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An addition to "logic of many-sorted theories"

by

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This article presents an extension of a theorem of Schmidt proved also in Wang [1]; familiarity with Wang's article will be presumed. Wang constructs theories $T_1^{(n)}$, formalized within the first order predicate calculus, equivalent to many sorted theories T_n. The theories T_n have the following restriction (top of page 106 in [1]): in each argument place of the primitive predicates of the many-sorted theories may occur only variables of one given sort. For example, in treating the simple theory of types as a many-sorted theory of this kind it would be necessary to introduce denumerably many primitive predicates ε_i, expressing membership between sets of type i and type $i+1$. In this paper it is shown that this restriction can be removed; in the argument places of the primitive predicates may occur any sort of variable, and in particular only certain sorts of variables, the choice of sort being possibly dependent upon the sort of variables occurring in other argument places of the primitive predicates. For example, the simple theory of types can be introduced as a many-sorted theory with primitive predicate ε for which the sort of the second argument place must be one "higher" than the sort of the first argument place. To achieve this extension a new proof of Wang's theorem 4.3 will be given. The present proof is inadequate since no proof is given of the statement on page 114 "Moreover, each line of Δ_2 is a formula of T_n which is either a truth-functional tautology or . . .". This statement is anything but obvious for the generalized theorem and its proof certainly not more trivial than the proof of the original theorem.

4.3. There is an effective process by which, given any proof in $L_1^{(n)}$ for a statement χ' of $T_1^{(n)}$ which has a translation χ in T_n, we can find a proof in L_n for χ.

We assume that χ is in prenex normal form. χ' will be provable in $L_1^{(n)}$ if and only if

$$(1) \qquad (E\alpha)S_1(\alpha) \;.\;.\;.\;.\;.\; (E\beta)S_k(\lambda) \cdot \supset \cdot \chi'$$

is provable in L_1, where more than one existence statement $(E\beta)S_i(\beta)$ arising from a given $S_i(\alpha)$ may occur, and where k is some finite number $\leqq n$. A prenex normal form of (1) is

$$(2) \qquad (\underline{\alpha})(Q\underline{\alpha})(S_1(\alpha) \supset (S_2(\beta) \supset \ldots (S_k(\lambda) \supset m(\chi')) \ldots))$$

where "$(\underline{\alpha})$" is a row of universal quantifiers one for each variable in $S_1(\alpha), \ldots, S_k(\lambda)$ and where "$(Q\underline{\alpha})$" is the row of quantifiers prefixing the matrix $m(\chi')$ for a prenex normal form of χ'. Hence χ' will be provable in $L_1^{(n)}$ if and only if (2) is provable in L_1. Any statement of the form (2) will be called a *proof-end* of χ'.

Consider a proof of (2) in Herbrand's normal form. We can assume without loss of generality that no variable of (2) is quantified twice. For each variable α of (2), assign to it the number i of the unique formula $S_i(\alpha)$ in which it occurs in (2), and assign to each variable in the proof the number obtained by applying 4.6, 4.6.1, 4.6.2, and 4.6.3, ignoring that these rules are for assigning a number to a particular occurrence of a variable in the proof.

4.3.1. If χ' is provable in $L_1^{(n)}$ there can be found a proof in L_1 of a proof-end of χ' in Herbrand normal form in which every variable occurring in the proof is assigned one and only one number.

That a proof in Herbrand normal form can be found follows from Herbrand's theorem. That there is one of this particular kind will follow from other lemmas.

A proof in Herbrand normal form of (2) begins with a formula of the form $\psi_1 \vee \psi_2 \vee \ldots \vee \psi_n$, to be called the *first formula* of the proof, where each ψ_i is obtained from the matrix of (2) by certain changes of variables. There is a part of each ψ_i, called the *core* and denoted by C_i, which is obtained from ψ_i by removing from ψ_i all occurrences of formulae $S_k(\alpha)$ for any α and k (S-formulae). The S-formulae which have been dropped from ψ_i to obtain C_i will be called the *prefixes* of ψ_i. A prefix $S_k(\alpha)$ of ψ_i will be called a *universal prefix* of ψ_i (or be said to occur universally in ψ_i) if it occurs in a part $(S_k(\alpha) \supset \phi)$ of ψ_i and will be called an existential prefix of ψ_i (or be said to occur existentially in ψ_i) if it occurs in a part $(S_k(\alpha) \cdot \phi)$ of ψ_i. The universal prefixes of ψ_i which have been obtained from the formulae $S_1(\alpha), \ldots, S_k(\lambda)$ explicitly indicated in (2) will be called the *axiom prefixes* of ψ_i. An occurrence of $S_k(\alpha)$ will be said to *immediately precede* an occurrence of $S_m(\beta)$ in ψ_i if for these occurrences $(S_k(\alpha) \supset (S_m(\beta) \supset \phi))$ or $(S_k(\alpha) \supset \supset (S_m(\beta) \cdot \phi))$ or $(S_k(\alpha) \cdot (S_m(\beta) \supset \phi))$ or $(S_k(\alpha) \cdot S_m(\beta) \cdot \phi)$ is a part of ψ_i. Finally, an occurrence of $S_k(\alpha)$ will be said to *precede* an occurrence of $S_m(\beta)$ in ψ_i if either it immediately precedes

$S_m(\beta)$, or there are prefixes $S_1(\alpha_1), \ldots, S_r(\alpha_r)$ of ψ_i such that $S_k(\alpha)$ immediately precedes $S_1(\alpha_1)$, etc. and $S_r(\alpha_r)$ immediately precedes $S_m(\beta)$ in ψ_i.

4.3.2. Given a proof of a proof-end of χ' in Herbrand normal form, a proof in Herbrand normal form of another proof-end of χ' can be found for which every S-formula occurring in the first formula of the proof occurs universally somewhere in the first formula.

Let $\psi_1 \vee \ldots \vee \psi_n$ be the first statement of a proof of (2) in Herbrand normal form. Consider truth value assignments to the atomic statements of the alternation clauses. Any alternation clause can be made false either by assigning truth (t) to all of its prefixes and assigning truth values to make its core false, or by assigning falsehood (f) to some existential prefix of ψ_i and t to all universal prefixes preceding it. Any alternation clause can be made true either by assigning t to all its prefixes and truth values to make its core t or by assigning f to some universal prefix and t to all existential prefixes preceding it.

Let $S_k(\alpha)$ be a formula only occurring existentially in $\psi_1 \vee \ldots \vee \psi_n$ and such that for some ψ_i it is not preceded in ψ_i by any other S-formula which does not occur universally somewhere. We can assume without loss of generality that ψ_1 is such an alternation clause. Consider the following truth value assignments:

(3) $S_k(\alpha)$ is assigned f, and all S-formulae preceding it are assigned t.

Any assignment (3) makes ψ_1 f and therefore makes $\psi_2 \vee \ldots \vee \psi_n$ t. We will show that $\psi_2 \vee \ldots \vee \psi_n$ can be modified to become the first formula of a proof in Herbrand normal form of a new proof-end of χ'.

Let the conjunction of the universal prefixes of ψ_1 preceding $S_k(\alpha)$ be φ_1. The existential prefixes of ψ_1 preceding $S_k(\alpha)$ occur universally somewhere in the first statement; let φ_2 be the conjunction of all such prefixes which also occur universally in ψ_1. Then replace the conjunction $S^2(\underline{\beta})$ of the axiom prefixes of ψ_2 in ψ_2 by $\varphi_1 \cdot \varphi_2 \cdot S^2(\underline{\beta})$ to become ψ_2^*. Let $(\varphi_1 \cdot \varphi_2)'$ be a formula formed from $\varphi_1 \cdot \varphi_2$ by replacing distinct variables by distinct variables which do not occur anywhere in the proof. Then replace the conjunction $S^3(\underline{\beta})$ of the axiom prefixes of ψ_3 in ψ_3 by $(\varphi_1 \cdot \varphi_2)' \cdot S^3(\underline{\beta})$ to become ψ_3^*. Repeat this for each of $\psi_4, \psi_5, \ldots$ ensuring each time that the variables being substituted for the variables of $(\varphi_1 \cdot \varphi_2)$ do not occur anywhere in the given proof nor in the preceding ψ^*'s.

4.3.3. $\psi_2^* \vee \ldots \vee \psi_n^*$ is a tautology.

It is immediate that $\psi_2 \vee \ldots \vee \psi_n \supset \psi_2^* \vee \ldots \vee \psi_n^*$ is a tautology and hence that $\sim(\psi_2^* \vee \ldots \vee \psi_n^*) \supset \sim(\psi_2 \vee \ldots \vee \psi_n)$ is a tautology. $\psi_2^* \vee \ldots \vee \psi_n^*$ will be false if either (i) every prefix is true and the cores are all false or (ii) for each alternation clause there is an existential prefix which is false and for which every prefix preceding it is true.

In case (i) the truth value of the formula will not be changed if $S_k(\alpha)$ is made false since it can occur only existentially (by the assumption that it does not occur universally) giving that $\psi_2 \vee \ldots \vee \psi_n$ would be made false by an assignment of the type (3) and contradicting that $\psi_1 \vee \ldots \vee \psi_n$ is a tautology.

In case (ii), no existential prefix of ψ_1 preceding $S_k(\alpha)$ can be false. For consider the first such prefix; this prefix occurs universally in $\psi_1 \vee \ldots \vee \psi_n$. If it occurs universally in ψ_1 then it has already been assigned t since it occurs in φ_2. If it occurs universally in $\psi_2 \vee \ldots \vee \psi_n$ and it has not already been assigned t, then ψ_1 could be assigned f together with $\psi_2^* \vee \ldots \vee \psi_n^*$ and therefore together with $\psi_2 \vee \ldots \vee \psi_n$ which is impossible. Therefore in case (ii) every prefix preceding $S_k(\alpha)$ will be true. But since $S_k(\alpha)$ can occur only existentially in $\psi_2^* \vee \ldots \vee \psi_n^*$ to assign it f will not change the truth value of this formula making it false together with ψ_1 and giving again a contradiction.

With $\psi_2^* \vee \ldots \vee \psi_n^*$ as first formula of a proof in Herbrand normal form, all the quantifiers introduced to the alternation clauses of $\psi_2, \ldots, \psi_n$ of the original proof can also be introduced in exactly the same order to the corresponding clauses here. For every existential quantifier can be introduced at any time and one could only be prevented from introducing a universal quantifier to an alternation clause by the fact that the variable to be quantified occurs free in another clause, which can only be the case if in the original proof one was prevented from introducing some universal quantifier into the first alternation clause. Further, all the variables of $\varphi_1 \cdot \varphi_2$ in ψ_2^*, of $(\varphi_1 \cdot \varphi_2)'$, etc., can be universally quantified in any order after all other variables of the clauses have been quantified. By choice of the latter quantified variables, we have a proof in Herbrand normal form of a new proof-end of χ' in which there is at least one less S-formula occurring existentially but not universally than in the original proof. Hence by repeating the above outlined process a sufficient number of times a proof-end of χ' and a proof of it can be found answering to lemma 4.3.2.

We are now able to prove 4.3.1. for consider a proof of a proof-end of χ' as given by 4.3.2. If a variable in this proof is assigned two numbers, then a variable of the first formula of the proof is assigned two numbers. If α is a variable that has been assigned both i and j, $(i \neq j)$, then at some point in the proof it has been existentially quantified to be replaced by a variable β which has been assigned the number i, and at some later point in the proof either the remaining occurrences of α have been universally quantified to be replaced by a variable λ, or again existentially quantified to be replaced by a variable λ which has been assigned a number j. But this cannot be so since then both $S_i(\alpha)$ and $S_j(\alpha)$ would occur universally, meaning that α would have to be universally quantified twice in the proof, which is impossible.

To prove 4.3 then consider a proof in which every variable is assigned one and only one number. Replace each occurrence of each variable in the proof by a new variable formed by adding the number assigned to the variable as a subscript of the variable. Thus for every α which has been assigned the number i, every occurrence of α is replaced by α_i. Further, replace each prefix $S_i(\alpha_i)$ in the proof by $\alpha_i = \alpha_i$. From the resulting scheme it is clear that a proof of χ in L_n can be constructed.

It should be stated that Schmidt has published a simplified version of his original proof in [2].

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