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Algebraic Local Invariants of Topological Spaces

by

Sze-Tsen Hu

1. Introduction [1])

Local homology groups were first introduced by E. R. van Kampen into singular homology theory in his Leyden thesis [14][2]); see also [21, p. 120 and p. 319]. In other homology theories, the local Betti numbers were studied by Čech [5], Alexandroff [1], Vaughan [24], Wilder [27] and others. In recent years, local homology groups were also defined and studied by H. B. Griffiths [8] and T. R. Brahana [4] both by limiting processes.

Local fundamental groups of locally triangulable spaces were defined by Seifert and Threlfall in [21, p. 177], and analogous definition of higher dimensional local homotopy groups is obvious. In this case, since both the local homology groups, [21, p. 120], and the local homotopy groups are defined as their global counterparts of the boundary of an open star, it follows that every result in the global theory automatically gives rise to a local version of the result.

For more general topological spaces, a method to define the local fundamental groups and, therefore, the higher dimensional local homotopy groups was implicitly suggested by O. G. Harrold, [9, p. 122]. Later, an explicit definition of the local homotopy groups was introduced by H. B. Griffiths, [8, p. 357]. However, under his definition, local homotopy groups may fail to exist. Besides, it is by no means obvious that all results in the global theory hold in the local theory; in fact, Griffiths proved elaborately the local Hurewicz theorem, [8, pp. 360—366].

In the present paper, we propose to define the local homology groups and the local homotopy groups of a topological space X at a point $x_0 \in X$ to be the (global) homology groups and the (global) homotopy groups of the tangent space $T(X, x_0)$ which is constructed in § 2. If X is locally triangulable at x_0, or more generally,

[1]) This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research & Development Command, under contract No. AF49 (638—179).

[2]) Numbers in square brackets refer to the bibliography at the end of the paper.

if x_0 is a conic point of X, then these groups are isomorphic to those of Griffiths [8] and hence to those of Seifert-Threlfall [21, p. 120 and p. 177].

As applications of these local invariants, we shall first study the homotopy classification of local maps, and a local version of the Hopf theorem will be proved. One can also apply these local invariants to study the fibre spaces with singularities in the sense of Montgomery and Samelson [18] or in the original sense of H. Seifert [20]. In the final section, we shall give an application to topological semi-groups.

2. The tangent space.

Let X be a given topological space. By a *path* in X, we mean a continuous map $\sigma : I \to X$ of the closed unit interval $I = [0,1]$ into X. The set $W(X)$ of all paths in X forms a topological space with the usual compact-open topology, [15, p. 221].

By the *total tangent space* $T(X)$ of X, we mean the subspace of $W(X)$ which consists of the totality of paths σ in X such that $\sigma(t) = \sigma(0)$ if and only if $t = 0$, in other words, a path $\sigma \in W(X)$ is in $T(X)$ if and only if it does not recross the initial point $\sigma(0)$. This space $T(X)$ was introduced by John Nash [19] in his proof of the topological invariance of the Stiefel-Whitney classes for a differentiable manifold.

Now, let x_0 be any given point in X. The subspace $T(X, x_0)$ of $T(X)$ which consists of the set of all paths $\sigma \in T(X)$ with $\sigma(0) = x_0$ will be called the *tangent space* of X at the point x_0. Therefore, $T(X, x_0)$ is the subspace of the path space $W(X)$ defined by the formula:

$$T(X, x_0) = \{\sigma \in W(X) : \sigma(t) = x_0 \text{ if and only if } t = 0\}.$$

Let C denote the arc-component of X which contains x_0. Then, obviously we have

$$T(C, x_0) = T(X, x_0).$$

The arc-component C is said to be *degenerate* if x_0 is the only point in C; otherwise, C is said to be *non-degenerate.* Then it follows immediately that *the tangent space $T(X, x_0)$ is empty if and only if the arc-component C is degenerate.*

Let A be any given subspace of X which contains x_0. Then $T(A, x_0)$ is a subspace of $T(X, x_0)$. As in [10, p. 491], we shall call (X, A, x_0) a *triplet*; then the pair (X^*, A^*), where $X^* = T(X, x_0)$ and $A^* = T(A, x_0)$, will be called the *tangent pair* of the given triplet (X, A, x_0).

3. Local homology and cohomology groups.

In this section, we shall define the local homology and cohomology groups by means of the tangent spaces constructed in the previous section. For this purpose, one may choose any (global) homology and cohomology theory defined on a suitable category of spaces and satisfying the axioms of Eilenberg and Steenrod, [6, pp. 10—15]. However, since we have already used paths in the construction of the tangent spaces and since we also plan to study the relations between local homology groups and local homotopy groups in the present paper, we have to use the singular theory, [6, pp. 185—211].

Let (X, A, x_0) be any given triplet. For each integer n and any abelian group G, the singular homology group $H_n(X^*, A^*; G)$ of the tangent pair (X^*, A^*) will be denoted by the symbol $L_n(X, A, x_0; G)$ and called the *n-dimensional local* (singular) *homology group of X modulo A at the point x_0 over the coefficient group G; in symbols, we have*

$$L_n(X, A, x; G) = H_n(X^*, A^*; G), X^* = T(X, x_0), A^* = T(A, x_0).$$

If the subspace A contains only a single point x_0, then this group will be denoted simply by $L_n(X, x_0; G)$ and called the *n-dimensional local homology group of X at x_0 over G*. On the other hand, if the coefficient group is the group Z of integers, we shall use, as usual, the simpler notation:

$$L_n(X, A, x_0) = L_n(X, A, x_0; Z), \ L_n(X, x_0) = L_n(X, x_0; Z).$$

Similarly, we may define the *n-dimensional local cohomology group*

$$L^n(X, A, x_0; G) = H^n(X^*, A^*; G), X^* = T(X, x_0), A^* = T(A, x_0),$$

and its special cases $L^n(X, x_0; G)$, $L^n(X, A, x_0)$ and $L^n(X, x_0)$.

As an immediate consequence of this definition, every operation which is available in the global homology or cohomology groups is also available in the local groups. We shall give two examples as follows.

Firstly, for each triplet (X, A, x_0), the following *boundary homomorphism* ∂ and *coboundary homomorphism* δ

$$\partial : L_n(X, A, x_0; G) \to L_{n-1}(A, x_0; G),$$
$$\delta \ : L^{n-1}(A, x_0; G) \to L^n(X, A, x_0; G)$$

are defined for every integer n and every coefficient group G.

Secondly, if the coefficient group is a ring R, then the *cup* *products* are defined in the local cohomology groups and the direct sum

$$L^*(X, x_0; R) = \sum_n L^n(X, x_0; R)$$

forms an algebra over R which will be called the *local cohomology algebra* of X at x_0 over R.

A direct description, without explicitly using the tangent spaces, of the local groups defined above can be given as follows.

By a *local singular n-simplex* in X at x_0, we mean a continuous map

$$\tau : \Delta_{n+1} \to X, \quad (n \geqq 0)$$

of the unit $(n+1)$-simplex Δ_{n+1} in euclidean $(n+2)$-space, [6, p. 55], into X such that the inverse image $\tau^{-1}(x_0)$ is the last vertex d^{n+1} of Δ_{n+1}. If $n > 0$ and i is an integer with $0 \leqq i \leqq n$, then the composed map

$$\tau e^i_{n+1} : \Delta_n \to X,$$

where $e^i_{n+1} : \Delta_n \to \Delta_{n+1}$ denotes the simplicial map defined in [6, p. 185], is a local singular $(n-1)$-simplex in X at x_0 which will be called the i-th *face* of τ. Thus, the local singular simplexes in X at x_0 form a semi-simplicial complex $S(X, x_0)$ called the *local singular complex of* X *at* x_0. Since $A \subset X$, $S(A, x_0)$ is a subcomplex of $S(X, x_0)$. Then $L_n(X, A, x_0; G)$ and $L^n(X, A, x_0; G)$ are respectively the n-dimensional homology and cohomology group of the complex $S(X, x_0)$ modulo $S(A, x_0)$ over G. In fact, one can easily see that $S(X, x_0)$ is essentially the singular complex of the tangent space $T(X, x_0)$.

4. Admissible maps.

By an *admissible map* of a triplet (X, A, x_0) into a triplet (Y, B, y_0), we mean a continuous function

$$f : (X, A, x_0) \to (Y, B, y_0)$$

such that $f^{-1}(y_0) = x_0$ or, equivalently, $f(X/x_0) \subset Y/y_0$. For example, if $(X, A, x_0) \subset (Y, B, y_0)$, i.e. if $X \subset Y$, $A \subset B$, $x_0 = y_0$, then the inclusion map of (X, A, x_0) into (Y, B, y_0) is admissible.

Two admissible maps $f, g : (X, A, x_0) \to (Y, B, y_0)$ are said to be *admissibly homotopic* if there exists a homotopy

$$h_t : (X, A, x_0) \to (Y, B, y_0), \quad 0 \leqq t \leqq 1,$$

such that $h_0 = f$, $h_1 = g$, and that, for each t, h_t is an admissible map. Such a homotopy will be called an *admissible homotopy*.

Every admissible map $f : (X, A, x_0) \to (Y, B, y_0)$ *induces a* (continuous) map

$$\hat{f} : (X^*, A^*) \to (Y^*, B^*)$$

of the corresponding tangent pairs defined by $\hat{f}(\sigma) = f\sigma$ for each $\sigma \in X^*$. If two admissible maps f, g are admissibly homotopic, then their induced maps $\hat{f}, \hat{g}$ are homotopic.

Therefore, we may define *induced homomorphisms*

$$f_* : L_n(X, A, x_0; G) \to L_n(Y, B, y_0; G),$$
$$f^* : L^n(Y, B, y_0; G) \to L^n(X, A, x_0; G)$$

by taking $f_* = \hat{f}_*$ and $f^* = \hat{f}^*$ for each integer n and each coefficient group G.

The following properties of the induced homomorphisms are obvious.

(4.1) *If f is the identity map of (X, A, x_0), then f_* and f^* are the identity automorphisms of $L_n(X, A, x_0; G)$ and $L^n(X, A, x_0; G)$ respectively.*

(4.2) *For any two admissible maps*

$$f : (X, A, x_0) \to (Y, B, y_0), \quad g : (Y, B, y_0) \to (Z, C, z_0),$$

the composed map $gf : (X, A, x_0) \to (Z, C, z_0)$ is also admissible and we have

$$(gf)_* = g_* f_*, \quad (gf)^* = f^* g^*.$$

(4.3) *If $f : (X, A, x_0) \to (Y, B, y_0)$ is an admissible map, then the map $g : (A, x_0) \to (B, y_0)$ defined by f is also admissible and the following rectangles are commutative:*

$$\begin{array}{ccccccc} L_n(X,A,x_0;G) & \xrightarrow{f_*} & L_n(Y,B,y_0;G) & \quad & L^n(X,A,x_0;G) & \xleftarrow{f^*} & L^n(Y,B,y_0;G) \\ \downarrow \partial & & \downarrow \partial & & \uparrow \delta & & \uparrow \delta \\ L_{n-1}(A,x_0;G) & \xrightarrow{g_*} & L_{n-1}(B,y_0;G) & & L^{n-1}(A,x_0;G) & \xleftarrow{g^*} & L^{n-1}(B,y_0;G). \end{array}$$

(4.4) *For any triplet (X, A, x_0), the inclusion maps $i : (A, x_0) \subset (X, x_0)$ and $j : (X, x_0) \subset (X, A, x_0)$ are admissible, and the following two sequences*

$$\cdots \xleftarrow{i_*} L_{n-1}(A,x_0;G) \xleftarrow{\partial} L_n(X,A,x_0;G) \xleftarrow{j_*} L_n(X,x_0;G) \xleftarrow{i_*} L_n(A,x_0;G) \xleftarrow{\partial} \cdots$$
$$\cdots \xrightarrow{i^*} L^{n-1}(A,x_0;G) \xrightarrow{\delta} L^n(X,A,x_0;G) \xrightarrow{j^*} L^n(X,x_0;G) \xrightarrow{i^*} L^n(A,x_0;G) \xrightarrow{\delta} \cdots$$

are exact. These will be called the local homology sequence and the local cohomology sequence respectively.

(4.5) *If two admissible maps f, $g : (X, A, x_0) \to (Y, B, y_0)$ are* *admissibly homotopic, then* $f_* = g_*$ *and* $f^* = g^*$ *for every integer* n *and every coefficient group* G.

(4.6) *If the coefficient group is a ring* R, *then the induced homomorphisms* f^* *of an admissible map* $f:(X, x_0) \to (Y, y_0)$ *preserve cup products and hence define an algebra homomorphism* $f^*: L^*(Y, y_0, R) \to L^*(X, x_0; R)$.

5. The local characterization.

To justify the definition of local homology and cohomology groups given in § 3, we have to show that the groups $L_n(X, A, x_0; G)$ and $L^n(X, A, x_0; G)$ are really *local invariants* of the pair (X, A) at the point x_0; in other words, we must prove that these groups will remain unchanged after deleting a part of the space X or the subspace A outside of an open neighborhood of the point x_0. This will be proved for completely regular spaces in the present section.

Let X be a given topological space, x_0 a given point in X, and U a given open neighborhood of x_0 in X. Then the tangent space $U^* = T(U, x_0)$ is a subspace of $X^* = T(X, x_0)$.

LEMMA 5.1. *If* X *is completely regular at* x_0, *then there exists a homotopy*

$$d_t: (X^*, U^*) \to (X^*, U^*), \qquad 0 \leqq t \leqq 1,$$

such that d_0 *is the identity map and* d_1 *send* X^* *into* U^*.

PROOF. Since X is completely regular at x_0, there exists a continuous real function $\chi: X \to I$ such that $\chi(X\backslash U) = 0$ and $\chi(x_0) = 1$. Next, define a continuous real function $\phi: X^* \times I \to I$ by setting

$$\phi(\sigma, t) = \mathrm{Inf}_{s \leqq t} \quad \chi[\sigma(s)]$$

for every $\sigma \in X^*$ and every $t \in I$. For a fixed σ in X^*, $\phi(\sigma, t)$ is a non-increasing function of t with $\phi(\sigma, 0) = 1$. Hence the equation $\phi(\sigma, t) = t$ has a unique solution $\psi(\sigma)$ in the variable t which depends continuously on σ. Thus, we obtain a continuous real function $\psi: X^* \to I$. Since $\phi(\sigma, 0) = 1$, it follows that $\psi(\sigma) > 0$ for every $\sigma \in X^*$.

By means of the continuous real function ψ, we may define a homotopy $d_t: X^* \to X^*$, $(0 \leqq t \leqq 1)$, as follows. For each path $\sigma \in X^*$ and each $t \in I$, let $d_t(\sigma)$ denote the path in X defined by

$$[d_t(\sigma)](s) = \sigma[s - st + st\,\psi(\sigma)], \quad (s \in I).$$

Intuitively speaking, $d_t(\sigma)$ is obtained from σ by omitting the part of σ beyond the point $\sigma[1 - t + t\psi(\sigma)]$. Since $\psi(\sigma) > 0$, it follows that $d_t(\sigma) \in X^*$. This completes the construction.

By the construction of the homotopy d_t, it is obvious that d_0 is the identity map on X^* and that $d_t(U^*) \subset U^*$. It remains to verify that d_1 sends X^* into U^*. For this purpose, let $\sigma \in X^*$ and $t \in I$ be arbitrarily given. Let $r = t\psi(\sigma)$. Then we have

$$[d_1(\sigma)](t) = \sigma(r).$$

Since $r \leqq \psi(\sigma)$, it follows from the definition of the number $\psi(\sigma)$ that $\phi(\sigma, r) \geqq r$. Since $\psi(\sigma) > 0$, this implies that $\chi[\sigma(r)] > 0$. Hence $[d_1(\sigma)](t) \in U$. Since $\sigma \in X^*$ and $t \in I$ are arbitrary, this proves $d_1(X^*) \subset U^*$ and completes the proof of the lemma.

By a *completely regular triplet*, we mean a triplet (X, A, x_0) in which X is a completely regular at x_0.

THEOREM 5.2. *If (X, A, x_0) is a completely regular triplet and (U, D, u_0) is a triplet such that U is an open neighborhood of x_0 in x, $D = U \cap A$, and $u_0 = x_0$, then the inclusion map $f : (U, D, u_0) \subset (X, A, x_0)$ induces the isomorphisms*

$$f_* : L_n(U, D, u_0; G) \approx L_n(X, A, x_0; G),$$
$$f^* : L^n(X, A, x_0; G) \approx L^n(U, D, u_0; G)$$

for every integer n and every coefficient group G.

PROOF. Let $g : (U, u_0) \subset (X, x_0)$ and $h : (D, u_0) \subset (A, x_0)$ denote the inclusion maps. Then g induces the inclusion map $\hat{g} : U^* \subset X^*$ of the tangent spaces. By the homotopy axiom of the (global) singular homology theory, Lemma 5.1 implies that

$$H_n(X^*, U^*; G) = 0, \qquad H^n(X^*, U^*; G) = 0$$

for every integer n and every coefficient group G. Then it follows from the exactness axiom that the inclusion map $\hat{g}$ induces isomorphisms

$$\hat{g}_* : H_n(U^*; G) \approx H_n(X^*; G), \quad \hat{g}^* : H^n(X^*; G) \approx H^n(U^*; G).$$

Hence the inclusion map g induces isomorphisms

$$g_* : L_n(U, u_0; G) \approx L_n(X, x_0; G), \quad g^* : L^n(X, x_0; G) \approx L^n(U, u_0; G).$$

As a subspace of X, A is completely regular at x_0. Since D is an open neighborhood of x_0 in A, the inclusion map h induces isomorphisms

$$h_* : L_n(D, u_0; G) \approx L_n(A, x_0; G), \quad h^* : L^n(A, x_0; G) \approx L^n(D, u_0; G).$$

Then our theorem follows from (4.2)—(4.4) and the famous „five" lemma, [6, p. 16].

COROLLARY 5.3. *Let (Y, B, y_0) be completely regular and $(X, A, x_0) \subset (Y, B, y_0)$. If there exists an open neighborhood*

U of $x_0 = y_0$ in Y such that $U \subset X$ and $U \cap B \subset A$, then the *inclusion map $f : (X, A, x_0) \subset (Y, B, y_0)$ induces the isomorphisms*

$$f^* : L_n(X, A, x_0 ; G) \approx L_n(Y, B, y_0; G),$$
$$f_* : L^n(Y, B, y_0; G) \approx L^n(X, A, x_0; G)$$

for each integer n and each coefficient group G.

Proof. Let $D = U \cap B$ and $u_0 = x_0$. Consider the inclusion maps $g : (U, D, u_0) \subset (X, A, x_0)$ and $h : (U, D, u_0) \subset (Y, B, y_0)$. Then we have $fg = h$. By (4.2), we obtain $f_* g_* = h_*$ and $g^* f^* = h^*$. According to (5.2), g_*, h_*, g^*, h^* are isomorphisms. Hence, $f_* = h_* g_*^{-1}$ and $f^* = g^{*-1} h^*$ are also isomorphisms. *Q.E.D.*

The complete regularity assumed in this section is used only in the proof of (5.1) which implies (5.2) and (5.3) no matter what global homology theory might have been chosen in § 3 provided that the axioms used in the proof of (5.2) and (5.3) are satisfied. However, for the singular homology theory which has been chosen in § 3, this condition is inessential; in fact, one can prove (5.2) and (5.3) without assuming the complete regularity by the methods used in § 14 below. See also (14.9). Hence, hereafter, we shall drop all conditions about complete regularity.

6. Local maps.

By a *local map* of a triplet (X, A, x_0) into a triplet (Y, B, y_0), we mean an admissible map

$$f : (U, U \cap A, x_0) \to (Y, B, y_0),$$

where U is an open neighborhood of x_0 in X. Let

$$g : (V, V \cap A, x_0) \to (Y, B, y_0)$$

be another local map of (X, A, x_0) into (Y, B, y_0). The local maps f, g are said to be *congruent*, $f \equiv g$, if there exists an open neighborhood W of x_0 in X such that $W \subset U \cap V$ and $f \mid W = g \mid W$. The local maps f, g are said to be *locally homotopic*, $f \simeq g$, if there exists an open neighborhood W of x_0 in X and an admissible homotopy

$$h_t : (W, W \cap A, x_0) \to (Y, B, y_0), \quad 0 \leqq t \leqq 1,$$

such that $h_0 \equiv f$ and $h_1 \equiv g$. Thus, the totality of local maps of (X, A, x_0) into (Y, B, y_0) are divided into disjoint *congruence classes* and into disjoint *homotopy classes.*

To define the induced homomorphisms of a local map, let us consider the following diagram:

$$(X, A, x_0) \xleftarrow{i} (U, U \cap A, x_0) \xrightarrow{f} (Y, B, y_0)$$

where f is a given local map of (X, A, x_0) into (Y, B, y_0) and i denotes the inclusion map. By (5.2), the induced homomorphisms i_* and i^* are isomorphisms. Hence we may define homomorphisms

$$f_\natural = f_* i_*^{-1} : L_n(X, A, x_0; G) \to L_n(Y, B, y_0; G),$$
$$f^\natural = i^{*-1} f^* : L^n(Y, B, y_0; G) \to L^n(X, A, x_0; G)$$

for each integer n and each coefficient group G. These homomorphisms will be called the *induced homomorphisms of the local map* f.

The admissible maps of (X, A, x_0) into (Y, B, y_0) are special cases of local maps. By (4.1), we have the following property:

(6.1) *If* $f : (X, A, x_0) \to (Y, B, y_0)$ *is an admissible map, then we have* $f_\natural = f_*$ *and* $f^\natural = f^*$.

Next, let $f : (U, U \cap A, x_0) \to (Y, B, y_0)$ be a local map of (X, A, x_0) into (Y, B, y_0) and $g : (V, V \cap B, y_0) \to (Z, C, z_0)$ be a local map of (Y, B, y_0) into (Z, C, z_0). Let $W = f^{-1}(V) \subset U$. Then W is an open neighborhood of x_0 in X. Define a continuous map $h : (W, W \cap A, x_0) \to (Z, C, z_0)$ by taking $h(x) = gf(x)$ for every $x \in W$. Since h is clearly admissible, it is a local map of (X, A, x_0) into (Z, C, z_0). This local map h will be called the *composition* of f and g; in symbols, $h = gf$. Then, we have the following property:

(6.2) $$(gf)_\natural = g_\natural f_\natural \text{ and } (gf)^\natural = f^\natural g^\natural.$$

The following assertions about induced homomorphisms of local maps are obvious.

(6.3) *If* $f : (U, D, x_0) \to (Y, B, y_0)$ *is a local map of* (X, A, x_0) *into* (Y, B, y_0) *with* $D = U \cap A$, *then the map* $g : (D, x_0) \to (B, y_0)$ *defined by* f *is a local map of* (A, x_0) *into* (B, y_0) *and the following rectangles are commutative*:

$$\begin{array}{ccc} L_n(X, A, x_0; G) & \xrightarrow{f_\natural} & L_n(Y, B, y_0; G) \\ \downarrow \partial & & \downarrow \partial \\ L_{n-1}(A, x_0; G) & \xrightarrow{g_\natural} & L_{n-1}(B, y_0; G) \end{array} \qquad \begin{array}{ccc} L^n(X, A, x_0; G) & \xleftarrow{f^\natural} & L^n(Y, B, y_0; G) \\ \uparrow \delta & & \uparrow \delta \\ L^{n-1}(A, x_0; G) & \xleftarrow{g^\natural} & L^{n-1}(B, y_0; G) \end{array}$$

(6.4) *If two local maps* f, g *of* (X, A, x_0) *into* (Y, B, y_0) *are locally homotopic, then* $f_\natural = g_\natural$ *and* $f^\natural = g^\natural$ *for every integer* n *and every coefficient group* G.

(6.5) *If the coefficient group is a ring* R, *then the induced homomorphisms* $f^\natural$ *of a local map* $f : (X, x_0) \to (Y, y_0)$ *preserve cup products and hence define an algebra homomorphism*

$$f^{\natural} : L^*(Y, y_0; R) \to L^*(X, x_0; R).$$

7. Conic points.

Let F be any non-vacuous topological space. If, in the topological product $F \times I$, we identify the subset $F \times 1$ to a single point v, we obtain a quotient space Con F called *the cone over* F. The point v is called the *vertex* of Con F. Let

$$p : F \times I \to \text{Con } F$$

denote the natural projection. Then p maps $F \times 1$ onto v and $(F \times I)\backslash(F \times 1)$ homeomorphically onto $(\text{Con } F)\backslash v$. The space F will be considered as a subspace of Con F by identifying $x \in F$ with $p(x, 0) \in \text{Con } F$. If K is any non-vacuous subspace of F, then Con K is the subspace $p(K \times I)$ of Con F; if K is the empty subspace of F, then we define Con $K = v$.

Let (X, A, x_0) be a given triplet. The point x_0 is said to be a *conic point* of the pair (X, A) if there exists an open neighborhood U of x_0 in X such that the part of (X, A) within the closure $\overline{U}$ is the *join* of x_0 and the part of (X, A) within the frontier $F = \overline{U}\backslash U$, that is to say, there exists a homeomorphism

$$h : \overline{U} \to \text{Con } F$$

of $\overline{U}$ onto Con F satisfying the following three conditions:

(CP1) $h(x_0) = v$,
(CP2) $h(\overline{U} \cap A) = \text{Con}(F \cap A)$,
(CP3) $h(x) = p(x, 0)$ for every $x \in F$.

This neighborhood U will be called a *conic neighborhood* of x_0 in (X, A). The pair (X, A) is said to be *locally triangulable* at the point x_0 if there exists a conic neighborhood U of x_0 in (X, A) such that the pair $(F, F \cap A)$, where $F = \overline{U}\backslash U$, is finitely triangulable.

Assume that x_0 is a conic point of (X, A) and that U is a conic neighborhood of x_0 in (X, A). Let $F = \overline{U}\backslash U$ and choose a homeomorphism $h : \overline{U} \to \text{Con } F$ satisfying the conditions (CP1—3). We are going to establish the following

Lemma 7.1. *The pair $(F, F \cap A)$ is of the same homotopy type as the tangent pair (X^*, A^*).*

Proof. Define a map $\iota : (F, F \cap A) \to (X^*, A^*)$ as follows. For each $x \in F$, let $\iota(x)$ denote the path in X given by

$$[\iota(x)](t) = h^{-1}p(x, 1 - t), \qquad t \in I.$$

It is clear that $\iota(x) \in X^*$ and that $\iota(x) \in A^*$ if $x \in F \cap A$. One can also easily see that ι is a homeomorphism of $(F, F \cap A)$ into (X^*, A^*). By means of this embedding ι, we may consider $(F, F \cap A)$ as a sub-pair of (X^*, A^*).

Next, let us define a continuous real function $\chi : X \to I$ as follows. Let $q : F \times I \to I$ denote the natural projection. Then χ is given by

$$\chi(x) = \begin{cases} qp^{-1}h(x), & \text{if } x \in \bar{U}, \\ 0, & \text{if } x \in X \backslash U. \end{cases}$$

Hence we have $\chi^{-1}(1) = x_0$ and $\chi^{-1}(0) = X \backslash U$.

As in the proof of (5.1), define a continuous real function $\phi : X^* \times I \to I$ by setting

$$\phi(\sigma, t) = \mathrm{Inf}_{s \leqq t}\, \chi[\sigma(s)]$$

for every $\sigma \in X^*$ and every $t \in I$. Then, define a continuous real function $\psi : X^* \to I$ by taking $\psi(\sigma)$ to be the unique solution of the equation $\phi(\sigma, t) = t$ for each given $\sigma \in X^*$. Since $\phi(\sigma, t) = 1$ if and only if $t = 0$, it follows that $0 < \psi(\sigma) < 1$ for every $\sigma \in X^*$.

Define a homotopy $d_t : (X^*, A^*) \to (X^*, A^*)$, $0 \leqq t \leqq 1$, by taking

$$[d_t(\sigma)](s) = \sigma[s - st + st\,\psi(\sigma)], \qquad (s \in I),$$

for each $t \in I$ and each $\sigma \in X^*$. As in the proof of (5.1), one can verify that d_0 is the identity map on (X^*, A^*) and $[d_1(\sigma)](I) \subset U$ for every $\sigma \in X^*$.

Let $r : F \times I \to F$ denote the natural projection. There is natural contraction $e_t : U \to U$, $0 \leqq t \leqq 1$, of U to the point x_0 defined by

$$e_t(x) = \begin{cases} x_0, & \text{if } x = x_0, \\ h^{-1}p[rp^{-1}h(x), t + (1-t)qp^{-1}h(x)], & \text{if } x \in U \backslash x_0, \end{cases}$$

for every $t \in I$.

Now, let us construct a family of continuous maps $f_t : (X^*, A^*) \to (X^*, A^*)$, $(0 \leqq t \leqq 1)$, as follows. For $t = 0$, we define $f_0 = d_1$. We are going to define f_t for the case $0 < t \leqq 1$. Let $\sigma \in X^*$. Then $d_1(\sigma)$ is a path in U. Using the natural contraction e_t, we define the path $f_t(\sigma) \in X^*$ by taking

$$[f_t(\sigma)](s) = \begin{cases} e_{(t-s)/t}\{[d_1(\sigma)](t)\}, & \text{if } 0 \leqq s \leqq t, \\ [d_1(\sigma)](s), & \text{if } t \leqq s \leqq 1. \end{cases}$$

Since $e_t\{[d_1(\sigma)](0)\} = x_0$ for every $t \in I$, it can be verified that f_t, $(0 \leqq t \leqq 1)$, form a homotopy. Intuitively speaking, $f_t(\sigma)$ is obtained by replacing the part of the path $d_1(\sigma)$ up to the parametric value t by the segment joining x_0 to $[d_1(\sigma)](t)$. In particular, $f_1(\sigma)$ is the line segment joining x_0 to $[d_1(\sigma)](1)$.

Next, define a homotopy $g_t : (X^*, A^*) \to (X^*, A^*)$, $(0 \leqq t \leqq 1)$, as follows. Let $\sigma \in X^*$. Then the point $u = [f_1(\sigma)](1)$ is in $U\backslash x_0$. Let

$$x = rp^{-1}h(u) \in F, \quad k = qp^{-1}h(u) \in I.$$

We define the path $g_t(\sigma) \in X$ by taking

$$[g_t(\sigma)](s) = h^{-1}p(x, 1 - s + ks - kst), \qquad (s \in I).$$

Intuitively speaking, $g_t(\sigma)$ is obtained by extending the line segment $f_1(\sigma)$ to a position where the value of the function χ is $k - kt$. One can easily verify that $g_0 = f_1$ and that g_1 is a retraction of (X^*, A^*) onto $\iota(F, F \cap A)$.

Let $\kappa : (X^*, A^*) \to (F, F \cap A)$ denote the map defined by $\kappa(\sigma) = \iota^{-1}[g_1(\sigma)]$ for each $\sigma \in X^*$. Then it follows that $\kappa\iota$ is the identity map on $(F, F \cap A)$ and that $\iota\kappa = g_1$ is homotopic to the identity map on (X^*, A^*). This completes the proof.

Corollary 7.2. *The frontier $F = \bar{U}\backslash U$ of any conic neighborhood U of x_0 in X is homeomorphic with a free deformation retract of the tangent space $X^* = T(X, x_0)$.*

The following theorem is an immediate consequence of (7.1).

Theorem 7.3. *If U is a conic neighborhood of x_0 in (X, A) and $F = \bar{U}\backslash U$, then for each integer n and each coefficient group G we have isomorphisms*

$$L_n(X, A, x_0; G) \approx H_n(F, F \cap A; G),$$
$$L^n(X, A, x_0; G) \approx H^n(F, F \cap A; G).$$

Corollary 7.4. *If X is a simplicial complex, $x_0 \in X$, and F denotes the frontier of the star of x_0 in X, then*

$$L_n(X, x_0; G) \approx H_n(F; G), \quad L^n(X, x_0; G) \approx H^n(F; G)$$

for each integer n and each coefficient group G.

Hence, for triangulable spaces, our local homology groups reduce to those defined by Seifert-Threlfall, [21, p. 121]. On the other hand, the relation between our local homology groups and those of van Kampen [14] is given by the following

Theorem 7.5. *If x_0 is a conic point in X, then we have*

$$L_n(X, x_0; G) \approx H_{n+1}(X, X\backslash x_0; G),$$
$$L^n(X, x_0; G) \approx H^{n+1}(X, X\backslash x_0; G),$$

for each $n \neq 0$ and each coefficient group G.

PROOF. Let U be a conic neighborhood of x_0 in X and $F = \overline{U}\backslash U$. By the excision axiom, we have

$$H_{n+1}(X, X\backslash x_0; G) \approx H_{n+1}(\overline{U}, \overline{U}\backslash x_0; G)$$

for each n. Since F is a deformation retract of $\overline{U}\backslash x_0$, we have

$$H_{n+1}(\overline{U}, \overline{U}\backslash x_0; G) \approx H_{n+1}(\overline{U}, F; G)$$

for each n. Since $\overline{U}$ is contractible, it follows from the exactness axiom that

$$\partial : H_{n+1}(\overline{U}, F; G) \approx H_n(F; G)$$

for each $n \neq 0$. By (7.4) and these isomorphisms, we obtain

$$L_n(X, x_0; G) \approx H_{n+1}(X, X\backslash x_0; G)$$

for each $n \neq 0$. Similarly, one can prove the isomorphisms for cohomology. Q.E.D.

For the missing case $n = 0$, we may define the *reduced* local homology and cohomology groups

$$\tilde{L}_0(X, x_0; G) = \tilde{H}_0(X^*; G), \ \tilde{L}^0(X, x^0; G) = \tilde{H}_0(X^*; G).$$

Then we have

$$\tilde{L}_0(X, x_0; G) \approx H_1(X, X\backslash x_0; G),$$
$$\tilde{L}^0(X, x_0; G) \approx H^1(X, X\backslash x_0; G).$$

As a special case of (7.4), we state the following proposition which corresponds to the dimension axiom in the global homology theory.

PROPOSITION 7.6. *The local homology and cohomology groups of the unit interval $I = [0,1]$ at the point 0 are as follows:*

$$L_0(I, 0; G) \approx G, \quad L_n(I, 0; G) = 0, \quad n \neq 0;$$
$$L^0(I, 0; G) \approx G, \quad L^n(I, 0; G) = 0, \quad n \neq 0.$$

8. Remarks on excision.

In the previous sections, we established properties of our local homology theory which are analogous to the Eilenberg-Steenrod axioms for global homology theory except the excision axiom, [6, p. 11].

Naturally, we would expect a notion of *local excision* defined as follows. Let (X, A, x_0) be a given triplet and U be a subset of X satisfying the conditions:

(LE1) $x_0 \in U \subset A$,
(LE2) $U\backslash x_0$ is open in $X\backslash x_0$,
(LE3) $\overline{U}\backslash x_0$ is contained in the interior of A.

Denote $X' = X \backslash (U \backslash x_0)$ and $A' = A \backslash (U \backslash x_0)$. Then the inclusion map $e : (X', A', x_0) \subset (X, A, x_0)$ may be called the *local excision* of $U \backslash x_0$.

In this case, the tangent space $U^* = T(U, x_0)$ is a subspace of X^* contained in A^* but, in general, U^* is not necessarily open in X^*. Hence we cannot deduce from the excision axiom for global homology theory that the induced homomorphisms of a local excision on local homology groups are isomorphisms. On the other hand, if one intends to prove this by using the direct definition of the local homology group at the end of § 3 and the method given in [6, pp. 197—200], he will find that he cannot get through because the unit simplex with one of its vertices deleted is non-compact. The author does not know if the induced homomorphisms of a local excision are always isomorphisms.

If we are willing to lose the compact-open topology of X^*, we can certainly get the local excision property by enlarging the topology of X^* so that U^* is open for every subset U of X such that $x_0 \in U$ and $U \backslash x_0$ is open in $X \backslash x_0$. But, it seems to the author that the local excision property is not so important as to compensate the loss of the nice compact-open topology of X^*.

Finally, the following weaker result is an immediate consequence of (7.1).

(8.1) *The induced homomorphisms e_* and e^* of the local excision $e : (X', A', x_0) \subset (X, A, x_0)$ on the local homology and cohomology groups are isomorphisms if there exist an open neighborhood V of x_0 in X and a homeomorphism*

$$h : \bar{V} \to \text{Con } F$$

of $\bar{V}$ onto Con F, *where $F = \bar{V} \backslash V$, such that*

(i) $h(x_0) = v$,

(ii) $h(\bar{V} \cap A) = \text{Con } (F \cap A)$,

(iii) $h(\bar{V} \cap U) = \text{Con } (F \cap U)$,

(iv) $h(x) = p(x, 0)$ for *every* $x \in F$.

9. The degree of a local map.

Let us consider a given local map f of (X, x_0) into (Y, y_0) defined on an open neighborhood U of x_0 in X, where Y is locally homeomorphic to the n-dimensional euclidean space R^n at the point y_0 with $n > 1$.

According to (7.3), the local cohomology group $L^{n-1}(Y, y_0)$ is an infinite cyclic group. By a *local orientation* of Y at y_0, we mean a choice of a generator of the group $L^{n-1}(Y, y_0)$. There are two orientations of Y at y_0. Assume that a generator e of $L^{n-1}(Y, y_0)$ has been chosen; thus, Y is *locally oriented* at y_0.

By § 6, the local map f induces a homomorphism

$$f^{\natural} : L^{n-1}(Y, y_0) \to L^{n-1}(X, x_0)$$

which depends only the local homotopy class of f.

The element $f^{\natural}(e)$ in the local cohomology group $L^{n-1}(X, x_0)$ will be called the *degree* of the local map f and denoted by deg (f).

Since Y is locally homeomorphic to R^n at y_0, it makes sense to talk about the line-segment joining two points in a sufficiently small neighborhood of y_0 in Y. Then we have the following generalization of the Poincaré-Bohl theorem [2, p. 459].

(9.1) *If $f : (U, x_0) \to (Y, y_0)$ and $g : (V, x_0) \to (Y, y_0)$ are two local maps of (X, x_0) into (Y, y_0) and if there exists an open neighborhood W of x_0 in X such that $W \subset U \cap V$ and that, for each $x \in W \backslash x_0$, the line-segment which joins $f(x)$ to $g(x)$ in Y does not contain the point y_0, then we have*

$$\deg (f) = \deg (g).$$

PROOF. Define a homotopy $h_t : (W, x_0) \to (Y, y_0)$, $(0 \leqq t \leqq 1)$, by taking $h_t(x)$ to be the point which divides the line-segment joining $f(x)$ to $g(x)$ in the ratio $t : (1 - t)$ for each $x \in W$ and each $t \in I$. Then $h_0 = f$, $h_1 = g$, and $h_t(W \backslash x_0) \subset Y \backslash y_0$ for each $t \in I$. This implies that f and g are locally homotopic and hence deg $(f) =$ deg (g).

For example, let us consider a system of n continuous real functions $\{f_1, \ldots, f_n\}$ defined on a neighborhood V of x_0 in X such that the point x_0 is an *isolated zero* of the system. Thus there exists an open neighborhood $U \subset V$ of x_0 in X such that, for each $x \in U$, $f_i(x) = 0$ for all $i = 1, \ldots, n$ if and only if $x = x_0$. Let $Y = R^n$ and $y_0 = (0, \ldots, 0)$. Then we obtain a local map $f : (U, x_0) \to (Y, y_0)$ defined by

$$f(x) = (f_1(x), \ldots, f_n(x))$$

for each $x \in U$. The degree deg (f) of this local map f will be called the *characteristic* of the system $\{f_1, \ldots, f_n\}$ at its isolated zero x_0.

If the space X is also locally homeomorphic to R^n at x_0 and is locally oriented at x_0 by the choice of a generator d of the infinite cyclic group $L^{n-1}(X, x_0)$, then the degree deg (f) of a local map f of (X, x_0) into (Y, y_0) determines an integer k such that deg $(f) =$ kd. This integer k will be called the *arithmetic degree* of the local map f or the *index* of f and will be denoted by ind (f). In particular, if we have $(X, x_0) = (Y, y_0)$ and $d = e$, then ind (f) does not depend on the orientation $d = e$ of X at x_0; in this case, the integer ind (f) is the Poincaré-Brouwer index of the isolated fixed point x_0 of f.

In the preceding example of a system of n continuous real functions $\{f_1, \ldots, f_n\}$, if X is locally homeomorphic to R^n and locally oriented at x_0, then the index ind (f) of the local map f will be called the *arithmetic characteristic* of the system $\{f_1, \ldots, f_n\}$ at its isolated zero x_0 or the *multiplicity* of this zero x_0.

As an illustrating example, let Z denote the space of all complex numbers and $z_0 = 0$. Let us consider the local maps of (Z, z_0) into itself. First, let f be an analytic function with z_0 as an isolated zero. Then there exists an open neighborhood U of z_0 in Z such that

$$f(z) = az^p[1 + \lambda(z)], \qquad z \in U,$$

where, a is a non-zero complex number, p is a positive integer, and $\lambda : U \to Z$ is an analytic function satisfying $| \lambda(z) | < 1$ for each $z \in U$. Define a local homotopy $f_t : (U, z_0) \to (Z, z_0)$, $0 \leqq t \leqq 1$, by taking

$$f_t(z) = [t + (1 - t)r]e^{i(1-t)\theta} z^p[1 + (1 - t)\,\lambda(z)]$$

for each $z \in U$, where $r = | a |$ and $\theta = am(a)$. Then we have $f_0 = f$ and $f_1(z) = z^p$ for each $z \in U$. This implies that ind $(f) =$ ind $(f_1) = p$. Next, let us define

$$g, h : (Z, z_0) \to (Z, z_0)$$

by $g(z) = | z |$ and $h(z) = \bar{z}$. Then we obtain ind $(g) = 0$ and ind $(h) = -1$. Finally, let $k = hf$. Then we get

$$\text{ind } (k) = \text{ind } (h) \cdot \text{ind } (f) = -p.$$

10. Local classification theorems.

Let us consider the set Γ of all local maps of (X, x_0) into (Y, y_0). According to § 6, Γ is divided into disjoint (local) homotopy classes. The *local classification problem* is to enumerate these homotopy classes.

In the present section, we shall study the local classification problem for the case where X is locally triangulable at x_0 and Y is locally euclidean at y_0. Let m and n denote the dimensions of the spaces X and Y at the points x_0 and y_0 respectively. Let Y be locally oriented at y_0 by a generator e of infinite cyclic group $L^{n-1}(Y, y_0)$. Then, by § 9, every local map $f \in \Gamma$ determines an element deg (f) of the local cohomology group $L^{n-1}(X, x_0)$.

THEOREM 10.1. (The local Hopf theorem). *If $n > 1$ and $m \leqq n$, then the assignment $f \to \deg(f)$ establishes a one-to-one correspondence between the homotopy classes of the local maps of (X, x_0) into (Y, y_0) and the elements of the local cohomology group $L^{n-1}(X, x_0)$.*

PROOF. Since f^h depends only on the homotopy class of the local map f, it follows that the assignment $f \to \deg(f)$ defines a function

$$\kappa : C \to L^{n-1}(X, x_0)$$

where C denotes the set of homotopy classes of the local maps Γ. We are going to prove that κ sends C onto $L^{n-1}(X, x_0)$ in a one-to-one fashion.

Let U be a conic neighborhood of x_0 in X and V be a conic neighborhood of y_0 in Y. Denote $M = \bar{U} \backslash U$ and $N = \bar{V} \backslash V$. Then there are homeomorphisms

$$h : \bar{U} \to \text{Con } M, \quad k : \bar{V} \to \text{Con } N$$

satisfying the three condition (CP1—3) with obvious modifications. According to our assumptions, M is a finitely triangulable space of dimensions not exceeding $n - 1$ and N is an $(n - 1)$-sphere.

As in the first paragraph of the proof of (7.1), we have natural imbeddings

$$\iota : M \to X^*, \qquad \chi : N \to Y^*$$

of M and N into the tangent spaces. According to (7.1), the induced homomorphisms

$$\iota^* : L^{n-1}(X, x_0) \to H^{n-1}(M), \quad \chi^* : L^{n-1}(Y, y_0) \to H^{n-1}(N)$$

are isomorphisms. Hence $d = \chi^*(e)$ is a generator of the infinite cyclic group $H^{n-1}(N)$.

Now, let us prove that κ sends C onto $L^{n-1}(X, x_0)$. Let α be any element in $L^{n-1}(X, x_0)$. By the (global) Hopf theorem, there exists a map $\phi : M \to N$ such that the induced homomorphism

$$\phi^* : H^{n-1}(N) \to H^{n-1}(M)$$

carries the generator d into the element $\iota^*(\alpha)$. The map ϕ defines a map $\phi' :$ Con $M \to$ Con N in the obvious way. Then the composed map

$$f = k^{-1} \phi' h : (U, x_0) \to (Y, y_0)$$

is a local map of (X, x_0) into (Y, y_0). This local map f will be denoted by Con (ϕ). It can be verified that the following rectangle

$$\begin{array}{ccc} L^{n-1}(X, x_0) & \xleftarrow{f^\natural} & L^{n-1}(Y, y_0) \\ \downarrow \iota^* & & \downarrow \chi^* \\ H^{n-1}(H) & \xleftarrow{\phi^*} & H^{n-1}(N) \end{array}$$

is commutative. Hence

$$\deg (f) = f^\natural(e) = \iota^{*-1} \phi^* \chi^*(e) = \iota^{*-1} \phi^*(d) = \alpha.$$

This proves that κ send C onto $L^{n-1}(X, x_0)$.

It remains to prove that κ is one-to-one. In order to do this, we have to prepare some preliminary considerations.

First, let f be an arbitrary local map of (X, x_0) into (Y, y_0) defined on an open neighborhood W of x_0 in X. For each positive real number $r < 1$, let K_r denote the set of points $h^{-1}p(x, t)$ with $x \in M$ and $r \leqq t \leqq 1$, where $p : M \times I \to \operatorname{Con} M$ denotes the natural projection. Then it follows from the continuity of f that there exists a K_r such that $K_r \subset W$ and $f(K_r) \subset V$.

Let $q : N \times I \to \operatorname{Con} N$, $\pi : N \times I \to N$ and $\sigma : N \times I \to I$ denote the natural projections. Define a homotopy $\xi_s : K_r \to V$, $(0 \leqq t \leqq 1)$, as follows. For $s = 0$, we set $\xi_0 = f \mid K_r$. Assume $0 < s \leqq 1$ and define ξ_s by taking

$$\xi_s h^{-1}p(x,t) = \begin{cases} fh^{-1}p(x, t), & (x \in M, r \leqq t \leqq rs - s + 1), \\ k^{-1}q[\phi(x, s), \theta(x, s, t)], & (x \in M, rs - s + 1 \leqq t \leqq 1), \end{cases}$$

where

$$\phi(x, s) = \pi q^{-1} kfh^{-1} p(x, rs - s + 1) \in N,$$

$$\theta(x, s, t) = [(1 - t)\sigma q^{-1} kfh^{-1} p(x, rs - s + 1) + t + s - rs - 1] /(s - rs).$$

Intuitively speaking, ξ_s is obtained by replacing $f \mid K_{rs-s+1}$ with a linear map. In particular, ξ_1 is a linear map given by

$$\xi_1 h^{-1} p(x, t) = k^{-1} q[\phi(x), \theta(x, t)], \quad (x \in M, r \leqq t \leqq 1),$$

where $\phi(x) = \rho(x, 1)$ and $\theta(x, t) = \theta(x, 1, t)$. One can verify that $\xi_s(x_0) = y_0$ and $\xi_s(K_r \backslash x_0) \subset V \backslash y_0$. Thus we get an admissible homotopy $\xi_s : (K_r, x_0) \to (Y, y_0)$, $(0 \leqq s \leqq 1)$, in the sense of § 4.

Define a homotopy $\eta_s : K_r \to V$, $(0 \leqq s \leqq 1)$, by taking

$$\eta_s h^{-1} p(x, t) = k^{-1} q[\phi(x), \tau(x, s, t)], \quad (x \in M, r \leqq t \leqq 1),$$

where $\tau(x, s, t) = [(1 - s)(1 - t)\theta(x, t) + t - r + rs - rst]/(1-r)$. Then $\eta_0 = \xi_1$, $\eta_s(x_0) = y_0$ and $\eta_s(K_r \backslash x_0) \subset V \backslash y_0$. Thus we get an admissible homotopy $\eta_s : (K_r, x_0) \to (Y, y_0)$, $(0 \leqq s \leqq 1)$. The map η_1 is given by

$$\eta_1 h^{-1} p(x, t) = k^{-1} q[\phi(x), t], \quad (x \in M, r \leqq t \leqq 1).$$

Hence we obtain a map $\phi : M \to N$ and η_1 is the restriction on K_r of the map Con (ϕ). This proves that, *for any given local map* $f \in \Gamma$*, there exists a map* $\phi : M \to N$ *such that* f *is locally homotopic to the local map* Con (ϕ).

Now, let us prove that the function κ is one-to-one. For this purpose, it suffices to show that any two local maps $f, q \in \Gamma$ are locally homotopic if deg (f) = deg (g). By the assertion proved in the last paragraph, we may assume that f = Con (ϕ) and g = Con (ψ), where ϕ and ψ are maps of M into N. By the commutativity of the rectangle in the first part of this proof, we have

$$\phi^*(d) = \phi^* \chi^*(e) = i^* f^{\natural}(e) = i^* \deg (f).$$

Similarly, we have

$$\psi^*(d) = \psi^* \chi^*(e) = i^* g^{\natural}(e) = i^* \deg (g).$$

Hence deg (f) = deg (g) implies $\phi^*(d) = \psi^*(d)$. By the (global) Hopf theorem, the latter implies that ϕ, ψ are homotopic and, therefore, f, g are locally homotopic. This completes the proof of (10.1).

The preceding proof suggests that the relation $\phi \to$ Con (ϕ) may reduce the local classification problem to the corresponding global problem for more general cases. In fact, we have the following

THEOREM 10.2. *If* x_0, y_0 *are conic points of* X, Y *with conic neighborhoods* U, V *respectively* $M = \bar{U}\backslash U$, $N = \bar{V}\backslash V$, *then the assignment of the local map* Con (ϕ) *to each map* $\phi : M \to N$ *induces a one-to-one correspondence between the homotopy classes of the maps of* M *into* N *and those of the local maps of* (X, x_0) *into* (Y, y_0).

PROOF. It suffices to prove two assertions: (1) for each local map f of (X, x_0) into (Y, y_0), there exists a map $\phi : M \to N$ such that f is locally homotopic to Con (ϕ); and (2) if $\phi, \psi : M \to N$ are maps such that Con (ϕ), Con (ψ) are locally homotopic, then ϕ, ψ are homotopic. The assertion (1) has been proved in the proof of (10.1) and the assertion (2) can be proved by the same technique applied on a local homotopy. Q.E.D.

The local Hopf theorem (10.1) is an immediate consequence of (10.2) and the (global) Hopf theorem. Furthermore, one can easily deduce local versions of the more refined (global) classification theorems of Steenrod, [22, p. 318], and others.

11. Stability of local maps

A local map f of (X, x_0) into (Y, y_0) defined on an open neighborhood U of x_0 in X is said to be *unstable* if, for every open neighborhood V of x_0 in U, there exists a homotopy $f_t : U \to Y$, $(0 \leqq t \leqq 1)$ such that $f_0 = f$, $f_1(U) \subset Y \backslash y_0$, and $f_t(x) = f(x)$ for every $x \in U \backslash V$ and every $t \in I$; otherwise, f is said to be *stable*. By the *stability* of a local map f, we mean the answer to the question whether f is stable or unstable. For related notions of this terminology, see [2, p. 523], [13, p. 74], and [3].

Naturally, one would ask whether or not the stability of a local map f depends only on its (local) homotopy class. This is answered affirmatively by the following theorem for the case where X is a normal Hausdorff space.

THEOREM 11.1. *Let f and g be two local maps of (X, x_0) into (Y,y) which are locally homotopic. If X is a normal Hausdorff space and g is unstable, then so is f.*

PROOF. Let f and g be defined on the open neighborhood U and V of x_0 in X respectively. Then there exists an open neighborhood W of x_0 in $U \cap V$ together with a local homotopy

$$h_t : (W, x_0) \to (Y, y_0), \qquad (0 \leqq t \leqq 1),$$

such that $h_0 = f \mid W$ and $h_1 = g \mid W$.

To prove that f is unstable, let M be any open neighborhood of x_0 in U. As a normal Hausdorff space, X is regular. Hence there exist open neighborhoods N and Q of x_0 in X such that $\bar{N} \subset Q$ and $\bar{Q} \subset M \cap W$. By Urysohn's lemma [16, p. 27], there exists a continuous real function $\phi : X \to I$ such that $\phi(X \backslash Q) = 0$ and $\phi(\bar{N}) = 1$. On the other hand, since g is unstable and N is an open neighborhood of x_0 in V, there exists a homotopy $g_t : V \to Y$, $(0 \leqq t \leqq 1)$, such that $g_0 = g$, $g_1(V) \subset Y \backslash y_0$, and $g_t(x) = g(x)$ for every $x \in V \backslash N$ and every $t \in I$.

Define a local homotopy $\xi_t : (U, x_0) \to (Y, y_0)$, $(0 \leqq t \leqq 1)$, by setting

$$\xi_t(x) = \begin{cases} h_{t\phi(x)}(x), & (x \in \bar{Q},\ t \in I), \\ f(x), & (x \in U \backslash Q,\ t \in I). \end{cases}$$

Then $\xi_0 = f$, $\xi_1(x) = g(x)$ for each $x \in \bar{N}$, and $\xi_t(x) = f(x)$ for each $x \in U \backslash Q$ and each $t \in I$. Therefore, we may define a homotopy $\eta_t : U \to Y$, $(0 \leqq t \leqq 1)$, by taking

$$\eta_t(x) = \begin{cases} g_t(x), & (x \in \bar{N},\ t \in I), \\ \xi_1(x), & (x \in U \backslash N,\ t \in I). \end{cases}$$

Then $\eta_0 = \xi_1$, $\eta_1(U) \subset Y/y_0$ and $\eta_t(x) = f(x)$ for each $x \in U\backslash Q$ and each $t \in I$.

Let $f_t : U \to Y$, $(0 \leqq t \leqq 1)$, be the homotopy defined by

$$f_t(x) = \begin{cases} \xi_{2t}(x), & (x \in U,\ 0 \leqq t \leqq \frac{1}{2}), \\ \eta_{2t-1}(x), & (x \in U,\ \frac{1}{2} \leqq t \leqq 1). \end{cases}$$

Then $f_0 = f$, $f_1(U) \subset Y\backslash y_0$, and $f_t(x) = f(x)$ for every $x \in U\backslash Q$ and every $t \in I$. Since $Q \subset M$, this proves the theorem.

THEOREM 11.2. *Let x_0 and y_0 be conic points of the spaces X and Y respectively. If a local map f of (X, x_0) into (Y, y_0) is unstable, then the induced homomorphisms*

$$f_\natural : L_n(X, x_0; G) \to L_n(Y, y_0; G), \qquad n \neq 0,$$
$$f_\natural : \tilde{L}_0(X, x_0; G) \to \tilde{L}_0(Y, y_0; G),$$
$$f^\natural : L^n(Y, y_0; G) \to L^n(X, x_0; G), \qquad n \neq 0,$$
$$f^\natural : \tilde{L}^0(Y, y_0; G) \to \tilde{L}^0(X, x_0; G)$$

are zero homomorphisms for every coefficient group G.

PROOF. By (5.2), we may assume without loss of generality that the local map f is defined throughout X. Then, by (7.5), it suffices to prove that the induced homomorphisms

$$f_* : H_n(X, X\backslash x_0; G) \to H_n(Y, Y\backslash y_0; G),$$
$$f^* : H^n(Y, Y\backslash y_0; G) \to H^n(X, X\backslash x_0; G)$$

of the admissible map $f : (X, x_0) \to (Y, y_0)$ are zero homomorphisms for every coefficient group G and every integer n.

Let U be a conic neighborhood of x_0 in X. Since f is unstable, there exists a homotopy $f_t : X \to Y$, $(0 \leqq t \leqq 1)$, such that $f_0 = f$, $f_1(X) \subset Y\backslash y_0$, and $f_t(x) = f(x)$ for every $x \in X\backslash U$ and every $t \in I$.

Since $X\backslash U$ is a deformation retract of $X\backslash x_0$, the inclusion map $i : (X, X\backslash U) \subset (X, X\backslash x_0)$ induces isomorphisms

$$i_* : H_n(X, X\backslash U; G) \approx H_n(X, X\backslash x_0; G),$$
$$i^* : H^n(X, X\backslash x_0; G) \approx H^n(X, X\backslash U; G).$$

Since $f_t(x) = f(x) \in Y\backslash y_0$ for every $x \in X\backslash U$ and every $t \in I$, the homotopy $f_t : X \to Y$, $(0 \leqq t \leqq 1)$, defines a homotopy

$$g_t : (X, X\backslash U) \to (Y, Y\backslash y_0), \qquad (0 \leqq t \leqq 1).$$

Then we have $g_0 = fi$ and $g_1(X) \subset Y\backslash y_0$. It follows that

$$f_* i_* = (fi)_* = g_{0*} = g_{1*} = 0,$$
$$i^* f^* = (fi)^* = g_0^* = g_1^* = 0.$$

Since i_* and i^* are isomorphisms, these imply $f_* = 0$ and $f^* = 0$. This completes the proof.

THEOREM 11.3. *Let X be locally triangulable at x_0 and Y be locally euclidean at y_0. Let m and n denote the dimensions of X and Y at the points x_0 and y_0 respectively. If $n > 1$ and $m \leqq n$, then a local map f of (X, x_0) into (Y, y_0) is unstable if and only if* deg $(f) = 0$.

PROOF. *Necessity.* Let Y be locally oriented at y_0 by the generator $e \in L^{n-1}(Y, y_0)$. Then deg $(f) = f^{\natural}(e)$. Hence, by (11.2), we have deg $(f) = 0$ if f is unstable.

Sufficiency. Let us use the notations in the proof of (10.1). Since U is a normal Hausdorff space, we may apply (11.1). Hence, by (10.1) and (11.1), it suffices to construct an unstable local map f of (X, x_0) into (Y, y_0) with deg $(f) = 0$. For this purpose, let us pick an arbitrary point $y \in N$ and define a map $f : (X, x_0) \to (Y, y_0)$ by taking

$$f(x) = \begin{cases} k^{-1} q[y, \omega p^{-1} h(x)], & (x \in \bar{U}), \\ y, & (x \in X \backslash U), \end{cases}$$

where $\omega : M \times I \to I$ denotes the natural projection. Thus f is an admissible map which sends X onto the line-segment joining y_0 to y. Now it is obvious that deg $(f) = 0$ and that f is unstable. This completes the proof.

12. Pathwise connectedness around a point.

A space X is said to be *pathwise connected around a point* $x_0 \in X$ if its tangent space $X^* = T(X, x_0)$ at x_0 is pathwise connected. A direct description of this notion without using the tangent space can be given as follows. Let J denote the subspace of the unit 2-simplex Δ_2 in euclidean 3-space consisting of those points (t_0, t_1, t_2) of Δ_2 satisfying $t_1 t_2 = 0$. Let v_0 denote the leading vertex $(1, 0, 0)$ of Δ_2. Then J is the union of the two sides of Δ_2 containing v_0 as a vertex. Now, one can easily see that a space X is pathwise connected around a point $x_0 \in X$ if and only if every admissible map $f : (J, v_0) \to (X, x_0)$ in the sense of § 4 has a continuous extension $F : (\Delta_2, v_0) \to (X, x_0)$ which is also an admissible map.

The following theorem is an immediate consequence of the definition given above.

THEOREM 12.1. *A space X is pathwise connected around a point $x_0 \in X$ if and only if $\tilde{L}_0(X, x_0) = 0$ or, equivalently, $L_0(X, x_0) \approx Z$.*

If X is completely regular at x_0, then it follows from (12.1) that the pathwise connectedness around x_0 is a local property of X at x_0. One can also prove directly that this is true without assuming the complete regularity.

Next, let us give a sufficient condition for the pathwise connectedness around a point in terms of the notion of a *local homotopy non-r-cut* point analogous to a homology version introduced by Wilder [27, p. 228]. For related notions, see also [8, p. 353].

A space X is said to have $x_0 \in X$ as a *local homotopy non-r-cut point* if, for every open neighborhood U of x_0 in X, there exists an open neighborhood V of x_0 in U such that every map $f : S^r \to V \backslash x_0$ of the r-sphere S^r is homotopic to a constant in $U \backslash x_0$.

THEOREM 12.2. *If a space X has a countable basis at a point $x_0 \in X$ and if x_0 is a local homotopy non-r-cut point of X for $r = 0,1$, then X is pathwise connected around x_0.*

PROOF. Since X has a countable basis at x_0, there exists a sequence of open neighborhoods

$$\{U_i\} = U_1, U_2, \ldots, U_i, \ldots$$

such that $U_{i+1} \subset U_i$ for each i and that their intersection contains only the point x_0. Since x_0 is a local homotopy non-r-cut point of X for $r = 1$ and 0, there exist two sequences of open neighborhoods

$$\{V_i\} = V_1, V_2, \ldots, V_i, \ldots, \qquad \{W_i\} = W_1, W_2, \ldots, W_i, \ldots$$

of x_0 in X satisfying the following conditions for every $i = 1, 2, \ldots$;

(1) $V_{i+1} \subset V_i \subset U_i$ and $W_{i+1} \subset W_i \subset V_i$.

(2) Every loop in $V_i \backslash x_0$ is homotopic to a constant in $U_i \backslash x_0$.

(3) Every pair of points in $W_i \backslash x_0$ can be connected by a path in $V_i \backslash x_0$.

To prove that X is pathwise connected around x_0, let $f : (J, v_0) \to (X, x_0)$ be any admissible map. We are going to construct an admissible extension $F : (\Delta_2, v_0) \to (X, x_0)$ of f.

Let $J_i = f^{-1}(W_i)$. Then J_i is an open neighborhood of v_0 in J. Choose an increasing sequence

$$\{a_i\} = a_1, a_2, \ldots, a_i, \ldots$$

of positive real numbers $a_i < 1$ such that the line-segments joining v_0 to the points

$$p_i = (a_i, 1 - a_i, 0), \qquad q_i = (a_i, 0, 1 - a_i)$$

are contained in J_i. Since f is admissible, it follows that the sequences of points $\{p_i\}$ and $\{q_i\}$ both converge to v_0. Hence, $\{a_i\}$ converges to 1.

Since $f(p_i)$ and $f(q_i)$ are points in $W_i \backslash x_0$, it follows from (3) that there exists a path $\sigma_i : I \to V_i \backslash x_0$ such that $\sigma_i(0) = f(p_i)$ and $\sigma_i(1) = f(q_i)$.

Let $T_i \subset \Delta_2$ denote the trapezoid with p_i, q_i, p_{i+1} and q_{i+1} as vertices; in other words,

$$T_i = \{(t_0, t_1, t_2) \in \Delta_2 : a_i \leqq t_0 \leqq a_{i+1}\}.$$

Then the boundary B_i of T_i consists of the points (t_0, t_1, t_2) of T_i such that

$$t_1 t_2 (t_0 - a_i)(t_0 - a_{i+1}) = 0.$$

Define a map $\phi_i : B_i \to V_i \backslash x_0$ by taking

$$\phi_i(t_0, t_1, t_2) = \begin{cases} f(t_0, t_1, t_2) & \text{if } t_1 t_2 = 0, \\ \sigma_i[t_2/(1 - a_i)], & \text{if } t_0 = a_i, \\ \sigma_{i+1}[t_2/(1 - a_{i+1})], & \text{if } t_0 = a_{i+1}. \end{cases}$$

It follows from (2) that ϕ_i has a continuous extension $\Phi_i : T_i \to U_i \backslash x_0$.

Finally, let T_0 denote the trapezoid consisting of the points $(t_0, t_1, t_2) \in \Delta_2$ such that $0 \leqq t_0 \leqq a_1$ and C the subspace of T_0 consisting of the points $(t_0, t_1, t_2) \in T_0$ such that $t_1 t_2 (t_0 - a_1) = 0$. Then C is a retract of T_0. Let $\rho : T_0 \to C$ be a retraction of T_0 onto C. Define a map $\phi_0 : C \to X \backslash x_0$ by taking

$$\phi_0(t_0, t_1, t_2) = \begin{cases} f(t_0, t_1, t_2) & \text{if } t_1 t_2 = 0, \\ \sigma_1[t_2/(1 - a_1)], & \text{if } t_0 = a_1. \end{cases}$$

Then ϕ_0 has a continuous extension $\Phi_0 : T_0 \to X \backslash x_0$ given by $\Phi_0 = \phi_0 \rho$.

Then, an admissible extension $F : (\Delta_2, v_0) \to (X, x_0)$ of f may be constructed by setting $F(p) = \Phi_i(p)$ if $p \in T_i$, $(i = 0, 1, 2, \ldots)$. The continuity of F at the point v_0 follows from the facts that $\Phi_i(T_i) \subset U_i$ for each $i > 0$ and that $\{U_i\}$ is a basis at x_0. The admissibility of F follows from the fact that $\Phi_i(T_i) \subset X \backslash x_0$ for each $i \geqq 0$. This completes the proof of (12.2).

The condition in (12.2) is obviously not necessary. For, if $X = \text{Con } S^1$ and $x_0 = v$, then x_0 is not a local homotopy non-1-cut point of X while X is pathwise connected around x_0 by (7.1). Because of this, (12.2) is rather unsatisfactory. To improve (12.2), we have to introduce another notion.

A space X is said to be *locally r-shrinkable* at x_0 if, for every open neighborhood U of x_0 in X, there exists an open neighborhood V of x_0 in U such that, for every open neighborhood V' of x_0 in V, there exists an open neighborhood W of x_0 in V' such that every map

$$f : (E^r, S^{r-1}) \to (V \backslash x_0, W \backslash x_0)$$

is homotopic to a constant in $(U \backslash x_0, W \backslash x_0)$, where E^r denotes the unit r-cell in the euclidean r-space and S^{r-1} denotes the boundary sphere of E^r. Applying the homotopy extension theorem twice, one can prove the existence of a homotopy

$$f_t : (E^r, S^{r-1}) \to (U\backslash x_0, W\backslash x_0), \qquad (0 \leqq t \leqq 1),$$

such that $f_0 = f$, $f_1(E^r) \subset W\backslash x_0$, and $f_t(p) = f(p)$ for every $p \in S^{r-1}$ and every $t \in I$.

If x_0 is a conic point of X, then it is clear that X is locally r-shrinkable at x_0 for every $r \geqq 1$.

THEOREM 12.3. *If a space X has a countable basis at a point $x_0 \in X$, if x_0 is a local homotopy non-0-cut point of X, and if X is locally 1-shrinkable at x_0, then X is pathwise connected around x_0.*

PROOF. Let $\{U_i\}$ be a decreasing sequence of open neighborhoods of x_0 in X such that their intersection contains only the point x_0. Since x_0 is a local homotopy non-0-cut point of X and X is locally 1-shrinkable at x_0, there exist two sequences of open neighborhoods $\{V_i\}$ and $\{W_i\}$ of x_0 in X satisfying the following conditions for every $i = 1, 2, \ldots$:

(1) $V_{i+1} \subset V_i \subset U_i$ and $W_{i+1} \subset W_i \subset V_{i+1}$.

(2) Every path in $V_i\backslash x_0$ whose end points are in $W_i\backslash x_0$ is homotopic in $U_i\backslash x_0$ to a path in $W_i\backslash x_0$ with end points held fixed during the homotopy.

(3) Every pair of points in $W_1\backslash x_0$ can be connected by a path in $V_1\backslash x_0$.

The remainder of the proof proceeds as in the proof of (12.2) with obvious modifications and hence is omitted.

As a partial converse of (12.2) and (12.3), we have the following

THEOREM 12.4. *If a space X is locally arcwise connected at a point $x_0 \in X$ and pathwise connected around x_0, then x_0 is a local homotopy non-0-cut point of X.*

PROOF. Let U be any given open neighborhood of x_0 in X. Since X is locally arcwise connected at x_0, there exists an open neighborhood V of x_0 in U such that every pair of points in V can be connected by an arc in U.

Let a and b be any two points in $V\backslash x_0$. Then there exists a pair of homeomorphisms $\xi, \eta : I \to U$ of the unit interval I into U such that $\xi(0) = a$, $\xi(1) = x_0$, $\eta(0) = b$, and $\eta(1) = x_0$. Hence we may define an admissible map $f : (J, v_0) \to (X, x_0)$ by taking

$$f(t_0, t_1, t_2) = \begin{cases} \xi(t_0), & (\text{if } t_1 = 0), \\ \eta(t_0), & (\text{if } t_2 = 0). \end{cases}$$

Since X is pathwise connected around x_0, the map f has an admissible extension $F : (\Delta_2, v_0) \to (X, x_0)$.

Since $F(v_0) = x_0$, it follows from the continuity of F that there exists a positive real number $k < 1$ such that $F(t_0, t_1, t_2) \in U$ for every point (t_0, t_1, t_2) of Δ_2 with $k \leqq t_0 \leqq 1$. Now define a path $\sigma : I \to X$ by taking

$$\sigma(t) = \begin{cases} \xi(3kt), & (0 \leqq t \leqq \frac{1}{3}), \\ F[k, 3(1-k)t-(1-k), 2(1-k)-3(1-k)t], & (\frac{1}{3} \leqq t \leqq \frac{2}{3}), \\ \eta(3k-3kt), & (\frac{2}{3} \leqq t \leqq 1). \end{cases}$$

Then one can easily verify that $\sigma(0) = a$, $\sigma(1) = b$, and $\sigma(I) \subset U \backslash x_0$. Hence x_0 is a local homotopy non-0-cut point. Q.E.D.

Finally, let us establish the fact that the pathwise connectedness around a point x_0 of a space X is a local property at x_0. In fact, we have the following

THEOREM 12.5. *Let U be any given open neighborhood of x_0 in X. Then X is pathwise connected around x_0 if and only if U is pathwise connected around x_0.*

PROOF. *Sufficiency.* Let $f : (J, v_0) \to (X, x_0)$ be any admissible map. Then, there exists a non-negative real number $k < 1$ such that $f(t_0, t_1, t_2) \in U$ for every $(t_0, t_1, t_2) \in J$ with $k \leqq t_0 \leqq 1$. Let S and T denote the subspaces of Δ_2 defined by

$$S = \{(t_0, t_1, t_2) \in \Delta_2 : k \leqq t_0 \leqq 1\},$$
$$T = \{(t_0, t_1, t_2) \in \Delta_2 : 0 \leqq t_0 \leqq k\}.$$

Since U is pathwise connected around x_0, there exists an admissible map $G : (S, v_0) \to (U, x_0)$ such that $G(p) = f(p)$ whenever $p \in J \cap S$. Then G can be extended throughout $J \cup S$ by setting $G(p) = f(p)$ for every $p \in J$. On the other hand, there exists a retraction $\rho : T \to T \cap (J \cup S)$. Define an admissible map $F : (\Delta_2, v_0) \to (X, x_0)$ by taking

$$F(p) = \begin{cases} G(p), & (\text{if } p \in S), \\ G\rho(p), & (\text{if } p \in T). \end{cases}$$

Then F is an extension of f and hence X is pathwise connected around x_0.

Necessity. Let $\phi : (J, v_0) \to (U, x_0)$ be any given admissible map. Since X is pathwise connected around x_0, ϕ has an admissible extension $\Psi : (\Delta_2, v_0) \to (X, x_0)$. Let us use the notation in the sufficiency proof. There exists a non-negative real number $k < 1$ such that $\Psi(S) \subset U$. Define an admissible map $\Phi : (\Delta_2, v_0) \to (U, x_0)$

by taking

$$\Phi(p) = \begin{cases} \Psi(p), & (\text{if } p \in S), \\ \Psi\rho(p), & (\text{if } p \in T). \end{cases}$$

Then Φ is an extension of ϕ and hence U is pathwise connected around x_0. This completes the proof.

13. Local homotopy groups.

Let X be a given topological space and x_0 a given point in X. Assume that the arc-component C of X which contains x_0 is non-degenerate and that X is pathwise connected around x_0. Then, by definition, the tangent space $X^* = T(X, x_0)$ is non-empty and pathwise connected. Therefore, for each $n \geqq 1$, the homotopy group $\pi_n(X^*)$ is well defined and does not depend on the choice of the basic point in X^*. This group will be called the *n-dimensional local homotopy group* of X at x_0; in symbols,

$$\lambda_n(X, x_0) = \pi_n[T(X, x_0)].$$

For a geometrical representation of the elements of $\lambda_n(X, x_0)$, let us pick a path $\sigma \in X^*$ to serve as the basic point of $\pi_n(X^*)$. Let J_{n+1} denote the subspace of the unit $(n+2)$-simplex Δ_{n+2} consisting of all faces of Δ_{n+2} except the one opposite to leading vertex v_0. Thus, J_{n+1} is given by the formula

$$J_{n+1} = \{(t_0, t_1, \ldots, t_{n+2}) \in \Delta_{n+2} : t_1 t_2 \ldots t_{n+2} = 0\}.$$

Then each element of $\lambda_n(X, x_0)$ is represented by an admissible map $f : (J_{n+1}, v_0) \to (X, x_0)$ such that the leading edge $v_0 v_1$ is mapped as the basic path σ, that is to say,

$$f(t_0, t_1, 0, \ldots, 0) = \sigma(t_1).$$

Two of these admissible maps f and g represent the same element of $\lambda_n(X, x_0)$ if and only if there exists an admissible homotopy h_t such that $h_0 = f$, $h_1 = g$, and h_t maps the leading edge $v_0 v_1$ as the basic path σ for each $t \in I$. In particular, f represents the neutral element of $\lambda_n(X, x_0)$ if and only if it has an admissible extension $F : (\Delta_{n+2}, v_0) \to (X, x_0)$.

If the space X is not pathwise connected around the point x_0, then the homotopy group $\pi_n(X^*, \sigma)$ depends on the choice of the basic path σ and, therefore, we have to indicate the path σ in the notation of the local homotopy group, namely

$$\lambda_n(X, x_0; \sigma) = \pi_n(X^*, \sigma).$$

We may also define the *relative local homotopy groups* as follows. Let (X, A, x_0) be a given triplet where the arc-component of A

which contains x_0 is non-degenerate. Then the tangent space $A^* = T(A, x_0)$ is non-empty. Choose a path $\sigma \in A^*$ as basic point and define

$$\lambda_n(X, A, x_0; \sigma) = \pi_n(X^*, A^*, \sigma).$$

One can also define the *boundary homomorphisms*

$$\partial : \lambda_n(X, A, x_0; \sigma) \to \lambda_{n-1}(A, x_0; \sigma)$$

in the obvious way.

If the subspace A is pathwise connected around the point x_0, then the group $\pi_n(X^*, A^*, \sigma)$ does not depend on the choice of the basic point σ and we may drop the symbol σ from the notation.

Now, let $f : (X, A, x_0) \to (Y, B, y_0)$ be an admissible map of (X, A, x_0) into another triplet (Y, B, y_0). By § 4, f induces a map

$$\hat{f} : (X^*, A^*, \sigma) \to (Y^*, B^*, f\sigma).$$

Therefore, we may define induced homomorphisms

$$f_* : \lambda_n(X, A, x_0; \sigma) \to \lambda_n(Y, B, y_0; f\sigma),$$
$$f_* : \lambda_n(X, x_0; \sigma) \to \lambda_n(Y, y_0; f\sigma),$$
$$f_* : \lambda_n(A, x_0; \sigma) \to \lambda_n(B, y_0; f\sigma),$$

as those induced by $\hat{f}$ on the relative and absolute (global) homotopy groups.

The following properties of the induced homomorphisms are obvious.

(13.1) *If f is the identity map of (X, A, x_0), then the induced homomorphisms f_* are the identity automorphisms.*

(13.2) *If $f : (X, A, x_0) \to (Y, B, y_0)$, $g : (Y, B, y_0) \to (Z, C, z_0)$ are admissible maps, then $(gf)_* = g_* f_*$.*

(13.3) *The following rectangle is commutative*:

$$\begin{array}{ccc} \lambda_n(X, A, x_0; \sigma) & \xrightarrow{f_*} & \lambda_n(Y, B, y_0; f\sigma) \\ \downarrow \partial & & \downarrow \partial \\ \lambda_{n-1}(A, x_0; \sigma) & \xrightarrow{f_*} & \lambda_{n-1}(B, y_0; f\sigma). \end{array}$$

Unlike their global counterparts, $\lambda_n(X, x_0; \sigma)$ cannot be considered as a special case of the relative group $\lambda_n(X, A, x_0; \sigma)$ by taking $A = x_0$ because σ is not a path in the subspace x_0. However, the inclusion map $j : (X^*, \sigma) \subset (X^*, A^*, \sigma)$ induces a homomorphism

$$j_* : \lambda_n(X, x_0; \sigma) \to \lambda_n(X, A, x_0; \sigma)$$

for each $n \geqq 2$. On the other hand, the inclusion map $i :(A, x_0) \subset (X, x_0)$ induces a homomorphism

$$i_* : \lambda_n(A, x_0; \sigma) \to \lambda_n(X, x_0; \sigma)$$

for each $n \geqq 1$. The following property is now obvious.

(13.4) *For any triplet (X, A, x_0), where the arc-component of A containing x_0 is non-degenerate, the sequence*

$$\lambda_1(X, x_0; \sigma) \xleftarrow{i_*} \lambda_1(A, x_0; \sigma) \xleftarrow{\partial} \lambda_2(X, A, x_0; \sigma) \xleftarrow{j_*} \lambda_2(X, x_0; \sigma) \xleftarrow{i_*} \cdots$$
$$\cdot \xleftarrow{\partial} \lambda_n(X, A, x_0; \sigma) \xleftarrow{j_*} \lambda_n(X, x_0; \sigma) \xleftarrow{i_*} \lambda_n(A, x_0; \sigma) \xleftarrow{\partial} \lambda_{n+1}(X, A, x_0; \sigma) \xleftarrow{j_*} \cdots$$

is exact. This will be called the local homotopy sequence of $(X, A, x_0;\sigma)$.

Finally, an admissible map $f : (X, A, x_0) \to (Y, B, y_0)$ of (X, A, x_0) into another triplet (Y, B, y_0) gives a *commutative ladder of induced homomorphisms* of the local homotopy sequence of $(X, A, x_0; \sigma)$ into that of $(Y, B, y_0; f\sigma)$.

14. Properties of local homotopy groups.

To justify the definition of local homotopy groups given in the preceding section, we have to show that the local homotopy groups are really local invariants. For this purpose, we are going to establish the following

THEOREM 14.1. *If U is an open neighborhood of a point x_0 in a space X, then the inclusion map $f : (U, x_0) \subset (X, x_0)$ induces an isomorphism*

$$f_* : \lambda_n(U, x_0; \sigma) \approx \lambda_n(X, x_0; \sigma)$$

for every $n \geqq 1$ and every basic path $\sigma \in T(U, x_0)$.

PROOF. First, let us prove that f_* is an epimorphism. For this purpose, let $\alpha \in \lambda_n(X, x_0; \sigma)$ be represented by an admissible map $\phi : (J_{n+1}, v_0) \to (X, x_0)$ such that $\phi(t_0, t_1, 0, \ldots, 0) = \sigma(t_1)$. By continuity of ϕ, there exists a non-negative real number $k < 1$ such that $\phi(t_0, t_1, \ldots, t_{n+2}) \in U$ for every point $(t_0, t_1, \ldots, t_{n+2})$ of J_{n+1} satisfying $k \leqq t_0 \leqq 1$. Let S and T denote the subspaces of J_{n+1} defined by

$$S = \{(t_0, t_1, \ldots, t_{n+2}) \in J_{n+1} : k \leqq t_0 \leqq 1\},$$
$$T = \{(t_0, t_1, \ldots, t_{n+2}) \in J_{n+1} : 0 \leqq t_0 \leqq k\}$$

and let C denote the subspace of T defined by

$$C = \{(t_0, t_1, \ldots, t_{n+2}) \in T : (t_0 - k)(t_2^2 + \ldots + t_{n+2}^2) = 0\}.$$

Then C is a deformation retract of T and hence there exists a homotopy $h_t : T \to T$, $(0 \leqq T \leqq 1)$, such that h_0 is the identity map, $h_1(T) \subset C$, and $h_t(p) = p$ for every $p \in C$ and $t \in I$. Define an admissible homotopy $\phi_t : (J_{n+1}, v_0) \to (X, x_0)$, $(0 \leqq t \leqq 1)$, by taking

$$\phi_t(p) = \begin{cases} \phi(p), & (p \in S,\ t \in I), \\ \phi h_t(p), & (p \in T,\ t \in I). \end{cases}$$

Then $\phi_0 = \phi$, $\phi_1(J_{n+1}) \subset U$, and $\phi_t(t_0, t_1, 0, \ldots, 0) = \sigma(t_1)$ for every $t \in I$. ϕ_1 represents an element β of $\lambda_n(U, x_0; \sigma)$ and the homotopy ϕ_t implies that $f_*(\beta) = \alpha$. Hence f_* is an epimorphism.

Next, let us prove that f_* is a monomorphism. For this purpose, let γ denote any element of $\lambda_n(U, x_0; \sigma)$ such that $f_*(\gamma) = 0$. Then γ is represented by an admissible map $\psi : (J_{n+1}; v_0) \to (U, x_0)$ satisfying $\psi(t_0, t_1, 0, \ldots, 0) = \sigma(t_1)$. Since $f_*(\gamma) = 0$, ψ has an admissible extension $\Psi : (\Delta_{n+2}, v_0) \to (X, x_0)$. By continuity of Ψ, there exists a non-negative real number $k < 1$ such that $\Psi(t_0, t_1, \ldots, t_{n+2}) \in U$ for every point of Δ_{n+2} satisfying $k \leqq t_0 \leqq 1$. Let S and T denote the subspaces of Δ_{n+2} defined by $k \leqq t_0 \leqq 1$ and $0 \leqq t_0 \leqq k$ respectively. Let C denote the subspace of T defined by

$$C = \{(t_0, t_1, \ldots, t_{n+2}) \in T : (t_0 - k)\, t_1 t_2 \ldots t_{n+2} = 0\}.$$

Then C is a retract of T and hence there exists a retraction $\rho : T \to C$ of T onto C. Define an admissible map $\Phi : (\Delta_{n+2}, v_0) \to (U, x_0)$ by taking

$$\Phi(p) = \begin{cases} \Psi(p), & (\text{if } p \in S), \\ \Psi\rho(p), & (\text{if } p \in T). \end{cases}$$

Then Φ is an extension of ψ and, therefore, $\gamma = 0$. This completes the proof.

Corollary 14.2. *If X is pathwise connected around x_0, then the local homotopy groups $\lambda_n(X, x_0)$, $n = 1, 2, \ldots$, are local invariants of X at the point x_0.*

This is an immediate consequence of (12.5) and (14.1).

Corollary 14.3. *If (X, A, x_0) and (U, D, u_0) are triplets such that U is an open neighborhood of x_0 in X, $D = U \cap A$, and $u_0 = x_0$, then the inclusion map $f : (U, D, u_0) \subset (X, A, x_0)$ induces an isomorphism*

$$f_* : \lambda_n(U, D, u_0; \sigma) \approx \lambda_n(X, A, x_0; \sigma)$$

for every $n \geqq 2$ and every basic path $\sigma \in T(D, u_0)$.

This follows immediately from (14.1) and an application of the „five" lemma, [6, p. 16].

Corollary 14.4. *If $(X, A, x_0) \subset (Y, B, y_0)$ and if there exists an open neighborhood U of $x_0 = y_0$ in Y such that $U \subset X$ and $U \cap B \subset A$*

then the inclusion map $f : (X, A, x_0) \subset (Y, B, y_0)$ *induces the isomorphisms*

$$f_* : \lambda_n(X, x_0; \sigma) \approx \lambda_n(Y, y_0; \sigma), \qquad n \geqq 1,$$
$$f_* : \lambda_n(X, A, x_0; \sigma) \approx \lambda_n(Y, B, y_0; \sigma), \qquad n \geqq 2$$

for every basic path $\sigma \in T(U \cap B, x_0)$.

The proof of this is similar to that of (5.3).

THEOREM 14.5. *If* U *is a conic neighborhood of* x_0 *in* (X, A), $F = \overline{U}\backslash U$, *and* σ *is the path joining* x_0 *to a point* $x_1 \in F \cap A$ *along the line segment* $x_0 x_1$, *then we have*

$$\lambda_n(X, x_0; \sigma) \approx \pi_n(F, x_1), \qquad n \geqq 1,$$
$$\lambda_n(X, A, x_0; \sigma) \approx \pi_n(F, F \cap A, x_1), \qquad n \geqq 2.$$

This is a direct consequence of (7.1). In fact, the homotopy equivalence $\iota : (F, F \cap A) \to (X^*, A^*)$ in the proof of (7.1) induces isomorphisms

$$\iota_* : \pi_n(F, x_1) \approx \pi_n(X^*, \sigma), \qquad n \geqq 1,$$
$$\iota_* : \pi_n(F, F \cap A, x_1) \approx \pi_n(X^*, A^*, \sigma), \qquad n \geqq 2.$$

COROLLARY 14.6. *If* U *is a conic neighborhood of* x_0 *in* X *with pathwise connected frontier* $F = \overline{U}\backslash U$, *then* X *is pathwise connected around* x_0 *and* $\lambda_n(X, x_0) \approx \pi_n(F)$ *for every* $n \geqq 1$.

This is an immediate consequence of (7.2) and (14.5). Hence, in this case, our local homotopy groups reduce to those of H. B. Griffiths, [8, p. 357]. As an important special case of (14.6), let X be a simplicial complex, $x_0 \in X$, and F the frontier of the star of x_0 in X. Then X is pathwise connected around x_0 if and only if F is connected. In this case, we have $\lambda_n(X, x_0) \approx \pi_n(F)$ for every $n \geqq 1$.

The properties stated in the remainder of this section are obvious consequences of the definition of local homotopy and homology groups.

THEOREM 14.7. *If* X *is pathwise connected around* x_0, *then there is a natural homomorphism*

$$h_n : \lambda_n(X, x_0) \to L_n(X, x_0)$$

for each $n \geqq 1$. *The homomorphism* h_1 *is an epimorphism and its kernel is the commutator subgroup of the local fundamental group* $\lambda_1(X, x_0)$. *If* $\lambda_q(X, x_0) = 0$ *for every* $q < n$, *then* h_n *is an isomorphism.*

THEOREM 14.8. *Every path* $\tau : I \to T(X, x_0)$ *induces an isomorphism*

$$\tau_* : \lambda_n(X, x_0; \sigma_1) \approx \lambda_n(X, x_0; \sigma_0)$$

where $\sigma_0 = \tau(0)$ *and* $\sigma_1 = \tau(1)$; τ_* *depends only on the homotopy class of* τ. *In particular, the local fundamental group* $\lambda_1(X, x_0; \sigma)$ *operates on* $\lambda_n(X, x_0; \sigma)$ *as a group of automorphisms; if* $n = 1$, *then* $\alpha(\beta) = \alpha\beta\alpha^{-1}$.

A space X is said to be *locally n-simple at* x_0 if, for every basic path $\sigma \in T(X, x_0)$, the local fundamental group $\lambda_1(X, x_0; \sigma)$ operates simply on $\lambda_n(X, x_0; \sigma)$, that is to say, $\alpha(\beta) = \beta$ for every $\alpha \in \lambda_1(X, x_0; \sigma)$ and every $\beta \in \lambda_n(X, x_0; \sigma)$. Thus, X is locally n-simple at x_0 if $\lambda_1(X, x_0; \sigma) = 0$ or $\lambda_n(X, x_0; \sigma) = 0$ for every basic path $\sigma \in T(X, x_0)$. Also, X is locally 1-simple at x_0 if and only if $\lambda_1(X, x_0; \sigma)$ is abelian for every basic path $\sigma \in T(X, x_0)$.

If X is pathwise connected around x_0 and is locally n-simple at x_0, then the elements of the local homotopy group $\lambda_n(X, x_0)$ may be considered as the (admissible) homotopy classes of the admissible maps of (J_{n+1}, v_0) into (X, x_0). A better geometrical representation of $\lambda_n(X, x_0)$ will be given in § 17.

Now let us consider a given admissible map

$$f : (X, x_0) \to (Y, y_0),$$

where X, Y are pathwise connected around x_0, y_0 respectively. Choose a basic path $\sigma \in T(X, x_0)$. Then f induces the homomorphisms

$$\xi_q : \lambda_q(X, x_0; \sigma) \to \lambda_q(Y, y_0; f\sigma), \eta_q : L_q(X, x_0) \to L_q(Y, y_0).$$

Then we have the following local version of the Whitehead theorem.

THEOREM 14.9. *For each positive integer n, we have the following two assertions*:

(1) *If* ξ_q *is an isomorphism for every* $q \leqq n$, *then so is* η_q. *If* ξ_q *is an isomorphism for every* $q < n$ *and is an epimorphism for* $q = n$, *then so is* η_q.

(2) *Assume that* $\lambda_1(X, x_0) = 0 = \lambda_1(Y, y_0)$. *If* η_q *is an isomorphism for every* $q < n$ *and is an epimorphism for* $q = n$, *then so is* ξ_q. *Furthermore, if* η_n *is also an isomorphism, then the kernel of* ξ_n *is contained in the kernel of the natural homomorphism* $h_n : \lambda_n(X, x_0; \sigma) \to L_n(X, x_0)$.

Hence, if X is pathwise connected around x_0, then (5.2) and (5.3) are consequences of (14.1) and (14.9) without assuming complete regularity. By considering the path-components of $T(X, x_0)$, one can also remove the condition that Y be pathwise connected around x_0.

It is quite clear that every operation introduced in the global theory and every result proved in the global theory have obvious local versions. This is the advantage of our definition of the local invariants, for we don't have to toil at the developments parallel to those done in the global theory.

15. The vanishing of local homotopy groups.

In the present section, we shall study the relation between the vanishing of a certain local homotopy group and other local properties such as being a homotopy non-r-cut point.

THEOREM 15.1. *If a space X has a countable basis at a point $x_0 \in X$ and if x_0 is a local homotopy non-r-cut point of X for $r = n$ and $n + 1$, then $\lambda_n(X, x_0; \sigma) = 0$ for every basic path $\sigma \in T(X, x_0)$.*

This can be proved as (12.2). We omit the proof, as the changes required are obvious.

The condition that x_0 be a local homotopy non-$(n + 1)$-cut point is obviously not necessary for $\lambda_n(X, x_0; \sigma) = 0$. For, if $X = \text{Con } S^{n+1}$ and $x_0 = v$, then x_0 is not a local homotopy non-$(n + 1)$-cut point while $\lambda_n(X, x_0) \approx \pi_n(S^{n+1}) = 0$. To improve (15.1), we have the following

THEOREM 15.2. *If a space X has a countable basis at a point $x_0 \in X$, if x_0 is a local homotopy non-n-cut point of X, and if X is locally $(n + 1)$-shrinkable at x_0, then $\lambda_n(X, x_0; \sigma) = 0$ for every basic path $\sigma \in T(X, x_0)$.*

The proof of this theorem is similar to that of (12.3) with obvious changes and hence omitted. This theorem is satisfactory because it implies that, at a conic point $x_0 \in X$, $\lambda_n(X, x_0; \sigma) = 0$ for every $\sigma \in T(X, x_0)$ if x_0 is a local homotopy non-n-cut point of X.

To establish some inverse of (15.1) and (15.2), we have to introduce another local property.

A space X is said to be *r-convergent* at a point $x_0 \in X$ if, for every open neighborhood U of x_0 in X, there exists an open neighborhood V of x_0 in U such that, for every open neighborhood W of x_0 in V, every map $f : S^r \to V\backslash x_0$ is homotopic in $U\backslash x_0$ to a map of S^r into $W\backslash x_0$.

If x_0 is a conic point of X, then it is clear that X is r-convergent at x_0 for every $r \geqq 0$. A space X is 0-convergent at x_0 if and only if X is locally arcwise connected at x_0.

THEOREM 15.3. *If a space X has a countable basis at a point $x_0 \in X$, if X is n-convergent at x_0, and if $\lambda_n(X, x_0; \sigma) = 0$ for every basic path $\sigma \in T(X, x_0)$, then x_0 is a local homotopy non-n-cut point of X.*

PROOF. Let U be any open neighborhood of x_0 in X. Since X has a countable basis at x_0, there exists a sequence $\{U_i\}$ of open neighborhoods of x_0 in X such that

$$U_1 \subset U, \quad U_{i+1} \subset U_i, \ \bigcap_{i=l}^{\infty} U_i = x_0.$$

Since X is n-convergent at x_0, there exists a sequence $\{V_i\}$ with

$$V_i \subset U_i, \quad V_{i+1} \subset V_i$$

such that every map of S^n into $V_i\backslash x_0$ is homotopic in $U_i\backslash x_0$ to a map into $V_{i+1}\backslash x_0$.

Now let $V = V_1$ and $\phi : S^n \to V\backslash x_0$ be a given map. It remains to show that ϕ is homotopic in $U\backslash x_0$ to a constant.

Consider S^n as the subspace of J_{n+1} defined by $t_0 = 0$. We shall first prove that ϕ has an admissible extension $f : (J_{n+1}, v_0) \to (U, x_0)$. For each $i = 1, 2, \ldots$, let

$$S_i = \left\{(t_0, t_1, \ldots, t_{n+2}) \in J_{n+1} : t_0 = \frac{i-1}{i}\right\},$$

$$T_i = \left\{[(t_0, t_1, \ldots, t_{n+2}) \in J_{n+1} : \frac{i-1}{i} \leqq t_0 \leqq \frac{i}{i+1}\right\}.$$

We shall construct inductively two sequences of maps

$$\phi_i : S_i \to V_i\backslash x_0, \qquad f_i : T_i \to U_i\backslash x_0$$

such that $f_1 \mid S_1 = \phi_1 = \phi$ and $f_{i-1} \mid S_i = \phi_i = f_i \mid S_i$ for each $i > 1$. Assume $j > 1$ and that we have already constructed the maps ϕ_i for each $i \leqq j$ and f_i for each $i < j$. Since S_j is also an n-sphere and ϕ_j is a map of S_j into $V_j\backslash x_0$, it follows that ϕ_j is homotopic in $U_j\backslash x_0$ to a map into $V_{j+1}\backslash x_0$. This implies the existence of a map $f_j : T_j \to U_j\backslash x_0$ such that $f_j \mid S_j = \phi_j$ and $f_j(S_{k+1}) \subset V_{j+1}\backslash x_0$. Define ϕ_{j+1} by taking ϕ_{j+1} by taking $\phi_{j+1} = f_j \mid S_{j+1}$. This completes the inductive construction of $\{\phi_i\}$ and $\{f_i\}$. Then we may define an admissible extension $f : (J_{n+1}, v_0) \to (U, x_0)$ by taking

$$f(p) = \begin{cases} f_i(p), & (\text{if } p \in T_i), \\ x_0, & (\text{if } p = v_0). \end{cases}$$

By (14.1), $\lambda_n(U, x_0; \sigma) \approx \lambda_n(X, x_0; \sigma) = 0$ for every basic path $\sigma \in T(U, x_0)$. Hence f has an admissible extension $F : (\varDelta_{n+2}, v_0) \to (U, x_0)$. Let E^{n+1} denote the subspace of $\varDelta_{n+2}$ defined by $t_0 = 0$. Then $F|E^{n+1}$ is an extension of ϕ. Since $F(E^{n+1})$ is contained in $U\backslash x_0$, it follows that ϕ is homotopic in $U\backslash x_0$ to a constant. This completes the proof.

COROLLARY 15.4. *Let x_0 be a conic point of a space X. Then x_0*

is a local homotopy non-n-cut point if and only if $\lambda_n(X, x_0; \sigma) = 0$ *for every basic path* $\sigma \in T(X, x_0)$.

This is an immediate consequence of (15.2) and (15.3).

COROLLARY 15.5. *If a space X has a countable basis at a point* $x_0 \in X$ *and if X is n-convergent at* x_0*, then X is n-LC at* x_0.

We recall that X is said to be n-*LC* at x_0 if, for every open neighborhood U of x_0 in X, there exists an open neighborhood V of x_0 in U such that every map $\phi : S^n \to V$ is homotopic in U to a constant, [17, p. 79]. This corollary is proved by the existence of the extension f of ϕ in the proof of (15.3).

Let n be a positive integer. A space X is said to be *n-connected around* a point $x_0 \in X$ if X is pathwise connected around x_0 and $\lambda_q(X, x_0) = 0$ for every $q \leqq n$. In case $n = 1$, it is also said to be *simply connected around* x_0. The following theorem is an immediate consequence of (12.2), (15.1) and (15.2).

THEOREM 15.6. *A space X is n-connected around a point* $x_0 \in X$ *if the following conditions are satisfied*:

(1) *X has a countable basis at* x_0;

(2) x_0 *is a local homotopy non-r-cut point of X for every* $r \leqq n$;

(3) *X is locally* $(n+1)$*-shrinkable at* x_0.

COROLLARY 15.7. *Let* x_0 *be a conic point of a space X. Then X is n-connected around* x_0 *if and only if* x_0 *is a local homotopy non-r-cut point of X for every* $r \leqq n$.

16. Induced homomorphisms of local maps.

To define the induced homomorphisms on local homotopy groups of a local map, let us consider the following diagram:

$$(X, x_0) \xleftarrow{i} (U, x_0) \xrightarrow{f} (Y, y_0)$$

where f is a given local map of (X, x_0) into (Y, y_0) defined on an open neighborhood U of x_0 in X and i denotes the inclusion map. Pick a basic path $\sigma \in T(U, x_0) \subset T(X, x_0)$. By (14.1), the induced homomorphism i^* in the following diagram

$$\lambda_n(X, x_0; \sigma) \xleftarrow{i_*} \lambda_n(U, x_0; \sigma) \xrightarrow{f_*} \lambda_n(Y, y_0; f\sigma)$$

is an isomorphism for each $n \geqq 1$. Hence we may define a homomorphism

$$f_\natural = f_* i_*^{-1} : \lambda_n(X, x_0; \sigma) \to \lambda_n(Y, y_0; \sigma)$$

for each integer $n \geqq 1$. This homomorphism will be called the *induced homomorphism of the local map f* on the n-dimensional local homotopy group.

The admissible maps of (X, x_0) into (Y, y_0) are special cases of local maps. By (13.1), we have the following property:

(16.1) *If* $f : (X, x_0) \to (Y, y_0)$ *is an admissible map, then* $f_\natural = f_*$.

Let $f : (U, x_0) \to (Y, y_0)$ be a local map of (X, x_0) into (Y, y_0) and $g : (V, y_0) \to (Z, z_0)$ be a local map of (Y, y_0) into (Z, z_0). Let $W = f^{-1}(V) \subset U$. Then $gf : (W, x_0) \to (Z, z_0)$ is a local map of (X, x_0) into (Z, z_0). The following property is obvious.

(16.2) *For every basic path* $\sigma \in T(W, x_0)$, *we have* $(gf)_\natural = g_\natural f_\natural$.

Now, let us study the effect of a local homotopy on the induced homomorphisms. For this purpose, let $f : (U, x_0) \to (Y, y_0)$ and $g : (V, x_0) \to (Y, y_0)$ be two local maps of (X, x_0) into (Y, y_0), and pick a basic path $\sigma \in T(U \cap V, x_0)$. Assume that f and g are local homotopic with a local homotopy

$$h_t : (W, x_0) \to (Y, y_0), \quad (0 \leqq t \leqq 1),$$

defined on an open neighborhood W of x_0 in $U \cap V$ such that $h_0 = f \mid W$ and $h_1 = g \mid W$. By continuity of the path σ, there exists a positive real number $k \leqq 1$ such that $\sigma(t) \in W$ whenever $0 \leqq t \leqq k$. Define a path $\tau : I \to T(U \cap V, x_0)$ by taking $\tau(s)$, $s \in I$, to be the path in Y defined by

$$[\tau(s)](t) = \begin{cases} f\sigma(t - 3st + 3kst), & (\text{if } 0 \leqq s \leqq \frac{1}{3}), \\ h_{3s-1}\sigma(kt), & (\text{if } \frac{1}{3} \leqq s \leqq \frac{2}{3}), \\ g\sigma(3st - 3kst + 3kt - 2t), & (\text{if } \frac{2}{3} \leqq s \leqq 1). \end{cases}$$

Then $\tau(0) = f\sigma$ and $\tau(1) = g\sigma$. Therefore, τ induces an isomorphism τ_* of $\lambda_n(Y, y_0; g\sigma)$ onto $\lambda_n(Y, y_0; f\sigma)$. One can verify that τ_* does not depend on the choice of the real number k. According to the global homotopy theory, we have the following property:

(16.3) *For each* $n \geqq 1$, *the following triangle is commutative*

$$\begin{array}{ccc} & \lambda_n(X, x_0; \sigma) & \\ {}^{f_\natural}\swarrow & & \searrow^{g_\natural} \\ \lambda_n(Y, y_0; f\sigma) & \xleftarrow{\ \tau_*\ } & \lambda_n(Y, y_0; g\sigma). \end{array}$$

The following theorem covers an important special case of what has been established above.

THEOREM 16.4. *If* X, Y *are pathwise connected around* x_0, y_0 *respectively and if* Y *is locally* n*-simple at* y_0, *then every local map* f *of* (X, x_0) *into* (Y, y_0) *induces a homomorphism*

$$f_\natural : \lambda_n(X, x_0) \to \lambda_n(Y, y_0)$$

satisfying (16.1), (16.2) *and that* $f_\natural = g_\natural$ *for any pair of local maps* f, g *which are locally homotopic.*

So far, we have established all properties of local homotopy groups corresponding to the seven axioms of the global homotopy theory [10, p. 493] except the most important „Fibering Axiom," a detailed study of which is to be given in a forthcoming paper of the author [12].

THEOREM 16.5. *If* X, Y *are pathwise connected around* $x_0 \in X$, $y_0 \in Y$ *respectively then, for the induced homomorphisms*

$$\xi_q : \lambda_q(X, x_0; \sigma) \to \lambda_q(Y, y_0; f\sigma),\ \eta_q : L_q(X, x_0) \to L_q(Y, y_0)$$

of a local map $f : (U, x_0) \to (Y, y_0)$ *with* $\sigma \in T(U, x_0)$, *then assertions* (1) *and* (2) *of* (14.9) *still hold.*

This follows immediately from (14.9) and the definition of the induced homomorphisms of a local map. Hereafter, (16.5) will be called the local Whitehead theorem.

THEOREM 16.6. *Let* x_0 *and* y_0 *be conic points of the spaces* X *and* Y *respectively. If a local map* $f : (U, x_0) \to (Y, y_0)$ *of* (X, x_0) *into* (Y, y_0) *is unstable, then the induced homomorphism*

$$f_\natural : \lambda_n(X, x_0; \sigma) \to \lambda_n(Y, y_0; f\sigma)$$

is a zero homomorphism for every $n \geqq 1$ *and every basic path* $\sigma \in T(U, x_0)$.

This can be proved as in (11.2) by using (14.5), (10.2) and the fact that

$$\pi_n(F, x_1) \approx \pi_{n+1}(\bar{U}, F, x_1) \approx \pi_{n+1}(\bar{U}, \bar{U}\backslash x_0, x_1)$$

for a conic neighborhood U of x_0 with $F = \bar{U}\backslash U$ and $x_1 \in F$.

17. Local homotopy groups as classes of local maps.

Now, we are in a position to give a better geometrical representation of the elements of the local homotopy group $\lambda_n(X, x_0; \sigma)$ as homotopy classes of the local maps of $(R^{n+1}, 0)$ into (X, x_0), where R^{n+1} denotes euclidean $(n+1)$-space and 0 denotes the origin of R^{n+1}, $n \geqq 1$.

Since the local homotopy group $\lambda_n(R^{n+1}, 0)$ is infinite cyclic, we have two different choices of a generator of $\lambda_n(R^{n+1}, 0)$. Each choice of a generator of $\lambda_n(R^{n+1}, 0)$ is called an *orientation* of R^{n+1} at 0. When a generator ι of $\lambda_n(R^{n+1}, 0)$ has been chosen, we say that R^{n+1} is *oriented* at 0. Hereafter, we assume that R^{n+1} is oriented at 0 by the choice of a generator ι of $\lambda_n(R^{n+1}, 0)$.

Consider any given space X, a given point $x_0 \in X$, and a basic path $\sigma \in T(X, x_0)$. Then σ is a path $\sigma : I \to X$ such that $\sigma(t) = x_0$ if and only if $t = 0$. Imbed the unit interval I as a subspace of R^{n+1} by identifying the point $t \in I$ with the point $(t, 0, \ldots, 0) \in R^{n+1}$. Consider the totality Ω of local maps

$$f : (U, 0) \to (X, x_0)$$

defined on an open neighborhood U of 0 in R^{n+1}, which depends on f, and satisfying $f(t) = \sigma(t)$ for every $t \in I \cap U$. Two local maps $f, g \in \Omega$ are said to be *locally homotopic relative to* σ if there exist an open neighborhood W of 0 in R^{n+1} and a homotopy

$$h_t : (W, 0) \to (X, x_0), \qquad 0 \leqq t \leqq 1,$$

such that $h_0 \equiv f$, $h_1 \equiv g$, and $h_t \in \Omega$ for every $t \in I$. For the meaning of the congruence, see § 6. Thus, the local maps Ω are divided into disjoint (local) *homotopy classes* relative to σ. Throughout the present section, we shall simply call them classes. We are going to identify these classes with the elements of $\lambda_n(X, x_0; \sigma)$.

Let $f : (U, 0) \to (X, x_0)$ be any local map in Ω. There exists a positive real number $k \leqq 1$ such that $t \in U$ whenever $0 \leqq t \leqq k$. Let σ_k denote the path defined by $\sigma_k(t) = \sigma(kt)$ for each $t \in I$. Then f induces a homomorphism

$$f_\natural : \lambda_n(R^{n+1}, 0) \to \lambda_n(X, x_0; \sigma_k).$$

Define a path $\tau : I \to T(X, x_0)$ by taking

$$[\tau(s)](t) = \sigma(t - st + kst), \quad (s \in I, t \in I).$$

Then $\tau(0) = \sigma$ and $\tau(1) = \sigma_k$. Therefore, by (14.8), τ induces an isomorphism

$$\tau_* : \lambda_n(X, x_0; \sigma_k) \approx \lambda_n(X, x_0; \sigma).$$

One can verify that the composed homomorphism $\tau_* f_\natural$ does not depend on the real number k chosen in the construction. Hence the local map f determines uniquely an element $\deg(f) = \tau_* f_\natural(\iota)$ of the local homotopy group $\lambda_n(X, x_0; \sigma)$ which will be called the *degree* of the local map f.

If two local maps $f, g \in \Omega$ are locally homotopic relative to σ, then it follows easily from (16.3) that $\deg(f) = \deg(g)$. Hence the assignment $f \to \deg(f)$ induces a function

$$\chi : K \to \lambda_n(X, x_0; \sigma)$$

of the set K of all classes of the local maps Ω into $\lambda_n(X, x_0; \sigma)$. We shall prove that χ carries K onto $\lambda_n(X, x_0; \sigma)$ in a one-to-one fashion.

Since ι is a generator of $\lambda_n(R^{n+1}, 0)$, it can be represented by an admissible map $\theta : (J_{n+1}, v_0) \to (R^{n+1}, 0)$ which maps J_{n+1} homeomorphically onto the unit $(n+1)$-cell E^{n+1} defined by

$$E^{n+1} = \{(y_0, y_1, \ldots, y_n) \in R^{n+1} : y_0^2 + y_1^2 + \ldots + y_n^2 \leqq 1\}$$

in such a way that $\theta(t_0, t_1, 0, \ldots, 0) = (t_1, 0, \ldots, 0)$.

Now, let us prove that χ is onto. For this purpose, let α be an arbitrary element of the local homotopy group $\lambda_n(X, x_0; \sigma)$. Then α is represented by an admissible map $\phi : (J_{n+1}, v_0) \to (X, x_0)$. Let U denote the interior of E^{n+1}. Then

$$f = \phi\theta^{-1} : (U, 0) \to (X, x_0)$$

is a local map in Ω. By the definition given above, we have $\deg(f) = \alpha$. Hence χ is onto.

Next, let us prove that χ is one-to-one. For this purpose, let $f : (U, 0) \to (X, x_0)$ and $g : (V, 0) \to (X, x_0)$ be any two local maps in Ω with $\deg(f) = \deg(g)$. We have to prove that f and g are locally homotopic relative to σ. Choose a positive real number $k \leqq 1$ such that the $(n+1)$-cell

$$E_k^{n+1} = \{(y_0, y_1, \ldots, y_n) \in R^{n+1} : y_0^2 + y_1^2 + \ldots + y_n^2 \leqq k^2\}$$

is contained in $U \cap V$. The generator ι of $\lambda_n(R^{n+1}, 0)$ can be represented by an admissible $\theta_k : (J_{n+1}, v_0) \to (R^{n+1}, 0)$ which sends J_{n+1} homeomorphically onto E_k^{n+1} in such a way that $\theta_k(t_0, t_1, 0, \ldots, 0) = (kt_1, 0, \ldots, 0)$. Since $\deg(f) = \deg(g)$, it follows that $f\theta_k$ and $g\theta_k$ are admissibly homotopic relative to σ_k; precisely, there exists an admissible homotopy

$$\phi_t : (J_{n+1}, v_0) \to (X, x_0), \qquad (0 \leqq t \leqq 1),$$

such that $\phi_0 = f\theta_k$, $\phi_1 = g\theta_k$, and $\phi_t(t_0, t_1, 0, \ldots, 0) = \sigma(kt_1)$ for each $t \in I$. Denote the interior of E_k^{n+1} by W. Then $W \subset U \cap V$ and we obtain a local homotopy

$$h_t : \phi_t\theta_k^{-1} : (W, 0) \to (X, x_0), \qquad (0 \leqq t \leqq 1),$$

satisfying $h_0 \equiv f$, $h_1 \equiv g$, and $h_t \in \Omega$ for every $t \in I$. Hence f and g are locally homotopic relative to σ. This proves that χ carries K onto $\lambda_n(X, x_0; \sigma)$ in a one-to-one fashion.

Thus, we have established the following

THEOREM 17.1 *The elements of the local homotopy group $\lambda_n(X, x_0; \sigma)$ may be identified with the classes of the local maps of $(R^{n+1}, 0)$ into (X, x_0) relative to σ.*

This having been done, one can easily see that the group operation, the neutral element, and the inverse of a given element of $\lambda_n(X, x_0; \sigma)$ can be represented as follows. As in the global theory, we shall use the additive notation although $\lambda_1(X, x_0; \sigma)$ is usually non-abelian.

Let α and β be arbitrary elements of $\lambda_n(X, x_0; \sigma)$. Then they can be represented by local maps $f : (U, 0) \to (X, x_0)$ and $g : (V, 0) \to (X, x_0)$ respectively such that

$$f(y_0, y_1, \ldots, y_n) = \sigma\left(\sqrt{y_0^2 + y_1^2 + \ldots + y_n^2}\right), \quad \text{if } y_1 \geqq 0,$$
$$g(y_0, y_1, \ldots, y_n) = \sigma\left(\sqrt{y_0^2 + y_1^2 + \ldots + y_n^2}\right), \quad \text{if } y_1 \leqq 0.$$

Let $W = U \cap V$ and define a local map $h : (W, 0) \to (X, x_0)$ by taking

$$h(y_0, y_1, \ldots, y_n) = \begin{cases} f(y_0, y_1, \ldots, y_n), & \text{if } y_1 \leqq 0, \\ g(y_0, y_1, \ldots, y_n), & \text{if } y_1 \geqq 0. \end{cases}$$

Then h is in Ω and represents the element $\alpha + \beta$ of $\lambda_n(X, x_0; \sigma)$; in other words, $\deg(h) = \alpha + \beta$.

Now, let us consider the neutral element 0 of $\lambda_n(X, x_0; \sigma)$. A local map $f : (U, 0) \to (X, x_0)$ has $\deg(f) = 0$ if and only if f is locally homotopic (relative to σ) to the local map

$$\omega : (\text{Int } E^{n+1}, 0) \to (X, x_0)$$

defined by $\omega(y_0, y_1, \ldots, y_n) = \sigma\left(\sqrt{y_0^2 + y_1^2 + \ldots + y_n^2}\right)$.

Finally, let us consider the inverse $-\alpha$ of a given element α of $\lambda_n(X, x_0; \sigma)$. Pick a local map $f : (U, 0) \to (X, x_0)$ in Ω with $\deg(f) = \alpha$. Let

$$V = \{(y_0, y_1, y_2, \ldots, y_n) \in R^{n+1} : (y_0, -y_1, y_2, \ldots, y_n) \in U\}.$$

Then $-\alpha$ is represented by the local map $g : (V, 0) \to (X, x_0)$ in Ω defined by

$$g(y_0, y_1, y_2, \ldots, y_n) = f(y_0, -y_1, y_2, \ldots, y_n).$$

If the space X is pathwise connected around x_0 and locally n-simple at x_0, then we may omit the relativity with respect to the given path σ throughout the previous study. Thus, the elements of the local homotopy group $\lambda_n(X, x_0)$ can be considered as the (local) homotopy classes of all local maps of $(R^{n+1}, 0)$ into (X, x_0).

Since the local map ω is obviously unstable, the following theorem is an immediate consequence of (11.1).

THEOREM 17.2. *A local map f of $(R^{n+1}, 0)$ into (X, x_0) is unstable if* $\deg(f) = 0$.

This and (16.6) imply the following

THEOREM 17.3. *Let x_0 be a conic point of X. Then a local map f of $(R^{n+1}, 0)$ into (X, x_0) is unstable if and only if* $\deg(f) = 0$.

18. Local maps of a euclidean space into another.

Let n, q be given positive integers and consider the local maps of $(R^{n+1}, 0)$ into $(R^{q+1}, 0)$. According to the previous section, the homotopy classes of these local maps are the elements of the local homotopy group

$$\lambda_n(R^{q+1}, 0) \approx \pi_n(S^q).$$

Hence every information about the structure of the (global) homotopy group $\pi_n(S^q)$ gives a corresponding result on the local maps. For example, we have the following assertions:

(18.1) *If $n < q$, then every local map of $(R^{n+1}, 0)$ into $(R^{q+1}, 0)$ is unstable.*

(18.2) *If $q = 1$ and $n > q$, then every local map of $(R^{n+1}, 0)$ into $(R^{q+1}, 0)$ is unstable.*

(18.3) *If $n = q$, then $\lambda_n(R^{n+1}, 0)$ is infinite cyclic with the homotopy class e of the identity local map on $(R^{n+1}, 0)$ as generator.*

One can easily verify that the homotopy class of an arbitrary local map f of $(R^{n+1}, 0)$ into itself is $\operatorname{ind}(f) \cdot e$, where $\operatorname{ind}(f)$ denotes the index of f defined in § 9.

Now let us give a local version of the celebrated notion of suspension introduced by Freudenthal [7]. Let f be any given local map of $(R^n, 0)$ into $(R^q, 0)$ defined on an open neighborhood U of 0 in R^n. Consider $R^{n+1} = R^n \times R$ and $R^{q+1} = R^q \times R$. Then, by the *suspension* of f, we mean the local map Sf of $(R^{n+1}, 0)$ into $(R^{q+1}, 0)$ defined on the open neighborhood $V = U \times R$ of 0 in R^{n+1} by

$$(Sf)(x \times t) = f(x) \times t, \qquad (x \in U, t \in R).$$

This local version of suspension appears simpler and more natural than its global counter-part.

It is illustrated in § 9 that, for each integer $m \neq 0$, the local map f_m of $(R^2, 0)$ into itself defined by $f_m(z) = z^m$ for each point $z \in R^2$ considered a complex number is of index m and therefore represents me of the group $\lambda_1(R^2, 0)$. One can also verify that, for $n > 1$, the $(n-1)$-fold iterated suspension of f_m is of index m and hence represents the element me of the group $\lambda_n(R^{n+1}, 0)$.

In general, the assignment $f \to Sf$ induces a homomorphism

$$S : \lambda_{n-1}(R^q, 0) \to \lambda_n(R^{q+1}, 0)$$

called the *suspension*. Then the local version of the suspension theorem can be stated as follows:

(18.4) *The suspension* S *is an isomorphism if* $q > 2$ *and* $n < 2q - 2$ *and is an epimorphism if* $q \geqq 2$ *and* $n = 2q - 2$.

A generator of the infinite cyclic group $\lambda_2(R_3, 0)$ is represented by the local Hopf map $f : (R^4, 0) \to (R^3, 0)$ defined as follows. Consider $R^4 = R^2 \times R^2$ as the space of all pairs (x, y) of complex numbers x and y. Let S^1 denote the unit circle in R^2 consisting of the complex numbers z with $|z| = 1$. Then S^1 is a topological transformation group of R^4 by the operation $z(x, y) = (zx, zy)$. Then the orbit space may be identified with R^3 and the natural projection $f : (R^4, 0) \to (R^3, 0)$ will be called the local Hopf map. One also define the local Hopf maps of $(R^8, 0)$ into $(R^5, 0)$ and of $(R^{16}, 0)$ into $(R^9, 0)$ by using quaternions and Cayley number as in [23, p. 108]. These local Hopf maps are all stable.

Since these local Hopf maps are fiberings with 0 as the only singularity in the sense of Montgomery and Samelson [18], it suggests the application of our local invariants to the study of fiberings with isolated singularities. By pinching a singular fiber into a point, one can also apply the local invariants to study the fiber spaces with singular fibers as originally considered by Seifert [20]. A detailed paper is under preparation [12].

19. An application.

To conclude the present paper, we shall give an application of the tangent space which led the author to the study of our local invariants.

A main problem on topological semi-groups is to answer the following question formulated by A. D. Wallace, [26, p. 96]: What compact connected Hausdorff spaces admit a continuous associative multiplication with two-sided unit? Since there have already been numerous results on the structure of topological groups, we may restrict our interest only to those continuous associative multiplications with two-sided unit which fail to be topological group operations. These multiplications are called *essential multiplications*, [11].

Throughout this final section, let X be a pathwise connected compact Hausdorff space in which there is given an essential multiplication with a point $u \in X$ as its two-sided unit.

Let H denote the set of all points of X which have right inverses. According to Wallace [26, p. 99], H is also the set of all points of X which have left inverses. Furthermore, H is a compact topological group under the given multiplication, namely, *the maximal* *subgroup* of X containing u, [25, p. 333]. The complement $J = X\backslash H$ is *the maximal two-sided proper ideal* of X, [26, p. 103].

A point a of a space X is said to be *pathwise accessible* from a subspace B of X if, for every point $b \in B$, there exists a path $\sigma : I \to X$ such that $\sigma(0) = a$, $\sigma(1) = b$, and $\sigma(t) \in B$ for every $t > 0$ in I. A subspace A of X is said to be *pathwise accessible* from B if every point of A is pathwise accessible from B. If X is a Hausdorff space, then pathwise accessibility is equivalent to arcwise accessibility, [27, p. 66].

LEMMA 19.1. *The point u is pathwise accessible from the set J.*

PROOF. Let v be any point in J. Since X is pathwise connected, there exists a path $\xi : I \to X$ with $\xi(0) = u$ and $\xi(1) = v$. The inverse image $\xi^{-1}(H)$ is a closed set of the unit interval I. Let k denote the least upper bound of $\xi^{-1}(H)$. Then $0 \leqq k < 1$ since $\xi(1) = v \in J$. Also, $\xi(k) \in H$ and $\xi(t) \in J$ whenever $k < t \leqq 1$.

Let $a = \xi(k)$. Since $a \in H$, there exists a point $b \in X$ such that $ab = u$. Define a path $\eta : I \to X$ by setting

$$\eta(t) = [\xi(t - kt + k)]b, \qquad (t \in I).$$

Then $\eta(0) = ab = u$, $\eta(1) = vb$, and $\eta(t) \in J$ for every $t > 0$ in I.

Since X is pathwise connected, there exists a path $\rho : I \to X$ with $\rho(0) = u$ and $\rho(1) = b$. Define a path $\sigma : I \to X$ by taking

$$\sigma(t) = \begin{cases} \eta(2t), & (\text{if } 0 \leqq t \leqq \tfrac{1}{2}), \\ v\rho(2 - 2t), & (\text{if } \tfrac{1}{2} \leqq t \leqq 1). \end{cases}$$

Then $\sigma(0) = u$, $\sigma(1) = vu = v$, and $\sigma(t) \in J$ for each $t > 0$ of I. This completes the proof.

LEMMA 19.2. *The tangent space $T(X, u)$ is contractible.*

PROOF. Using the path $\sigma : I \to X$ constructed in the proof of (19.1), we define a homotopy $h_t : T(X, u) \to T(X, u)$, $(0 \leqq t \leqq 1)$, by taking

$$[h_t(\tau)](s) = \begin{cases} \sigma(s), & (\text{if } 0 \leqq s \leqq t), \\ \sigma(t)\tau(s - t), & (\text{if } t \leqq s \leqq 1), \end{cases}$$

for each $\tau \in T(X, u)$. Then h_0 is the identity map on $T(X, u)$ and $h_1[T(X, u)] = \sigma$. This completes the proof.

If u is a conic point of X, then (19.2) and (7.1) imply the main theorem in [11] and hence the theorems of Wallace, [26, pp. 96, 97], as indicated in [11].

By the *boundary* ∂X of a space X, we mean the set of all points w of X with contractible tangent spaces $T(X, w)$.

THEOREM 19.3. *The maximal subgroup H is contained in the boundary ∂X of X and is pathwise accessible from the maximal ideal J.*

PROOF. Let $w \in H$. The assignment $x \to xw$ defines a homeomorphism of X which carries u into w, H onto H, and J onto J. Hence, it follows that $T(X, w)$ is contractible and that w is pathwise accessible from J. This completes the proof.

COROLLARY 19.4. *The space X is pathwise connected around each $w \in H$ and $\lambda_n(X, w) = 0$ for every $n \geqq 1$.*

COROLLARY 19.5. *For each $w \in H$ and each coefficient group G,* we have

$$\tilde{L}_0(X, w; G) = 0, \quad \tilde{L}^0(X, w; G) = 0,$$

and

$$L_n(X, w; G) = 0, \quad L^n(X, w; G) = 0.$$

for every $n \geqq 1$.

COROLLARY 19.6. *The maximal ideal J is everywhere dense in X.*

THEOREM 19.7. *Any pair of points a, b of X can be connected by a path $\sigma : I \to X$ such that $\sigma(t) \in J$ whenever $0 < t < 1$.*

PROOF. Since X is pathwise connected, there exists a path $\tau : I \to X$ with $\tau(0) = a$ and $\tau(1) = b$. By (19.1), there exists a path $\xi : I \to X$ such that $\xi(0) = u$ and $\xi(t) \in J$ for each $t > 0$ in I. Define a path $\sigma : I \to X$ by setting

$$\sigma(t) = \begin{cases} \xi(2t)\tau(t), & (\text{if } 0 \leqq t \leqq \frac{1}{2}), \\ \xi(2 - 2t)\tau(t), & (\text{if } \frac{1}{2} \leqq t \leqq 1). \end{cases}$$

Then $\sigma(0) = a$, $\sigma(1) = b$, and $\sigma(t) \in J$ whenever $0 < t < 1$. This completes the proof.

COROLLARY 19.8. *The maximal ideal J is pathwise connected.*

COROLLARY 19.9. *Every subset of the maximal subgroup H fails to separate the space X.*

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