$ we may take the functions recursive in predicates of $\mathrm{An}_u$, i.e. by 3.2 the functions of $\mathrm{An}_u$. Universal quantifiers are handled dually. Upon carrying out the replacements for all the predicate quantifiers in the given expression under 3.6, we obtain an expression under 3.1 for $R(a_1, \ldots, a_n)$ as a predicate of $\mathrm{An}_y$.

(b) Consider any expression under 3.1 for a predicate $R(a_1, \ldots, a_n)$ of $\mathrm{An}_y$. Bringing it to prenex form, with $(Et)$ or $(t)$ innermost (inserted redundantly if necessary), and using [4, Theorem IV* p. 292], we obtain an equivalent $(Q\mathfrak{b})T(\tilde{\alpha}_1(t), \ldots, \tilde{\alpha}_m(t), \mathfrak{f}, a_1, \ldots, a_n, \mathfrak{c}, t)$, where $T$ is primitive recursive, and $(Q\mathfrak{b})$ are quantifiers on the function variables $\alpha_1, \ldots, \alpha_m$ and the number variables $\mathfrak{c}, t$. Adapting [5, top p. 318], for $P$ a 1-ary predicate variable, let $C(P, a, b) \doteqdot [P(a) \rightarrow a = 2^{(a)_0} \cdot 3^{(a)_1} \ \& \ (r)[r < (a)_1 \rightarrow \bar{P}(2^{(a)_0} \cdot 3^r)] \ \& \ (s)[s < a \rightarrow P(2^s \cdot (3 \exp \Pi_{i<s} p_i^{(b)_i}))]$. Then $(a)(Eb)C(P, a, b) \equiv \{P \text{ is } \lambda a \; a = 2^{(a)_0} \cdot 3^{\tilde{\alpha}((a))_0}$ for some function $\alpha\}$, in which case $\alpha = \lambda a \, (\mu b P(2^{a+1} \cdot 3^b))_a$. Now $(E\alpha)(Q\mathfrak{d}, t) A(\tilde{\alpha}(t), \mathfrak{d}, t) \equiv (EP)\{(a)(Eb)C(P, a, b) \ \& \ (Q\mathfrak{d}, t)(Eb)[P(2^t \cdot 3^b) \ \& \ A(b, \mathfrak{d}, t)]\})$. Given any $\alpha$ for the left member here, there is a $P$ recursive in $\alpha$ for the right; and vice versa. Using 3.2 and the hyp. ind., if the function variable $\alpha$ being replaced ranges over $\mathrm{An}_u$ for a $u <_O y$, the resulting 1-place predicate variable $P$ may be taken to range over $\mathrm{A}^2_u$. We deal with $(\alpha)$ dually. To our equivalent of $R(a_1, \ldots, a_n)$ we apply this method of replacing a function quantifier by a 1-place predicate quantifier repeatedly, each time to the outermost function quantifier not yet replaced. [10]) Then the primitive recursive functions occurring as arguments of a predicate variable $P$ can be replaced by their representing predicates ($P(2^{(a)_0} \cdot 3^r)$ becoming $(Eq)[2^{(a)_0} \cdot 3^r = q \ \& \ P(q)]$, etc.), and the primitive recursive predicates can be expressed in the familiar way in terms of $a+b=c$, $a \cdot b = c$ and the predicate calculus with number quantifiers only.[11]) Thus we obtain an expression under 3.6 for $R(a_1, \ldots, a_n)$ as a predicate of $\mathrm{A}^2_u$.

3.9 Remark 6. Utilizing the final remarks of Lemmas 4 and 5, the above treatment shows that for expressing predicates of $\mathrm{An}_y$ under 3.1 (of $\mathrm{A}^2_y$ under 3.6) it suffices to use besides number quantifiers only a single function quantifier (1-place predicate quantifier), either existential or universal as we choose, with its range also a basis.

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[10]) This is in lieu of making Lemma 5 depend on § 2 via Remark 6 below.

[11]) Use [4, Corollary Theorem I p. 242], replace $a+b$, $a \cdot b$, 0, 1 as functions by their representing predicates (cf. [4, Lemma 29 p. 411]), and use $0 = c \equiv (x)[x+c = x]$, $1 = c \equiv (x)[x \cdot c = x]$.

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