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On the topological Degree

by

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Let n be a positive integer > 1; E_i, for $i = n, n+1$, the i-dimensional Euclidean space; P_i^*, for $i = n, n+1$, an orientation of the i-dimensional finite Euclidean manifold P_i; U an open set in E_{n+1} and V an open set in P_{n+1}; A a simplicial 1-sphere in U and B such a one in V; A^* an orientation of A and B^* an orientation of B.

By g, g' denoting continuous maps of P_{n+1} in P_n, we call the set consisting of all points p of P_{n+1} with $g(p) = g'(p)$, the set of the coincidences of (g, g'), the "*singular form*" of (g, g'). The pair (g, g') be named "*normal*", if the singular form of (g, g') is either empty or composed of a finite number of pairwise disjoint simplicial 1-spheres.

Suppose φ_1, φ_2 are continuous maps of $\overline{U}$ in E_n; the set of the coincidences of (φ_1, φ_2), the "singular form" of (φ_1, φ_2), equal to A; moreover $B_1, \ldots, B_m$ mutually disjoint simplicial 1-spheres of P_{n+1}, $B_1 = B$, $B_i \cdot \bar{V} = 0$ for $i > 1$; and (γ_1, γ_2) a normal pair of maps γ_i: $P_{n+1} \to P_n$; ΣB_i the singular form of (γ_1, γ_2). Then we designate the B_i as the "*singularities*" of (γ_1, γ_2) and A as the "singularity" of (φ_1, φ_2).

The significance of n, E_n, E_{n+1}, P_n, P_{n+1}, U, V, A, B, A^*, B^*, P_n^*, P_{n+1}^*, φ_1, φ_2, γ_1, γ_2 thus defined remain till the end of this paper.

By the way, I shall prove the following approximation theorem elsewhere. If γ denotes a continuous map of P_{n+1} in P_n and ε a positive number, then there are simplicial maps γ^1 and γ^2 of P_{n+1} in P_n homotopic to γ and having the further properties: the set of the coincidences of (γ^1, γ^2) is either empty or the union of a finite number of mutually disjoint 1-spheres, $d(\gamma, \gamma^1) < \varepsilon$ and $d(\gamma, \gamma^2) < \varepsilon$. More shortly: *one can normally approximate* (γ, γ).

In Section 1, we associate with the orientated singularity A^* and just so with B^* an integer as its "degree" in such a way that degree of orientated singularities and classical degree are corresponding concepts. Some simpler theorems in Section 2 enumerate properties which both these degrees have in common: topological invariance, invariance at deformations, and a decomposition property.

A known property of coincidences, relative to which we will compare singularities and coincidences, is pronounced in the next paragraph; whereby P signifies an $(n+1)$-dimensional finite Euclidean manifold which possesses an orientation and lies in an Euclidean space.

Let c be a point of V and h_1, h_2 continuous maps of P_{n+1} in P; c the only coincidence of (h_1, h_2) on $\bar{V}$; the degree of c at (h_1, h_2) equal to zero. Then there exists a pair (h_1', h_2') homotopic to (h_1, h_2), consisting of maps $h_i' : P_{n+1} \to P$, and having the property: for $p \notin V$ hold the equations $h_1'(p) = h_1(p)$ and $h_2'(p) = h_2(p)$, on $\bar{V}$ there is no coincidence of (h_1', h_2').

Is there any property of singularities being apt to stand comparison with this property of coincidences? In this problem Section 3 engages. First the following theorem. The singularity B of (γ_1, γ_2) having the degree zero, there exists a point b in V and a pair (g_1, g_2) homotopic to (γ_1, γ_2), composed of maps $g_i : P_{n+1} \to P_n$, and of the fashion: $g_1(p) = \gamma_1(p)$ and $g_2(p) = \gamma_2(p)$ for $p \notin V$, the point b is the only coincidence of (g_1, g_2) on $\bar{V}$. Perhaps you may say in brief: *singularities of the degree zero can be contracted on a single point.* Yet, an example in Section 3 shows that the resting point cannot always be removed.

Some theorems used in the following easily result from known [1]) properties of the Brouwer degree.

1. The degree of a singularity.

If m is a positive integer and $q = (\alpha_1, \ldots, \alpha_m)$, $r = (\beta_1, \ldots, \beta_m)$ are points of the Euclidean m-space E_m, $q+r$ means the point $(\alpha_1+\beta_1, \ldots, \alpha_m+\beta_m)$ and $d(q, r)$ the Euclidean distance from q to r. "Simplexes" are Euclidean and open. If C signifies a 2-simplex in E_m and D the topological (topological and simplicial) image of $\bar{C}-C$, then D is said to be a "1-sphere" ("simplicial 1-sphere"). If just one point of the set M is attached to each point p of the set N by the map f, we denote the first point by $f(p)$. The pair (φ_1, φ_2) is said to be a pair of $\overline{U}$ in E_n. Let, for

[1]) See for instance: P. J. Hilton, "An introduction to homotopy theory", Cambridge Univ. Press, vol. 43 (1953).

$i = 1, 2, g_i$ be a map of P_{n+1} in P_n homotopic to γ_i, then (g_1, g_2) is called a pair "homotopic" to (γ_1, γ_2).

Let a be a point of A, then we will define an "*index*" of a under (φ_1, φ_2) relative to A^* as follows.

Be denoted by S an n-simplex in U with $a \in S$ and $A \cdot \bar{S} = a$, by E_n^* and E_{n+1}^* the natural orientations of E_n and E_{n+1}. Let $a_1, \ldots, a_{n+1}$ points of E_{n+1} with the properties: the points a, $a_1, \ldots, a_{n+1}$ are linearly independent; the orientation induced by $(aa_1, \ldots, aa_{n+1})$ into E_{n+1} concurs with the orientation E_{n+1}^*; the 1-simplex with the vertexes a and a_1 lies in A; the orientation induced by aa_1 into A and the orientation A^* agree; the points $a_2, \ldots, a_{n+1}$ lie in S. Let S^* be the orientation induced by $(aa_2, \ldots, aa_{n+1})$ into S. Furthermore let T be an n-simplex in E_n, T^* the orientation which E_n^* induces into T, t an affine map of $\bar{T}$ on $\bar{S}$ with $t(T^*) = S^*$, b the point in T determined by $t(b) = a$. Let f be defined by

$$f(p) = \varphi_1 t(p) - \varphi_2 t(p), \quad p \in \bar{T},$$

as map of $\bar{T}$ in E_n. Then b is the only fixed point of f, the index of b at f is said to be the index of a at (φ_1, φ_2) with respect to A^*.

You instantly verify that the last definition is unique and has the further property: if A^{**} means the orientation opposite to A^*, α^* and α^{**} are the indexes of a at (φ_1, φ_2) relative to A^* and A^{**} respectively, then $\alpha^* = -\alpha^{**}$. One easily sees:

There is an integer α such that, for each point p of A, the index of p at (φ_1, φ_2) referring to A^* is equal to α. Then we will define α to be the "*degree*" of A^* under (φ_1, φ_2), more exactly the degree of A under (φ_1, φ_2) with respect to (E_{n+1}^*, E_n^*). Correspondingly one may declare the "degree" of B^* under (γ_1, γ_2) with respect to (P_{n+1}^*, P_n^*).

2. Elementary properties of a singularity.

From the topological invariance of the fixed point index insues:

THEOREM 1. *The degree of A^* is topologically invariant, more precisely: Let t be a topological map of E_{n+1} onto itself such that $t(E_{n+1}^*) = E_{n+1}^*$, $t(A)$ a simplicial* 1*-sphere, $f_1 = t\varphi_1 t^{-1}$, and $f_2 = t\varphi_2 t^{-1}$. Then (f_1, f_2) represents a normal pair of mappings $f_i : t(U) \to E_n$, $t(A)$ is the only singularity of (f_1, f_2), and the degree of $t(A^*)$ at (f_1, f_2) is equal to the degree of A^* at (φ_1, φ_2).*

We will show:

THEOREM 2. *Let* $(\varphi_1^\tau, \varphi_2^\tau)$, $0 \leqq \tau \leqq 1$, *be normal pairs of maps* $\varphi_i^\tau : \overline{U} \to E_n$ *which continuously depend on* τ *and* A, *for* $0 \leqq \tau \leqq 1$, *the only singularity of* $(\varphi_1^\tau, \varphi_2^\tau)$. *Then the degree of* A^* *at* $(\varphi_1^0, \varphi_2^0)$ *is equal to the degree of* A^* *at* $(\varphi_1^1, \varphi_2^1)$.

PROOF. It suffices to show that, given a point a of A, the index of a at $(\varphi_1^0, \varphi_2^0)$ relative to A^* and the index of a at $(\varphi_1^1, \varphi_2^1)$ relative to A^* are equal.

To prove this, let the significance of S, T, t, and b be the one defined in the first section; moreover f^τ, for $0 \leqq \tau \leqq 1$, determined by

$$f^\tau(p) = \varphi_1^\tau t(p) - \varphi_2^\tau t(p), \quad p \in \overline{T},$$

as map of $\overline{T}$ in E_n. Then, for $0 \leqq \tau \leqq 1$, the point b is the only fixed point of f^τ, so the index of b under f^0 equal to the index of b under f^1. This already yields the assertion.

If A' is a 1-sphere in U, we denote A and A' as "*neighbouring*", provided the statements I and II are true. I. There is a homotopy $(t^\tau, 0 \leqq \tau \leqq 1)$ of topological maps $t^\tau : A \to U$ such that t^0 is the identity, $t^1(A) = A'$, and

$$d(p, t^\tau(p)) < 2d(p, t^1(p))$$

for all (p, τ) with $p \in A$ and $0 \leqq \tau \leqq 1$. II. The homotopies $(t_i^\tau, 0 \leqq \tau \leqq 1)$, $i = 1, 2$, being conditioned like $(t^\tau, 0 \leqq \tau \leqq 1)$, then the orientations $t_1^1(A^*), t_2^1(A^*)$ of A' agree. The orientation $t^1(A^*)$ of A' we name the orientation "*induced*" by A^* into A'.

In the last paragraph replacing A, A', A^*, U by B, B', B^*, V respectively, you obtain the definition of a 1-sphere B' such that B and B' are "neighbouring" and the definition of the orientation which B^* "induces" into B'.

Now let us establish:

THEOREM 3. *If* α *denotes the degree of* A^* *under* (φ_1, φ_2) *and* $\alpha_1, \ldots, \alpha_m$ *are integers with* $\Sigma\, \alpha_i = \alpha$, *then there are simplicial* 1-*spheres* $A_1, \ldots, A_m$ *in* U *by pairs disjoint and a normal pair* (f_1, f_2) *of maps* $f_i : \overline{U} \to E_i$ *with the properties:* $f_1(p) = \varphi_1(p)$ *and* $f_2(p) = \varphi_2(p)$ *for* $p \in \overline{U} - U$; *for* $i = 1, \ldots, m$, *the spheres* A_i *and* A *are neighbouring*; *the* A_i *are the singularities of* (f_1, f_2); A_i^* *being the orientation which* A^* *induces into* A_i, *the number* α_i *represents the degree of* A_i^* *at* (f_1, f_2).

PROOF. Let T be an n-simplex in E_n. Then you easily see that there are points a^τ, $0 \leqq \tau \leqq 1$, of A continuously dependent on τ and n-simplexes S^τ, $0 \leqq \tau \leqq 1$, continuously dependent on τ, too, and a homotopy $(t^\tau, 0 \leqq \tau \leqq 1)$ of affine maps $t^\tau : \overline{T} \to \overline{S}^\tau$

with the properties: $a^0 = a^1$, $S^0 = S^1$, and $t^0 = t^1$; for $0 < |\tau_1 - \tau_2| < 1$ there hold $a^{\tau_1} \neq a^{\tau_2}$ and $\bar{S}^{\tau_1} \cdot \bar{S}^{\tau_2} = 0$; $a^\tau \in S^\tau$ for all τ; if, for all τ, f^τ denotes the map defined by

$$f^\tau(p) = \varphi_1 t^\tau(p) - \varphi_2 t^\tau(p), \; p \in \bar{T},$$

and b^τ the point of T where $t^\tau(b^\tau) = a^\tau$, then the index of b^τ at f^τ is equal to α. Thus, Theorem 3 easily follows from

LEMMA 1. *Let S be an n-simplex in E_n; and $(a_i^\tau, 0 \leqq \tau \leqq 1)$, $i = 1, \ldots, m$, curves of points a_i^τ of S; for $0 \leqq \tau \leqq 1$, the points $a_1^\tau, \ldots, a_m^\tau$ mutually disjoint; $(f^\tau, 0 \leqq \tau \leqq 1)$ a homotopy of maps $f^\tau : \bar{S} \to E_n$; $p \neq f^\tau(p)$ for all (p, τ) with $p \in \bar{S} - S$ and $0 \leqq \tau \leqq 1$; further $a_1^i, \ldots, a_m^i$, for $i = 0, 1$, the fixed points of f^i; and, for $k = 1, \ldots, m$, the index of a_k^0 at f^0 equal to the index of a_k^1 at f^1. Then there exists a homotopy $(g^\tau, 0 \leqq \tau \leqq 1)$ of maps $g^\tau: \bar{S} \to E_n$ such that: $f^\tau(p) = g^\tau(p)$ for all (p, τ) where either $p \in \bar{S}$ and $\tau = 0, 1$ or $p \in \bar{S} - S$ and $0 \leqq \tau \leqq 1$; for $0 \leqq \tau \leqq 1$, the points $a_1^\tau, \ldots, a_m^\tau$ are the fixed points of g^τ.*

PROOF. It suffices to show the following simpler proposition.

Let a_1, a_2 be different points of S and f^0, f^1 continuous maps of $\bar{S}$ in E_n with the properties: $f^0(p) = f^1(p)$ for $p \in \bar{S} - S$; for $i = 0, 1$, the points a_1, a_2 are the fixed points of f^i; for $k = 1, 2$, the index of a_k at f^0 is equal to the index of a_k at f^1. Then there is a homotopy $(g^\tau, 0 \leqq \tau \leqq 1)$ of maps $g^\tau: \bar{S} \to E_n$ such that the following holds: $g^\tau(p) = f^0(p)$ for all (p, τ) with $p \in \bar{S} - S$ and $0 \leqq \tau \leqq 1$; $g^0 = f^0$ and $g^1 = f^1$; for $0 \leqq \tau \leqq 1$, the points a_1, a_2 are the only fixed points of g^τ.

To establish this, first let T denote an $(n-1)$-simplex with $T \subset S$, $\bar{T} - T \subset \bar{S} - S$, and the property: if S_1, S_2 are both the components of the set $S - T$, we have $a_1 \in S_1$ and $a_2 \in S_2$. Following a known theorem on the fixed point index, there is a homotopy $(g^\tau, 0 \leqq \tau \leqq 1/2)$ of maps $g^\tau: \bar{S} \to E_n$ which disposes of the properties: $g^\tau(p) = f^0(p)$ for all (p, τ) with $p \in \bar{S} - S$ and $0 \leqq \tau \leqq 1/2$; $g^0 = f^0$;

$$g^{1/2}(p) = f^1(p) \text{ for } p \in \bar{T};$$

for $0 \leqq \tau \leqq 1/2$, the points a_1 and a_2 are the only fixed points of g^τ.

So it remains to show:

Let a be a point of S and f, f' continuous maps of $\bar{S}$ in E_n: $f(p) = f'(p)$ for $p \in \bar{S} - S$, a the only fixed point of f and just so the only fixed point of f'. Then there is a homotopy $(h^\tau, 0 \leqq \tau \leqq 1$ of maps $h^\tau: \bar{S} \to E_n$ such that: $h^\tau(p) = f(p)$ for all (p, τ) with

$p \in \bar{S}-S$ and $0 \leqq \tau \leqq 1$; $h^0 = f$ and $h^1 = f'$; a represents the only fixed point of h^τ for $0 \leqq \tau \leqq 1$.

The last proposition, however, is true, as you may easily verify.

Like Theorem 1, 2, and 3 one can prove:

The degree of B is topologically invariant. If $(\gamma_1^\tau, \gamma_2^\tau)$, $0 \leqq \tau \leqq 1$, are normal pairs of maps γ_i^τ: $P_{n+1} \to P_n$ which continuously depend on τ and if B, for $0 \leqq \tau \leqq 1$, represents a singularity of $(\gamma_1^\tau, \gamma_2^\tau)$, then the degree of B^* at (γ_1^0, γ_2^0) and the degree of B^* at (γ_1^1, γ_2^1) are equal. Let β be the degree of B^* at (γ_1, γ_2) and $\beta_1, \ldots, \beta_m$ integers with $\Sigma\, \beta_i = \beta$; then there are mutually disjoint simplicial 1-spheres $B_1, \ldots, B_m$ in V and a normal pair (g_1, g_2) homotopic to (γ_1, γ_2), composed of maps g_i: $P_{n+1} \to P_n$, and provided with the following properties: $g_1(p) = \gamma_1(p)$ and $g_2(p) = \gamma_2(p)$ for $p \notin V$; for $i = 1, \ldots, m$, the spheres B_i and B are neighbouring; the B_i are the singularities of (g_1, g_2) on V; by B_i^* denoting the orientation which B^* induces into B_i, one obtains β_i to be the degree of B_i^* under (g_1, g_2).

3. Singularities of the degree zero.

The singularity A of (φ_1, φ_2) be called "*unessential*" if, for every open set U_1 of E_{n+1} with $A \subset U_1 \subset U$, there are continuous maps f_i: $\bar{U} \to E_n$ such that: $f_1(p) = \varphi_1(p)$ and $f_2(p) = \varphi_2(p)$ for $p \notin U_1$, $f_1(p) \neq f_2(p)$ for $p \in \bar{U}_1$. We designate A as "*essential*" singularity if it is not unessential. Correspondingly one defines the "essentiality" and "unessentiality" of B. Hereupon holds:

Theorem 4. *The singularity A of (φ_1, φ_2) being unessential, its degree is equal to zero.*

Proof. Let a be a point of A, S and n-simplex of U with $a \in S$ and $A \cdot \bar{S} = a$, T an n-simplex in E_n, and t an affine map of $\bar{T}$ onto $\bar{S}$. Let f be defined by $f(p) = \varphi_1 t(p) - \varphi_2 t(p)$, $p \in \bar{T}$, as map of $\bar{T}$ in E_n. To establish that the index of a at (φ_1, φ_2) and thus the degree of A at (φ_1, φ_2) is equal to zero, it is sufficient to show: there exists a continuous map f': $\bar{T} \to E_n$ which has no fixed point and agrees with f on $\bar{T}-T$.

May U_1 denote an open set in E_{n+1} with $A \subset U_1 \subset U$ and $(\bar{S}-S) \cdot \bar{U}_1 = 0$. Then the unessentiality of A yields continuous maps φ_i': $\bar{U} \to E_n$, $i = 1, 2$, such that $\varphi_1'(p) = \varphi_1(p)$ and $\varphi_2'(p) = \varphi_2(p)$ for $p \notin U_1$, $\varphi_1'(p) \neq \varphi_2'(p)$ for $p \in \bar{U}_1$. Now setting $f'(p) = \varphi_1' t(p) - \varphi_2' t(p)$ for $p \in \bar{T}$, we obtain a map f': $\bar{T} \to E_n$ of the desired kind.

Like Lemma 1 you can prove:

Lemma 2. *Let S be an n-simplex in E_n and a a point of S. Suppose $(f^\tau,\ 0 \leqq \tau \leqq 1)$ to be a homotopy of maps $f^\tau\colon \bar{S} \to E_n$ with the properties: $p \neq f^\tau(p)$ for all (p, τ) where $p \in \bar{S}-S$ and $0 \leqq \tau \leqq 1$; the point a is the only fixed point of f^0 and just so of f^1, the index of a at f^0 is equal to zero. Then there exists a homotopy $(g^\tau,\ 0 \leqq \tau \leqq 1)$ of maps $g^\tau\colon \bar{S} \to E_n$ which have the properties: $g^\tau(p) = f^\tau(p)$ for all (p, τ) where either $p \in \bar{S}$ and $\tau = 0, 1$ or $p \in \bar{S}-S$ and $0 \leqq \tau \leqq 1$; for $0 < \tau < 1$, the map g^τ has no fixed point.*

A modified inversion of Theorem 4 is given by

Theorem 5. *Let the degree of A at (φ_1, φ_2) be zero. Then there exists a point a in U and continuous maps $f_i\colon \bar{U} \to E_n$ with the properties: $f_1(p) = \varphi_1(p)$ and $f_2(p) = \varphi_2(p)$ for $p \in \bar{U}-U$, the point a is the only coincidence of (f_1, f_2).*

Proof. Let S^τ, $0 \leqq \tau \leqq 1$, be n-simplexes of U continuously dependent on τ such that: for all τ, the intersection $A \cdot S^\tau$ consists of a single point a^τ; for $\tau_1 \neq \tau_2$, the intersection $\bar{S}^{\tau_1} \cdot \bar{S}^{\tau_2}$ is empty. The union of all S^τ with $0 < \tau < 1$ we denote by S.

Following Lemma 2, there are continuous maps $g_i\colon \bar{U} \to E_n$, $i = 1, 2$, of the condition: $g_1(p) = \varphi_1(p)$ and $g_2(p) = \varphi_2(p)$ for $p \notin S$, $g_1(p) \neq g_2(p)$ for $p \in S$.

The set $A-S$ is homeomorph to a closed segment. Thus there exists an open set U_1 in E_{n+1} such that $A-S \subset U_1 \subset U$ and $\bar{U}_1$ is homeomorph to the closure of a simplex. Let a be a point of U_1. Then there exists a continuous map w of $\bar{U}_1 - a$ onto $\bar{U}_1 - U_1$ so that $w(p) = p$ for $p \in \bar{U}_1 - U_1$.

Hereupon we set $\lambda(p) = d(p, a)/(d(p, a)+d(p, \bar{U}_1-U_1))$ for all points $p \in \bar{U}_1 - a$, and $f_1 = g_1$, moreover

$$f_2(p) = g_1(p)+\lambda(p)(g_2w(p)-g_1w(p)) \text{ for } p \in \bar{U}_1-a,$$

further $f_2(p) = g_2(p)$ for $p \in \bar{U}-U_1$, and $f_2(a) = g_1(a)$.

For the sake of finishing the argumentation it suffices to show that a represents the only coincidence of (f_1, f_2) on $\bar{U}_1$: If p means a point of $\bar{U}_1 - a$, we have

$$f_2(p)-f_1(p) = \lambda(p)(g_2w(p)-g_1w(p)),$$

besides $\lambda(p) > 0$, and $g_2w(p) \neq g_1w(p)$, thus $f_2(p) \neq f_1(p)$.

Similarly as Theorem 4 and 5 one can prove:

The singularity B of (γ_1, γ_2) being unessential, its degree is equal to zero. The degree of B under (γ_1, γ_2) being zero, there is a point b in V and a pair (g_1, g_2) homotopic to (γ_1, γ_2), consisting of maps $g_i\colon P_{n+1} \to P_n$, and of the further condition: $g_1(p) = \gamma_1(p)$ and $g_2(p) = \gamma_2(p)$ for $p \notin V$, the point b is the only coincidence of (g_1, g_2) on $\bar{V}$.

The precise inversion of Theorem 4 is not correct:

There exist singularities of the degree zero which are essential.

Proof. Let S be a 4-simplex in E_4, T a 3-simplex in E_3, a a point of S, and b a point of T. Set $f_1(p) = b$ for $p \in \bar{S}$. Further, let f_2 be a continuous map of $\bar{S}$ onto $\bar{T}$ with the properties: $f_2(a) = b$,

$$f_2(p) \neq b \text{ for } p \neq a,$$

the map $f_2|\bar{S}-S$ represents an essential map of the 3-sphere $\bar{S}-S$ on the 2-sphere $\bar{T}-T$. Following a known theorem [2]), such a map exists.

Now denote by C a simplicial 1-sphere ϵ a in S, by R a 3-simplex in S with $a \in R$ and $C \cdot \bar{R} = a$. Let S_1 be an open set in E_4 such that

$$C-a \subset S_1, \ \bar{R} \cdot \bar{S}_1 = a, \text{ and } \bar{S}_1 \subset S;$$

further $\zeta(p) = d(p, C)/(d(p, C)+d(p, \bar{S}_1-S_1))$ for all points p of S_1; and $g_2(p) = f_2(p)$ for $p \in \bar{S}-S_1$,

$$g_2(p) = b+\zeta(p)(f(p)-b) \text{ for } p \in S_1.$$

The pair (f_1, g_2) thus defined is regular, and C represents its only singularity.

The assumption, C be an unessential singularity of (f_1, g_2), leads to a contradiction as follows. Then there would exist continuous maps f^i: $\bar{S} \to E_3$, $i = 1, 2$, so conditioned that: $f^1(p) = f_1(p)$ and $f^2(p) = g_2(p)$ for $p \in \bar{S}-S$, $f^1(p) \neq f^2(p)$ for all points p of $\bar{S}$.

We define f by $f(p) = b+(f^2(p)-f^1(p))$, $p \in \bar{S}$, as map of $\bar{S}$ in E_3, that disposes of the following properties: 1) the sphere $\bar{S}-S$ is essentially mapped on $\bar{T}-T$ by $f|\bar{S}-S$, 2) for all points p of $\bar{S}$ holds $b \neq f(p)$. Assertion 1) is true, since $f^1(p) = f_1(p) = b$ for $p \in \bar{S}-S$ and $f^2|\bar{S}-S = f_2|\bar{S}-S$ is an essential map of $\bar{S}-S$ on $\bar{T}-T$. From $f^1(p) \neq f^2(p)$, $p \in \bar{S}$, ensues the correctness of the second assertion. The affirmations 1) and 2), however, contradict to one another.

In order to prove, the degree of C at (f_1, g_2) be zero, it suffices to show: the index of a at (f_1, g_2) is zero. This to establish, let t be an affine map of $\bar{T}$ on $\bar{R}$. Determine h by $h(p) = f_1t(p)-g_2t(p)$, $p \in \bar{T}$, as map of $\bar{T}$ in E_3. The point b is the only fixed point of h. We will show that the index of b under h is equal to zero.

[2]) H. Hopf, „Zur Algebra der Abbildungen von Mannigfaltigkeiten", Journal f. reine und angewandte Math., vol. 163 (1930), pp. 71—88.

For this purpose let R^τ, $0 \leqq \tau \leqq 1$, be 3-simplexes of S continuously dependent on τ such that $R^0 = R$ and $a \notin R^\tau$ for $\tau > 0$; further (t^τ, $0 \leqq \tau \leqq 1$) a homotopy of affine maps t^τ: $\overline{T} \to \overline{R}^\tau$ with $t^0 = t$; besides h^τ, for $0 \leqq \tau \leqq 1$, defined by

$$h^\tau(p) = f_1 t^\tau(p) - f_2 t^\tau(p), \; p \in \overline{T},$$

as map of $\overline{T}$ in E_3.

On account of $\overline{R} \cdot \overline{S}_1 = a$ and $g_2(p) = f_2(p)$, $p \notin \overline{S}_1$, holds $f_2(p) = g_2(p)$ for $p \in \overline{R}$, hence $h^0 = h$. For all (p, τ) with $p \in \overline{T}$ and $0 < \tau \leqq 1$, one has $t^\tau(p) \neq a$, consequently $f_1 t^\tau(p) \neq f_2 t^\tau(p)$; from which it follows that, for $0 < \tau \leqq 1$, the map h^τ has no fixed point. Thus, the index of b under $h^0 = h$ is equal to zero.

And the proof is complete.

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