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Topology of *B*-metric spaces

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Topology of B-Metric Spaces [1]

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1. Introduction. Numerous studies (1, 2, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18, 23, 24) have been made concerning geometries and topologies induced in sets by general distance functions. A formulation of the notion „generalized metric space" has been given (12). In this paper we begin the elaboration of the topology induced in sets over σ-complete Boolean algebras by the Kantorovitch topologies of the respective algebras. The work bears mainly on fundamental questions in the topology of such spaces. Among the more interesting results presented are a discussion of Birkhoff's problem 77 (3), the autometrization of a σ-complete Boolean algebra in its Kantorovitch topology, the analogue for B-metric spaces of the Cantor-Hausdorff completion (20), and the Boolean metrization of zero-dimensional spaces. The work was suggested mainly by the interesting comparison between the distance geometries of ordinary metric spaces and the autometrized Boolean algebras studied by one of us (10, 11) and, more recently, by L. M. Blumenthal and others (5). There is a hint of the program, however, in a paper of Löwig around 1936 (22).

2. Preliminaries and property (††). In this paper, B shall always denote a σ-complete Boolean algebra. In B we denote the operations of join, meet, complement, and symmetric difference by $a \vee b$, $a \wedge b$, a', and $a \oplus b$, respectively.

If $\{x_i\}$ is a sequence of points of B, one defines $\overline{\lim_i}\, x_i = \bigwedge_{k=1}^{\infty} \bigvee_{i=k}^{\infty} x_i$ and $\underline{\lim_i}\, x_i = \bigvee_{k=1}^{\infty} \bigwedge_{i=k}^{\infty} x_i$. One also defines $\lim_i x_i = x$ if and only

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[2]) Part of the contributions of the senior author to this paper were made while he was at the Institute for Air Weapons Research, The University of Chicago.

[3]) The contributions of the junior author to this paper constitute part of his University of Florida doctoral dissertation.

if $\overline{\lim_i} x_i = x = \underline{\lim_i} x_i$. The K-topology (Kantorovitch topology (16, 21), sequential order topology (3)) of B is the derivative topology (25) of the sequential topology under the above definition of $\lim_i x_i$.

We shall need the following well-known and easily verified facts:

1. $\underline{\lim_i} x_i \leqq \overline{\lim_i} x_i$ for arbitrary sequences (3).
2. $\underline{\lim_i} (x_i \vee y_i) \leqq \underline{\lim_i} x_i \vee \overline{\lim_i} y_i$ (22).
3. $\lim_i x_i = 0$ is equivalent to $\overline{\lim_i} x_i = 0$ (this follows immediately, of course, from 1. above).
4. The Boolean operations of B are continuous in the K-topology of B (3, 22).

Consider now the following two properties relevant to sequential topologies:

(†) $\lim_j x_j^i = x_i$, for all i, and $\lim_i x_i = x$ imply the existence of a function $j(i)$ so that $\lim_i x_{j(i)}^i = x$.

(††) $\lim_j x_j^i = x_i$, for all i, and $\lim_i x_i = x$ imply the existence of a function $j(i)$ so that $k(i) \geqq j(i)$, for all i, implies $\lim_i x_{k(i)}^i = x$.

Garrett Birkhoff's *Lattice Theory* (3) proposes an unsolved problem, No. 77, which may be formulated as: Does every σ-complete Boolean algebra possess Property (†) in its K-topology? Clearly (††) implies (†).

The following Lemma seems to be well-known, but we include a proof for the sake of completeness.

LEMMA 1. *In the metric topology of a metric space,* (††) *subsists.*

Proof. Let us consider a metric space in which $\lim_j \delta(x_j^i, x_i) = 0$, for all i, and $\lim_i \delta(x_i, x) = 0$. For each i select $j(i)$ so that $k(i) \geqq j(i)$ implies $\delta(x_{k(i)}^i, x_i) < 2^{-i}$. Now, for each $\epsilon > 0$ there is k so that $\epsilon > 2^{-(k-1)}$ and N so that $i > N$ implies $\delta(x_i, x) < 2^{-k}$. Setting $M = \max(N, k)$, we find that for $i > M$ and $k(i) \geqq j(i)$, for all i, we have

$$\delta(x_{k(i)}^i, x) \leqq \delta(x_{k(i)}^i, x_i) + \delta(x_i, x) < 2^{-i} + 2^{-k} < \epsilon.$$

LEMMA 2. *If B is the set algebra of a countable set, it may be made into a normed lattice* (3).

Proof. Let B be the set algebra of a countable set. B may be represented (σ-isomorphically) as the direct product of countably many replicas of the two element Boolean algebra. Thus, if $x \epsilon B$ we set $x = ({}_1x, {}_2x, \ldots)$ where ${}_ix$ is either 0 or 1 according as the i^{th} point in a fixed enumeration of our countable set is not in x or is in x, respectively. Define $|x| = \sum_{i=1} {}_ix 2^{-i}$. This functional is obviously sharply monotone increasing and modular (3). Hence, Lemma 2 follows.

LEMMA 3. *If B is the set algebra of a countable set, the distance function induced by the functional of Lemma 2 is a metrization of the K-topology of B. Thus, B has Property (††) in its K-topology.*

Proof. Consider the function $d(x, y) = x \oplus y$ (10, 11) in B. From Property 4 at the beginning of this Section, $\lim_i x_i = x$ implies $\lim_i d(x_i, x) = d(x, x) = 0$. Conversely, suppose $\lim_i d(x_i, x) = 0$. Then, again using Property 4 $x \wedge \lim_i x'_i = x' \wedge \lim_i x_i = 0$. This yields $x \leqq \lim_i x_i$ and $\lim_i x_i \leqq x$, or $x = \lim_i x_i$. Thus, we have

$$(1) \qquad \lim_i x_i = x \text{ if and only if } \lim_i d(x_i, x) = 0.$$

Suppose again that $\lim_i x_i = x$ and observe that $\delta(x, x_i) = |x \vee x_i| - |x \wedge x_i| = |d(x, x_i)|$, by the modularity of the norm functional. In view of (1) above and this observation it will suffice to show

$$(2) \qquad \lim_i y_i = 0 \text{ if and only if } \lim_i |y_i| = 0$$

in order to complete the proof of Lemma 3. Suppose then that $\lim_i y_i = 0$. From the representation employed in Lemma 2, we have that for each k there is $N(k)$ so that $i > N(k)$ implies ${}_kx_i = 0$. Select $\epsilon > 0$ and select k so that $\epsilon > 2^{-(k-1)}$. Then for $i > \max_{j \leqq k} N(j)$, $|x_i| \leqq \sum_{i=k}^{\infty} 2^{-i} < \epsilon$. Conversely, if $|x_i|$ has limit 0 there is an $N(k)$ as described above for each k and $\lim_i x_i = 0$.

We observe that Lemma 3 contradicts Exercise 46 on Page 286 of (25).

LEMMA 4. *If B is an algebra with Property (††), and if J is a σ-ideal* (3) *in B, then B/J has Property (††)* (K-topologies understood).

Proof. Assume the hypotheses and denote elements a of B mod J by $\{a\}$.

Let $\lim_j \{x_j^i\} = \{x_i\}$, for all i, and $\lim_i \{x_i\} = \{x\}$. Then $\overline{\lim}_j x_j^i = x_i \vee u_i$ and $\underline{\lim}_j x_j^i = x_i \vee v_i$, for all i, and $\overline{\lim}_i x_i = x \vee z$, $\underline{\lim}_i x_i = x \vee w$, where u_i, v_i, w and z are in J. Let $t = (\bigvee_{i=1}^{\infty} u_i) \vee (\bigvee_{i=1}^{\infty} v_i) \vee z \vee w$. Now, t is in J and one sees immediately that $\lim_j (x_j^i \vee t) = x_i \vee t$, for all i, and $\lim_i (x_i \vee t) = x \vee t$, and since B has Property (††), there is $j(i)$ so that $k(i) \geqq j(i)$, for all i, implies $\lim_i (x_{k(i)}^i \vee t) = x \vee t$. Thus, $\lim_i \{x_{k(i)}^i\} = \{x\}$, since t is in J.

One calls a topological space *sequentially compact* provided any sequence in the space has a convergent subsequence. Clearly, any compact space whose topology is obtained as the derivative topology of a sequential topology is sequentially compact provided the sequential topology has Property (†). We shall call an algebra a (†)-algebra or an (††)-algebra if it has Property (†) or Property (††), respectively, in its K-topology. Clearly, any (††)-algebra is a (†)-algebra.

Lemma 5. *Any set algebra of an uncountable set fails to have Property* (†), *and, hence, is neither a* (†)*-algebra nor a* (††)*-algebra.*

Proof. Let B be the set algebra of an uncountable set. Then B contains as a closed subset a σ-isomorphic image (and, hence, a homeomorphic image) of the set algebra of the set algebra of the natural numbers, $2^{2^{\aleph_0}}$. Now, any set algebra is bicompact (25) and, hence, compact.

Thus, to prove Lemma 5, it suffices to show that $2^{2^{\aleph_0}}$ is not sequentially compact. The following example showing this is due to Mr. Alfred B. Lehman. Let a_n be the set of sets of integers which contain the integer n. (By integer in this discussion we mean positive integer or natural number). This sequence fails to have any convergent subsequence. This is clear since if a_{n_i} were a convergent subsequence, every set of integers which contained infinitely many of the n_i would have to contain almost all (all but a finite number) of the n_i.

Combining Lemmas 3, 4, and 5 we provide an answer to Birkhoff's Problem 77 (3) with

Theorem 1. $2^{\aleph_0}$ *is a* (††)*-algebra. Any σ-factor algebra of a*

(††)-algebra is a (††)-algebra. In general, however, Boolean σ-algebras fail to have Property (†).

One should note that Theorem 1 corrects a previously announced [4]) solution of Problem 77 which was based on a fallacious argument.

3. *B*-metric spaces and their distance topologies.

By a *B*-metric space we shall mean a set Σ together with a mapping, $d(\xi, \eta) : \Sigma\Sigma \to B$, of $\Sigma\Sigma$ into B with the properties:

1. *Vanishing.* $d(\xi, \eta) = 0$ if and only if $\xi = \eta$.
2. *Symmetry.* $d(\xi, \eta) = d(\eta, \xi)$; for all ξ, η.
3. *Triangle inequality.* $d(\xi, \zeta) \leqq d(\xi, \eta) \vee d(\eta, \zeta)$; for all ξ, η, ζ.

In the standardized terminology of the paper (12), a *B*-metric space is a generalized metric space over *B*.

If Σ is a *B*-metric space, we define $d\text{-}\lim_i \xi_i = \xi$ in Σ if and only if $\lim_i d(\xi_i, \xi) = 0$ in B. The resulting sequential topology and its derivative topology (25) are called the *d*-topology or metric topology of Σ.

A topological space Σ is said to be metrizable over B if there is a function $d(\xi, \eta) : \Sigma\Sigma \to B$ under which Σ forms a *B*-metric space such that $\lim_i \xi_i = \xi$ in the original topology of Σ if and only if $d\text{-}\lim_i \xi_i = \xi$.

It is to be recalled (10, 11) that B itself forms a *B*-metric space under the autometrization $d(x, y) = x \oplus y$. That is, with symmetric difference as distance function, B is a generalized metric normal ground space (12).

3. The autometrization of *B*.

THEOREM 2. *B in its K-topology is metrizable over itself by the autometrization.*

Proof. In view of the remark immediately preceding this Section, it suffices to show that $\lim_i x_i = x$ if and only if $\lim_i d(x_i, x) = 0$. This, however, was done in the proof of Lemma 3.

Theorem 2 implies, of course, that we shall obtain results about the *K*-topology of *B* as special cases of results dealing with general *B*-metric spaces. Some of these results have been obtained in this special case by Löwig (22), although our definitions do not all agree with his.

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[4]) See (13) in the bibliography.

4. Continuity of the distance function. Let Σ be a B-metric space. It is well-known that limits are unique, subsequences of a convergent sequence converge to the limit of the sequence, and a sequence almost all (all but a finite number) of whose elements coincide converges to that element in the K-topology of B (3, 16, 21). These facts clearly imply the analogous facts for the metric topology of Σ. Thus, (25), the d-topology of Σ makes Σ a Hausdorff space.

THEOREM 3. *If, in* Σ, $d\text{-}\lim_i \xi_i = \xi$ *and* $d\text{-}\lim_i \eta_i = \eta$, *then* $\lim_i d(\xi_i, \eta_i) = d(\xi, \eta)$.

Proof. Suppose the hypotheses. Then

$$(1) \qquad d(\xi, \eta) \leqq d(\xi, \xi_i) \vee d(\eta, \eta_i) \vee d(\xi_i, \eta_i); \text{ for all } i.$$

$$(2) \qquad d(\xi, \eta) \leqq \underline{\lim}_i \, (d(\xi, \xi_i) \vee d(\xi_i, \eta_i)) \vee \overline{\lim}_i \, d(\eta, \eta_i)$$
$$\leqq \underline{\lim}_i \, d(\xi_i, \eta_i) \vee \overline{\lim}_i \, d(\xi, \xi_i) = \underline{\lim}_i \, d(\xi_i, \eta_i).$$

Similarly, one shows

$$(3) \qquad \overline{\lim}_i \, d(\xi_i, \eta_i) \leqq d(\xi, \eta).$$

(2) and (3) together yield Theorem 3. Theorem 3 may, of course, be stated: *The distance function of a B-metric space is simultaneously continuous.*

As a Corollary to Theorem 3 we have

COROLLARY. *If B is a* (††)*-algebra* (*resp.* (†)*-algebra*) *then* (††) (*resp.* (†)) *subsists in the d-topology of any B-metric space.*

5. Cauchy sequences and completion of spaces over (††)-algebras. Throughout this Section, Σ will denote a B-metric space. We shall define a *Cauchy sequence* in Σ as a sequence $\{\xi_i\}$ for which

$$(\mathrm{C}) \qquad \lim_i \lim_j d(\xi_i, \xi_j) = 0.$$

One should note that this is weaker than Löwig's (22) definition of Cauchy sequence which requires existence of an independent double limit, rather than an iterated limit. One should also note that (C) is equivalent to $\overline{\lim}_i \lim_j d(\xi_i, \xi_j) = 0$.

One says, as usual, that Σ is *complete* provided every Cauchy sequence in Σ converges to a point of Σ.

THEOREM 4. *B is complete in the autometrization.*

Proof. Suppose x_i is a Cauchy sequence in B. Now,

$$(1)\quad d(x_i, \overline{\lim_j}\, x_j) = \underline{\lim_j}\, (x_i \wedge x_j') \vee \overline{\lim_j}\, (x_i' \wedge x_j)$$
$$\leqq \overline{\lim_j}\, d(x_i,\ x_j) = \lim_j d(x_i,\ x_j).$$

Thus,

$$(2)\quad \overline{\lim_i}\, d(x_i,\ \overline{\lim_j}\, x_j) = 0 \ \text{ and } \ d\text{-}\lim_i x_i = \overline{\lim_j}\, x_j.$$

Theorem 5. *If, in* Σ, d-$\lim_i \xi_i$ *exists, then* $\{\xi_i\}$ *is a Cauchy sequence.*

Proof. Let $\lim_i d(\xi,\ \xi_i) = 0$. By Theorem 3, $\lim_j d(\xi_i,\ \xi_j) = d(\xi,\ \xi_i)$; for all i. Hence, $\lim_i \lim_j d(\xi_i,\ \xi_j) = 0$, again by Theorem 3.

Having illustrated the methods of computation utilized in these limit proofs, we shall omit many of them in the sequel.

We now undertake the completion of a B-metric space. Although the first few Lemmas are unrestricted, it is apparently necessary for the final result that B be a (††)-algebra.

Lemma 6. *If, in* Σ, $\{\xi_i\}$ *is a Cauchy sequence and* $n \in \Sigma$, *then* $\lim_i d(\xi_i,\ \eta)$ *exists.*

Proof.

$$(1)\quad d(\xi_i,\ \eta) \leqq d(\xi_i,\ \xi_j) \vee d(\eta,\ \xi_j);\ \text{for all } i,\ j.$$

$$(2)\quad \overline{\lim_i}\, d(\xi_i,\ \eta) \leqq d(\xi_j,\ \eta) \vee \lim_i d(\xi_i,\ \xi_j);\ \text{for all } j.$$

$$(3)\quad \overline{\lim_i}\, d(\xi_i,\ \eta) \leqq \underline{\lim_j}\, (d(\xi_j,\ \eta) \vee \lim_i d(\xi_i,\ \xi_j))$$
$$\leqq \underline{\lim_j}\, d(\xi_j,\ \eta) \vee \overline{\lim_j} \lim_i d(\xi_i,\ \xi_j).$$

Thus,

$$(4)\quad \overline{\lim_i}\, d(\xi_i,\ \eta) \leqq \underline{\lim_i}\, d(\xi_i,\ \eta).$$

Lemma 7. *If* $\{\xi_i\}$ and $\{\eta_i\}$ *are Cauchy sequences in* Σ, *then* $\lim_i d(\xi_i,\ \eta_i)$ *exists.*

Proof. We shall apply Lemma 6 throughout this proof without further comment.

$$(1)\quad d(\xi_i,\ \eta_i) \leqq d(\xi_i,\ \xi_j) \vee d(\xi_j,\ \eta_i);\ \text{for all } i,\ j.$$

$$(2)\quad \overline{\lim_i}\, d(\xi_i,\ \eta_i) \leqq \lim_i d(\xi_i,\ \xi_j) \vee \lim_i d(\xi_j,\ \eta_i);\ \text{for all } j.$$

(3) $\overline{\lim_i}\, d(\xi_i, \eta_i) \leqq \underline{\lim_j} \lim_i d(\xi_j, \eta_i)$.

(4) $d(\xi_j, \eta_i) \leqq d(\xi_i, \eta_i) \vee d(\xi_i, \xi_j)$; for all i, j.

(5) $\lim_i d(\xi_j, \eta_i) \leqq \underline{\lim_i}\, d(\xi_i, \eta_i) \vee \lim_i d(\xi_i, \xi_j)$; for all j.

(6) $\underline{\lim_j} \lim_i d(\xi_j, \eta_i) \leqq \underline{\lim_i}\, d(\xi_i, \eta_i)$.

From (3) and (6),

(7) $\overline{\lim_i}\, d(\xi_i, \eta_i) \leqq \underline{\lim_i}\, d(\xi_i, \eta_i)$.

Let Σ_1 denote the class of all Cauchy sequences in Σ and define for $\{\xi_i\}$, $\{\eta_i\}$ in Σ_1, $\{\xi_i\} \sim \{\eta_i\}$ if and only if $d(\{\xi_i\}, \{\eta_i\}) \equiv \lim_i d(\xi_i, \eta_i) = 0$.

Lemma 8. *The relation $\sim$ is an equivalence relation in Σ_1.*

Proof. The relation $\sim$ is obviously reflexive and symmetric. To show transitivity, let $\{\xi_i\} \sim \{\eta_i\}$ and $\{\eta_i\} \sim \{\zeta_i\}$. Then

(1) $d(\xi_i, \zeta_i) \leqq d(\xi_i, \eta_i) \vee d(\eta_i, \zeta_i)$; for all i

so that

(2) $\overline{\lim_i}\, d(\xi_i, \zeta_i) \leqq \lim_i d(\xi_i, \eta_i) \vee \lim_i d(\eta_i, \zeta_i) = 0$.

Let Σ_2 denote the set of equivalence classes under $\sim$ in Σ_1. That is, $\Sigma_2 = \Sigma_1/\sim$. If $\{\xi_i\} \in \Sigma_1$, we denote by $[\{\xi_i\}]$ the member of Σ_2 containing $\{\xi_i\}$. We define in Σ_2, $d([\{\xi_i\}], [\{\eta_i\}]) = d(\{\xi_i\}, \{\eta_i\})$. It is clear that Σ_2 will be a B-metric space under this distance function provided that the distance function is independent of the representatives used to compute it.

Lemma 9. *If, in Σ_1, $\{\xi_i\} \sim \{\alpha_i\}$* and *$\{\eta_i\} \sim \{\beta_i\}$, then $d(\{\xi_i\}, \{\eta_i\}) = d(\{\alpha_i\}, \{\beta_i\})$.*

Proof.

(1) $d(\xi_i, \eta_i) \leqq d(\xi_i, \alpha_i) \vee d(\alpha_i, \beta_i) \vee d(\beta_i, \eta_i)$; for all i. Thus,

(2) $\lim_i d(\xi_i, \eta_i) \leqq \lim_i d(\alpha_i, \beta_i)$.

(2) with the inequality reversed is obtained in a similar fashion.

We are now ready to show that if B is a (††)-algebra, then Σ_2 is a completion of Σ of the Cantor-Hausdorff type (20).

Theorem 6. *If B is a (††)-algebra, Σ_2 is a complete B-metric space and Σ is congruent* (12) *to a dense subset of Σ_2.*

Proof. By our preceding Lemmas and remarks, Σ_2 is a B-metric space. It is clear that Σ is congruently imbedded (12) in Σ_2 by the mapping $\xi \to [\{\xi\}]$, where $\{\xi\}$ denotes a sequence all of whose members are ξ. The density in Σ_2 of the image of Σ under this mapping follows in the usual fashion (20) by selecting sequences diagonally since Σ_2 has Property (††). Thus, we shall only outline the proof that Σ_2 is complete making frequent use of what has been previously established. Let $\{[\{\xi^i_j\}_j]\}_i$ be a Cauchy sequence in Σ_2. Then,

(1) $\lim_i \lim_j \lim_k d(\xi^i_k, \xi^j_k) = 0.$

By the Corollary to Theorem 3 there is $b(j)$ so that $k(j) \geqq b(j)$; for all j implies

(2) $\lim_i \lim_j d(\xi^i_{k(j)}, \xi^j_{k(j)}) = 0.$

Also,

(3) $\lim_j \lim_k d(\xi^i_j, \xi^i_k) = 0$; for all i. Thus there is $a(i)$ so that $j(i) \geqq a(i)$; for all i implies

(4) $\lim_i \lim_k d(\xi^i_{j(i)}, \xi^i_k) = 0.$

Let $j(i) \geqq \max\,(a(i), b(i))$; for all i.

By (2), (4), and subsequence arguments, $\lim_v d(\xi^i_{j(i)}, \xi^i_{j(v)})$ and $\lim_n d(\xi^i_{j(n)}, \xi^n_{j(n)})$ exists and

(5) $\lim_i \lim_v d(\xi^i_{j(i)}, \xi^i_{j(v)}) = \lim_i \lim_v d(\xi^i_{j(v)}, \xi^v_{j(v)}) = 0.$

Thus,

(6) $d(\xi^i_{j(i)}, \xi^v_{j(v)}) \leqq d(\xi^i_{j(v)}, \xi^v_{j(v)}) \vee d(\xi^i_{j(i)}, \xi^i_{j(v)})$; for all i, v implying

(7) $\lim_v d(\xi^i_{j(i)}, \xi^v_{j(v)}) \leqq \lim_v d(\xi^i_{j(v)}, \xi^v_{j(v)}) \vee \lim_v d(\xi^i_{j(i)}, \xi^i_{j(v)})$; for all i,

together with (5) and a rather complicated sequence of applications of the triangle inequality and limiting processes yields the existence of $\lim_v d(\xi^i_{j(i)}, \xi^v_{j(v)})$ and

(8) $\lim_i \lim_v d(\xi^i_{j(i)}, \xi^v_{j(v)}) = 0.$

Thus, $[\{\xi^i_{j(i)}\}] \in \Sigma_2$.

Finally, by the triangle inequality,

(9) $\lim_k d(\xi^i_k, \xi^k_{j(k)}) \leqq \lim_k d(\xi^i_k, \xi^i_{j(i(}) \vee \lim_k d(\xi^i_{j(i)}, \xi^k_{j(k)})$; for all i.

From (5) and (9),

(10) $\lim_i \lim_k d(\xi^i_k, \xi^k_{j(k)}) = 0$

and

(11) $d\text{-}\lim_{i} [\{\xi^i_j\}] = [\{\xi^k_{j(k)}\}]$, completing the proof.

6. Application to zero-dimensional spaces. We assume familiarity of the reader with the use of uniform structures induced by filtres (26). We adopt the

Definition. A zero-dimensional space is a T_1 topological space (25) which has a basis for open sets consisting of sets which are both open and closed (ambiguous sets). By space we shall understand T_1 space.

We state without proof the easily established.

LEMMA 10. *A space is zero-dimensional if and only if its topology is compatible with a uniform structure defined by a filtre having a symmetric, idempotent (under relational product) base closed under finite intersection.*

THEOREM 7. *Let S be zero-dimensional space. Then S is metrizable over the Boolean algebra $2^{2^{SS}}$.*

Proof. Let $\boldsymbol{B}$ be the base of the filtre $\boldsymbol{F}$ assured by Lemma 10. Define $\boldsymbol{B}(x, y) = \{\boldsymbol{u} \in \boldsymbol{B} \mid (x, y) \in \boldsymbol{u}\}$ and define $d(x, y) = \boldsymbol{B} - \boldsymbol{B}(x, y)$. Since the diagonal of SS is in all elements of $\boldsymbol{F}$, $d(x, y)$ satisfies the vanishing condition one way. Since S is T_1, $d(x, y) = 0$ implies $x = y$. The symmetry condition for $d(x, y)$ is immediate from the symmetry of B. Now let $\boldsymbol{u} \in d(x, y)$. If $\boldsymbol{u} \notin d(x, z)$ and $\boldsymbol{u} \notin d(y, z)$, then $(x, y) \in \boldsymbol{u}^2 = \boldsymbol{u}$, which is a contradiction. Thus $d(x, y)$ satisfies the triangle inequality. Thus, S forms a $2^{2^{SS}}$-metric space under $d(x, y)$.

Let $\lim_n x_n = x$ in the topology of S. Suppose there is an element $\boldsymbol{u}$ in $\boldsymbol{B}$ so that $\boldsymbol{u} \in \overline{\lim_j}\, d(x, x_j)$. Then for a given integer k there is $j \geqq k$ so that $(x, x_j) \notin \boldsymbol{u}$ which is impossible since $\boldsymbol{u}$ contains almost all (x, x_j). Thus, the topology of S is stronger than (25) the induced distance topology.

Suppose next that $\overline{\lim_j}\, d(x, x_j) = 0$ and there is a neighborhood, $\boldsymbol{u}(x)$ which excludes infinitely many x_j. Then $\boldsymbol{u} \in \boldsymbol{d}(x, x_j)$ for infinitely many x_j. Since $\boldsymbol{B}$ is a base for $\boldsymbol{F}$ and is closed under finite intersection, this is a contradiction and the induced distance topology of S is stronger than the original topology. Thus the topologies coincide and Theorem 7 is proved.

We borrow from Vaidyanathaswami (25) the

LEMMA 11. *Any set algebra is zero-dimensional in its K-topology.*

THEOREM 8. *If B is a set algebra, any B-metric space is zero-dimensional.*

Proof. Let Σ be a B-metric space and select $\alpha \neq \beta$ in Σ. Now, $f(\xi) = d(\alpha, \xi)$ is a continuous mapping of Σ into B with $f(\alpha) = 0$ and $f(\beta) \neq 0$. By Lemma 11, $f(\alpha)$ and $f(\beta)$ may be separated in B by ambiguous sets. The counterimages of these sets in Σ will be ambiguous sets separating α and β. Thus, Σ is zero-dimensional.

Combining Theorems 7 and 8 we have

THEOREM 9. *Among T_1 spaces, the zero-dimensional spaces are precisely those spaces metrizable over set algebras.*

7. Weak topological products.

Let Σ_n be a sequence of B-metric spaces. Denote the combinatory product by $\prod_n \Sigma_n$ and the projection of $\prod_n \Sigma_n$ onto Σ_n by $\pi_n : \prod_n \Sigma_n \to \Sigma_n$. Define a distance function and, hence, an induced distance topology, in $\prod_n \Sigma_n$ by $d(\xi, \eta) = \bigvee_{i=1}^{n} d(\xi_i, \eta_i)$ where $\xi_i^n = -\pi_i(\xi)$.

LEMMA 12. *For each n, $\pi_n : \prod_n \Sigma_n \to \Sigma_n$ is continuous.*

Proof. If $\lim_m d(\xi^m, \xi) = 0$, then $\lim_m d(\xi_n^m, \xi_n) = 0$.

LEMMA 13. *For each n, $\pi_n : \prod_n \Sigma_n \to \Sigma_n$ is an open mapping.*

Proof. The proof is by diagonal selection and is left to the reader.

Combining Lemmas 6 and 7 we find

THEOREM 10. *$\prod_n \Sigma_n$ forms a B-metric space under the distance function defined above. The topology induced by this distance function is a weak product topology* (25).

A similar Theorem may be obtained for a distance function defined on a cardinal product of algebras for a combinatory product of spaces metrized over these respective algebras.

8. Sequential compactness in set algebras.

We have already remarked in Section 2 that set algebras are not, in general, sequentially compact. However, one may easily show that if one defines a subset of a set algebra to be *bounded* provided the elements of its distance set (12) are countable then set algebras are *finitely compact* in the sense that bounded sets are sequentially compact, when closed.

Another question of interest is the obtaining of an algebraic hold on the closed sets in a set algebra. This is close to Problem

76 of Birkhoff (3). One might suspect that the closed intervals form a sub-basis for open sets, but examples show that this is not the case. We offer, however, the

CONJECTURE. *The σ-sublattices form a sub-basis for closed sets in a set algebra.*

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