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F. BAGEMIHL

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by

F. Bagemihl

Princeton

Let us denote the linear continuum by C. Suppose that to every $x \in C$ there corresponds a set $P(x) \subset C$ such that $x \notin P(x)$ and x is not a limit point of $P(x)$. Two points x and y of C are said to be independent, provided that $x \notin P(y)$ and $y \notin P(x)$. A subset of C is said to be independent, provided that every pair of points of this subset is independent.

Fodor [1], [2] has obtained results concerning the Lebesgue measure of independent sets. The present note contains several theorems regarding the Baire category of independent sets, a few of which are somewhat analogous to Fodor's results. In the proof of Theorem 1 we make use of an idea due to Lázár [4]. Our theorems are valid for more general sets than C, as will be seen from the proofs.

THEOREM 1. *There always exists an independent set of second category.*

Proof: We have assumed that no point x of C is a limit point of $P(x)$, and therefore we can associate with every $x \in C$ an interval $J(x)$ with rational endpoints, such that $J(x) \cap P(x)$ is empty and $x \in J(x)$. The set of all intervals with rational endpoints is enumerable; denote these intervals by $J_1, J_2, \ldots, J_n, \ldots$. For every natural number n, let C_n be the set of points $x \in C$ with the property that $J(x) = J_n$. Then $C = \bigcup_{n=1}^{\infty} C_n$. If C_n were of first category for every n, C would also be of first category [3, p. 130], which is impossible [3, p. 136]. Hence, there exists a natural number k such that C_k is of second category. If x and y are any two points of C_k, then $P(x) \cap C_k$ and $P(y) \cap C_k$ are both empty, so that x and y are independent, and consequently C_k is an independent set. This completes the proof.

A consequence of Theorem 1 and a theorem [3, p. 134] on category, is that there always exists an independent set which is of second category in every subinterval of some interval of C. It is not true, however, that there always exists an independent set which is of second category in every interval of C. This follows immediately from

THEOREM 2. *There does not always exist an independent set which is everywhere dense in* C.

Proof: If $x \in C$, define $P(x)$ to be the set of all real numbers y satisfying the relation $[x]+2 \leqq y < [x]+3$, where $[x]$ denotes the greatest integer in x. Now suppose that D, a subset of C, is everywhere dense in C. Then D must contain a point x such that $0 \leqq x < 1$, and a point y such that $2 \leqq y < 3$; since $y \in P(x)$, x and y are not independent, and hence D cannot be an independent set.

A fortiori [3, p. 135] there does not always exist an independent set which is a residual subset of C. A sufficient condition for the nonexistence of a residual independent set is furnished by

THEOREM 3. *Let* M, *a subset of* C, *be of second category, and suppose that* $P(x)$ *is of second category for every* $x \in M$. *Then there does not exist a residual independent set.*

Proof: If R is a residual set, then [3, p. 134] $R \cap M$ is not empty; let $x \in R \cap M$. Since $P(x)$ is of second category, it again follows that $R \cap P(x)$ is not empty; let $y \in R \cap P(x)$. Now x and y are not independent, and hence R cannot be an independent set.

THEOREM 4. *There does not always exist an independent set which is residual in some interval of* C.

PROOF: It is possible (see, *e.g.*, [5, p. 208]) to express C as the union of enumerably many mutually exclusive sets $E_1, E_2, \ldots, E_n, \ldots$, each of which is of second category in every interval of C. If $x \in C$, let n be that natural number for which $x \in E_n$, and define $P(x)$ to be the set of all elements of E_n lying outside the interval of length $1/n$ with x as midpoint. Now suppose that S, a subset of C, is residual in some interval, K, of C. Since each set $E_n (n = 1, 2, 3, \ldots)$ is of second category in every subinterval of K, it follows [3, pp. 130, 134] that $S \cap K \cap E_n$ $(n = 1, 2, 3, \ldots)$ is everywhere dense in K. Hence, if n is sufficiently large, there exists an $x \in S \cap K \cap E_n$ such that the interval of length $1/n$ with x as midpoint has both endpoints in the interior of K. This implies the existence of a subinterval, L, of K such that $L \cap E_n \subset P(x)$, and since $S \cap K \cap E_n$ is everywhere dense in K, there exists a $y \in S \cap L \cap E_n$. Thus S contains two elements x and y which are not independent, and therefore S cannot be an independent set.

In the proof of Theorem 4, $P(x)$ was chosen to be of second category for every $x \in C$. Does Theorem 4 remain valid if $P(x)$ is required to be a "thinner" set for every $x \in C$? Of course if $P(x)$ is required to be empty for every $x \in C$, then Theorem 4 is trivially false. The next theorem indicates, however, that the "thinness" of $P(x)$ has very little effect on the truth of Theorem 4. The proof of Theorem 5 obviously constitutes an alternative proof of Theorem 4.

THEOREM 5. *The assumption that $P(x)$ contains at most one point for every $x \in C$ does not imply the existence of an independent set which is residual in some interval of C.*

PROOF: There are enumerably many closed (nondegenerate) intervals of C with rational endpoints. There are [3, p. 344] $2^{\aleph_0}$ G_δ-subsets of C that are everywhere dense in C, and likewise, for every closed interval, H, of C with rational endpoints, there are $2^{\aleph_0}$ G_δ-subsets of H that are everywhere dense in H; all together, then, this makes $2^{\aleph_0} . \aleph_0 = 2^{\aleph_0}$ subsets, which may be arranged in a transfinite sequence,

$$G_0, G_1, \ldots, G_\xi, \ldots \qquad (\xi < \omega_\gamma), \tag{1}$$

where ω_γ is the initial number [3, p. 43] of $Z(2^{\aleph_0})$. Every $G_\xi (\xi < \omega_\gamma)$ contains [3, pp. 135, 128] $2^{\aleph_0}$ points.

Now we define, by means of transfinite induction, a sequence of distinct points

$$x_0, x_1, \ldots, x_\xi, \ldots \qquad (\xi < \omega_\gamma)$$

and a sequence of points

$$y_0, y_1, \ldots, y_\xi, \ldots \qquad (\xi < \omega_\gamma)$$

as follows. Let x_0 be an arbitrary point of G_0, and let y_0 be any other point of G_0. Suppose that $0 < \alpha < \omega_\gamma$, and that we have defined x_β and y_β for every $\beta < \alpha$. There are fewer than $2^{\aleph_0}$ points x_β wit $\beta < \alpha$, whereas G_α contains $2^{\aleph_0}$ points. Let x_α be an arbitrary point of G_α such that $x_\alpha \neq x_\beta$ $(\beta < \alpha)$, and let y_α be any other point of G_α. This completes the induction.

For every $\xi < \omega_\gamma$, let $P(x_\xi) = \{y_\xi\}$; for every $x \in C$ such that $x \neq x_\xi$ $(\xi < \omega_\gamma)$, let $P(x)$ be the empty set.

Suppose that T, a subset of C, is residual in some interval of C; then T is residual in some closed subinterval, Q, of this interval, with rational endpoints. Hence [3, p. 135], there exists a $\xi < \omega_\gamma$ such that $G_\xi \subseteqq T \cap Q$, which implies that x_ξ and y_ξ belong to T. Since x_ξ and y_ξ are not independent, T cannot be independent.

THEOREM 6. *Let d be a positive number, and suppose that, for a residual set, R, of elements* $x \in C$, *the distance between* x *and* $P(x)$ *is not less than d. Then, if D is any interval of C of length d, there exists an independent set which is residual in D.*

PROOF: The set $R \cap D$ is residual in D, and if x and y are distinct points of $R \cap D$, both different from the possible end-points of D, then x and y are independent, because $P(x)$ and $P(y)$ cannot contain any points in the interior of D.

THEOREM 7. *Let d be a positive number, and suppose that, for a set, S, of elements* $x \in C$, *which is of second category in every interval of C, the distance between* x *and* $P(x)$ *is not less than d. Then, if D is any interval of C of length d, there exists an independent set which is of second category in every subinterval of D; but there does not always exist an independent set which is everywhere dense in C, even if* $S = C$.

PROOF: The first and second parts of the conclusion follow from arguments analogous to those used in proving Theorems 6 and 2, respectively.

THEOREM 8. *Let d be a positive number. Suppose that, for every* $x \in C$, $P(x)$ *consists of at most one point, and the distance between* x *and* $P(x)$ *is not less than d. Then there does not always exist a residual independent set.*

PROOF: It is only necessary to modify the proof of Theorem 5 in two essential respects: let the terms of the sequence (1) be the G_δ-subsets of C that are everywhere dense in C, and subject each y_ξ $(\xi < \omega_\gamma)$ to the additional condition that the distance between x_ξ and y_ξ be not less than d.

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Institute for Advanced Study

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