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Algebraic Logic, I

Monadic Boolean algebras

by

Paul R. Halmos

Preface. The purpose of the sequence of papers here begun is to make algebra out of logic. For the propositional calculus this program is in effect realized by the existing theory of Boolean algebras. An indication of how the program could be realized for the first-order functional calculus was published recently (3). In this paper the details will be carried out for the so-called first-order monadic functional calculus.

While the projected sequel will, in part at least, supersede some of the present discussion, the major part of this paper is an indispensable preliminary to that sequel. In order to be able to understand the algebraic versions of the intricate substitution processes that give the polyadic calculi their characteristic flavor, it is necessary first to understand the algebraic version of the logical operation of quantification. The latter is the subject matter of this paper, which, accordingly, could have been subtitled: "An algebraic study of quantification."

The theory of what will presently be called monadic (Boolean) algebras is discussed here not as a possible tool for solving problems about the foundations of mathematics, but as an independently interesting part of algebra. A knowledge of symbolic logic is unnecessary for an understanding of this theory; the language and the techniques used are those of modern algebra and topology.

It will be obvious to any reader who happens to be familiar with the recent literature of Boolean algebras that the results that follow lean heavily on the works of M. H. Stone and A. Tarski. Without the inspiration of Stone's representation theory and without Tarski's subsequent investigations of various Boolean algebras with operators, the subject of algebraic logic could not have come into existence. The present form of the paper was strongly influenced by some valuable suggestions of Mr. A. H. Kruse and Mr. B. A. Galler.

PART 1

Algebra

1. Functional monadic algebras. There is no novelty nowadays in the observation that propositions, whatever they may be, tend to band together and form a Boolean algebra. On the basis of this observation it is natural to interpret the expression "propositional function" to mean a function whose values are in a Boolean algebra. Accordingly, we begin our algebraic study of quantification with the consideration of a non-empty set X (the *domain*) and a Boolean algebra $\mathbf{B}$ (the *value-algebra*). The set $\mathbf{B}^X$ of all functions from X to $\mathbf{B}$ is itself a Boolean algebra with respect to the pointwise operations. Explicitly, if p and q are in $\mathbf{B}^X$, then the supremum $p \vee q$ and the complement p' are defined by

$$(p \vee q)(x) = p(x) \vee q(x) \text{ and } p'(x) = (p(x))'$$

for each x in X; the zero and the unit of $\mathbf{B}^X$ are the functions that are constantly equal to 0 and to 1, respectively.

The chief interest of $\mathbf{B}^X$ comes from the fact that it is more than just a Boolean algebra, What makes it more is the possibility of associating with each element p of $\mathbf{B}^X$ a subset $\mathbf{R}(p)$ of $\mathbf{B}$, where

$$\mathbf{R}(p) = \{p(x) : x \in X\}$$

is the range of the function p. With the set $\mathbf{R}(p)$, in turn, there are two obvious ways of associating an element of $\mathbf{B}$: we may try to form the supremum and the infimum of $\mathbf{R}(p)$. The trouble is that unless $\mathbf{B}$ is complete (in the usual lattice-theoretic sense), these extrema need not exist, and, from the point of view of the intended applications, the assumption that $\mathbf{B}$ is complete is much too restrictive. The remedy is to consider, instead of $\mathbf{B}^X$, a Boolean subalgebra $\mathbf{A}$ of $\mathbf{B}^X$ such that (i) for every p in $\mathbf{A}$ the supremum $\bigvee \mathbf{R}(p)$ and the infimum $\bigwedge \mathbf{R}(p)$ exist in $\mathbf{B}$, and (ii) the (constant) functions $\exists p$ and $\forall p$, defined by

$$\exists p(x) = \bigvee \mathbf{R}(p) \text{ and } \forall p(x) = \bigwedge \mathbf{R}(p)$$

belong to $\mathbf{A}$. Every such subalgebra $\mathbf{A}$ will be called a *functional monadic algebra*, or, to give it its full title, a $\mathbf{B}$-valued functional monadic algebra with domain X. The reason for the word "monadic" is that the concept of a monadic algebra (to be defined in appropriate generality below) is a special case of the concept of a polyadic algebra; the special case is characterized by the superimposition on the Boolean structure of exactly one additional operator.

A simple example of a functional monadic algebra is obtained by assuming that **B** is finite (or, more generally, complete), and letting $\mathbf{B}^X$ itself play the role of **A**. An equally simple example, in which **A** is again equal to $\mathbf{B}^X$, is obtained by assuming that X is finite. An example with **B** and X unrestricted is furnished by the set of all those functions from X to **B** that take on only a finite number of values.

If **B** happens to be the (complete) Boolean algebra of all subsets of a set Y, and if y is a point in Y, then a value $p(x)$ of a function p in **A** ($= \mathbf{B}^X$) corresponds in a natural way to the proposition "y belongs to $p(x)$." Since supremum in **B** is set-theoretic union, it follows that each value of $\exists p$ corresponds to "there is an x such that y belongs to $p(x)$," and, dually, each value of $\forall p$ corresponds to "for all x, y belongs to $p(x)$." For this reason, the operator $\exists$ on a functional monadic algebra is called a *functional existential quantifier*, and the operator $\forall$ is called a *functional universal quantifier.*

It is frequently helpful to visualize the example in the preceding paragraph geometrically. If X and Y are both equal to the real line, then **A** is naturally isomorphic to the algebra of all subsets of the Cartesian plane, via the isomorphism that assigns to each p in **A** the set $\{(x, y) : y \in p(x)\}$. The set that corresponds to $\exists p$ under this isomorphism is the union of all horizontal lines that pass through some point of the set corresponding to p; the set that corresponds to $\forall p$ is the union of all horizontal lines that are entirely included in the set corresponding to p.

In the definition of a functional monadic algebra it is not necessary to insist that for every p in **A** *both* $\exists p$ and $\forall p$ exist and belong to **A**: either one alone is sufficient. The reason for this is the validity of the identities

$$\forall p = (\exists p')' \text{ and } \exists p = (\forall p')'.$$

In more detail: if **A** is a Boolean subalgebra of $\mathbf{B}^X$ such that, for every p in **A**, the supremum $\bigvee \mathbf{R}(p)$ exists and the function $\exists p$, whose value at every point is that supremum, belongs to **A**, then, for every p in **A**, the infimum $\bigwedge \mathbf{R}(p)$ also exists and the function $\forall p$, whose value at every point is that infimum, also belongs to **A**. The converse of this assertion is, in an obvious sense, its dual, and is also true. The perfect duality between $\exists$ and $\forall$ justifies the asymmetric treatment in what follows; we shall study $\exists$ alone and content ourselves with an occasional comment on the behavior of $\forall$.

The functional existential quantifier $\exists$ on a functional monadic algebra **A** is *normalized, increasing,* and *quasi-multiplicative.* In other words

$$(Q_1) \qquad \exists 0 = 0,$$

$$(Q_2) \qquad p \leqq \exists p,$$

$$(Q_3) \qquad \exists(p \wedge \exists q) = \exists p \wedge \exists q,$$

whenever p and q are in **A**. The assertions (Q_1) and (Q_2) are immediate consequences of the definition or $\exists$. The proof of (Q_3) is based on the following distributive law (true and easy to prove for every Boolean algebra **B**): if $\{p_i\}$ is a family of elements of **B** such that $\bigvee_i p_i$ exists, then, for every q in **B**, $\bigvee_i (p_i \wedge q)$ exists and is equal to $(\bigvee_i p_i) \wedge q$. The corresponding assertions for a functional universal quantifier are obtained from (Q_1)—(Q_3) upon replacing $\exists$, 0, $\leqq$, and $\wedge$ by $\forall$, 1, $\geqq$, and $\vee$, respectively.

2. Quantifiers. A general concept of quantification that applies to any Boolean algebra is obtained by abstraction from the functional case. In the process of abstraction the domain X and the value-algebra **B** disappear. What remains is the following definition: a *quantifier* (properly speaking, an *existential quantifier*) is a mapping $\exists$ of a Boolean algebra into itself, satisfying the conditions (Q_1)—(Q_3). The concept of an existential quantifier occurs implicitly in a brief announcement of some related work of Tarski and Thompson (9). The concept of a *universal quantifier* is defined by an obvious dualization, or, if preferred, via the equation $\forall p = (\exists p')'$. Since we have agreed to refer to universal quantifiers only tangentially, the adjective "existential" will usually be omitted.

The following examples show that the conditions (Q_1)—(Q_3) are independent of each other. For (Q_1): **A** is arbitrary and $\exists p = 1$ for all p in **A**. For (Q_2): **A** is arbitrary and $\exists p = 0$ for all p in **A**. For (Q_3): **A** is the class of all subsets of a topological space that includes a non-closed open set and $\exists p$ is the closure of p for all p in **A**.

It is worth while to look at some quantifiers that are at least *prima facie* different from the functional examples of the preceding section. (i) The identity mapping of a Boolean algebra into itself is a quantifier; this quantifier will be called *discrete.* (ii) The mapping defined by $\exists 0 = 0$ and $\exists p = 1$ for all $p \neq 0$ is a quantifier; this quantifier will be called *simple.* (The reason for the terms "discrete", borrowed from topology, and "simple", borrowed from algebra, will become apparent later; cf. sections 3 and 5, respectively.) (iii) Suppose that **A** is the class of all subsets of some set and that G is a group of one-to-one transformations of that set onto itself. If $\exists p = \cup_{g \in G} gp$ for all p in **A**, then $\exists$ is a quantifier; $\exists p$ is the least set including p that is invariant under G. (Examples of this type are of some importance in ergodic theory.) (iv) Suppose that **A** is the algebra of all subsets of, say, the real line, modulo sets of Lebesgue measure zero. (Generalizations to other measure spaces are obvious.) If, for all p in **A**, $\exists p$ is the measurable cover of p (modulo sets of Lebesgue measure zero, of course), then $\exists$ is a quantifier.

To obtain insight into the algebraic properties of a quantifier, it is now advisable to derive certain elementary consequences of the definition. Several of these consequences are almost trivial and are stated formally for convenience of reference only. The important facts are that a quantifier is *idempotent* (Theorem 1) and *additive* (Theorem 2). Throughout the following statements it is assumed that **A** is a Boolean algebra and that $\exists$ is a quantifier on **A**.

Lemma 1. $\exists 1 = 1$.

Proof. Put $p = 1$ in (Q_2).

Theorem 1. $\exists\exists = \exists$.

Proof. Put $p = 1$ in (Q_3) and apply Lemma 1.

Lemma 2. *A necessary and sufficient condition that an element p of* **A** *belong to the range of $\exists$, i.e., that $p \in \exists(\mathbf{A})$, is that $\exists p = p$.*

Proof. If $p \in \exists(\mathbf{A})$, say $p = \exists q$, then $\exists p = \exists\exists q = \exists q$ (by Theorem 1), so that, indeed, $\exists p = p$. This proves necessity; sufficiency is trivial.

Lemma 3. *If $p \leqq \exists q$, then $\exists p \leqq \exists q$.*

Proof. By assumption $p \wedge \exists q = p$; it follows from (Q_3) that $\exists p = \exists(p \wedge \exists q) = \exists p \wedge \exists q$, so that, indeed, $\exists p \leqq \exists q$.

Lemma 4. *A quantifier is monotone; i.e., if $p \leqq q$, then $\exists p \leqq \exists q$.*

Proof. Note that $q \leqq \exists q$ by (Q_2) and apply Lemma 3.

Lemma 5. $\exists(\exists p)' = (\exists p)'$.

Proof. Since $(\exists p)' \wedge \exists p = 0$, it follows that

$$0 = \exists((\exists p)' \wedge \exists p) \qquad \text{(by } (Q_1)\text{)},$$
$$= \exists(\exists p)' \wedge \exists p \qquad \text{(by } (Q_3)\text{)},$$

and hence that $\exists(\exists p)' \leqq (\exists p)'$. The reverse inequality is immediate from (Q_2).

LEMMA 6. *The range* $\exists(\mathbf{A})$ *of the quantifier* $\exists$ *is a Boolean subalgebra of* $\mathbf{A}$.

PROOF. If p and q are in $\exists(\mathbf{A})$, then (by Lemma 2) $p = \exists p$ and $q = \exists q$, and consequently (by (Q_3)) $p \wedge q = \exists p \wedge \exists q = \exists(p \wedge \exists q)$. This proves that $\exists(\mathbf{A})$ is closed under the formation of infima. If $p \in \exists(\mathbf{A})$, then (again by Lemma 2) $p = \exists p$, and therefore (by Lemma 5) $p' = (\exists p)' = \exists(\exists p)'$. This proves that $\exists(\mathbf{A})$ is closed under the formation of complements.

Theorem 2. $\exists(p \vee q) = \exists p \vee \exists q$.

PROOF. Since $p \leqq p \vee q$ and $q \leqq p \vee q$, it follows from Lemma 4 that $\exists p \leqq \exists(p \vee q)$ and $\exists q \leqq \exists(p \vee q)$, and hence that $\exists p \vee \exists q \leqq \exists(p \vee q)$. To prove the reverse inequality, observe first that both $\exists p$ and $\exists q$ belong to $\exists(\mathbf{A})$ and that therefore (by Lemma 6) $\exists p \vee \exists q$ belongs to $\exists(\mathbf{A})$. It follows from Lemma 2 that $\exists(\exists p \vee \exists q) = \exists p \vee \exists q$. Since $p \leqq \exists p \vee \exists q$ (by (Q_2)), and, similarly, $q \leqq \exists p \vee \exists q$, so that $p \vee q \leqq \exists p \vee \exists q$, Lemma 4 implies that $\exists(p \vee q) \leqq \exists(\exists p \vee \exists q)$; this, together with what was just proved about $\exists(\exists p \vee \exists q)$, completes the proof of the theorem.

It is sometimes necessary to know the relation between quantification and relative complementation (where the relative complement of q in p is defined by $p - q = p \wedge q'$) and the relation between quantification and Boolean addition (where the Boolean sum, or symmetric difference, of p and q is defined by $p + q = (p - q) \vee (q - p)$). The result and its proof are simple.

LEMMA 7. $\exists p - \exists q \leqq \exists(p - q)$ and $\exists p + \exists q \leqq \exists(p + q)$.

PROOF. Since $p \vee q = (p - q) \vee q$, it follows (by Theorem 2) that $\exists p \vee \exists q = \exists(p - q) \vee \exists q$. Forming the infimum of both sides of this equation with $(\exists q)'$, we obtain

$$\exists p - \exists q = \exists(p - q) - \exists q \leqq \exists(p - q).$$

The result for Boolean addition follows from two applications of the result for relative complementation.

3. Closure operators. A *closure operator* is a normalized, increasing, idempotent, and additive mapping of a Boolean algebra into itself; in other words, it is an operator $\exists$ on a Boolean algebra $\mathbf{A}$, such that the conditions stated in (Q_1), (Q_2), Theorem 1, and Theorem 2 are satisfied. The first systematic investigation of the algebraic properties of closure operators was carried out by McKinsey and Tarski (5). A typical example of a closure operator is obtained by taking $\mathbf{A}$ to be the class of all subsets of a topological space and defining $\exists p$ to be the closure of p for every p in $\mathbf{A}$. Included among the results of the preceding section is the fact that every quantifier is a closure operator. In the converse direction, the only obvious thing that can be said is that the closure operator on a discrete topological space is a quantifier. It is, in fact, a discrete quantifier; this is the reason for the use of the word "discrete" in connection with quantifiers.

Despite the apparently promising connection between quantification and topology, it turns out that the topological point of view is almost completely valueless in the study of quantifiers. Not only is it false that every closure operator is a quantifier, but, in fact, the discrete (and therefore topologically uninteresting) closure operators are essentially the only ones that are quantifiers. The precise statement of the facts is as follows. The closure operator on a topological space is a quantifier if and only if, in that space, every open set is closed, or, equivalently, every closed set is open. (The proof is an easy application of Lemma 5.) In such a space the relation R, defined by writing $x\,R\,y$ whenever x belongs to the closure of the one-point set $\{y\}$, is an equivalence whose associated quotient space is discrete. Conversely, every space with this latter property has a quantifier for its closure operator. It follows that such spaces are as nearly discrete as a space not satisfying any separation axioms can ever be; in particular the T_1-spaces among them are discrete. Since these results border on pathology, and are of no importance for the theory of quantification, the details are omitted.

Nevertheless, closure operators play a useful role in quantifier algebra. The point is that it is frequently necessary to define a Boolean operator by certain algebraic constructions, and then to prove that the operator so constructed is a quantifier. It is usually easy to prove that the construction leads to a closure operator; the proof of quasi-multiplicativity, however, is likely to be more intricate. For this reason, it is desirable to have at hand a usable answer to the question: when is a closure operator a quantifier?

Theorem 3. *If $\exists$ is a closure operator on a Boolean algebra* $\mathbf{A}$, *then the following conditions are mutually equivalent.*

(i) *$\exists$ is a quantifier.*

(ii) *The range of $\exists$ is a Boolean subalgebra of* $\mathbf{A}$.

(iii) $\exists(\exists p)' = (\exists p)'$ *for all p in* $\mathbf{A}$.

Proof. The implication from (i) to (ii) is the statement of Lemma 6. To derive (iii) from (ii), note first (cf. Lemma 2) that $p \in \exists(\mathbf{A})$ if and only if $\exists p = p$. It follows that (iii) is equivalent to the assertion that $(\exists p)' \in \exists(\mathbf{A})$ for all p, and this in turn is an immediate consequence (via (ii)) of the fact that $\exists p \in \exists(\mathbf{A})$

for all p. It remains only to prove that if (iii) is satisfied, then $\exists$ is quasi-multiplicative.

Since $p \wedge \exists q \leqq p \leqq \exists p$, it follows that $\exists(p \wedge \exists q) \leqq \exists\exists p = \exists p$. (The reasoning here depends on the fact that an additive operator, and hence in particular a closure operator, is monotone; cf. Lemma 4.) Similarly, since $p \wedge \exists q \leqq \exists q$, it follows that $\exists(p \wedge \exists q) \leqq \exists q$ and hence that $\exists(p \wedge \exists q) \leqq \exists p \wedge \exists q$. To prove the reverse inequality, note that

$$p = (p \wedge \exists q) \vee (p \wedge (\exists q)') \leqq (p \wedge \exists q) \vee (\exists q)'$$

and that, therefore, $\exists p \leqq \exists(p \wedge \exists q) \wedge (\exists q)'$. (This is where (iii) is used.) Forming the infimum of both sides of this relation with $\exists q$, we obtain

$$\exists p \wedge \exists q \leqq \exists(p \wedge \exists q) \wedge \exists q = \exists(p \wedge \exists q),$$

and the proof is complete.

4. Relative completeness. There are certain similarities between Boolean homomorphisms and quantifiers. A homomorphism is a mapping, satisfying certain algebraic conditions, from one Boolean algebra into another; a quantifier is a mapping, satisfying certain algebraic conditions, from a Boolean algebra into itself. A homomorphism uniquely determines a subset of its domain (namely, the kernel). The homomorphism theorem can be viewed as a characterization of kernels; it asserts that a subset of a Boolean algebra is the kernel of a homomorphism if and only if it is a proper ideal. Similarly, a quantifier uniquely determines a subset of its domain (namely, the range). The purpose of this section is to point out that the range uniquely determines the quantifier and to characterize the possible ranges of quantifiers.

A Boolean subalgebra **B** of a Boolean algebra **A** will be called *relatively complete* if, for every p in **A**, the set $\mathbf{B}(p)$, defined by

$$\mathbf{B}(p) = \{q \in \mathbf{B}\colon\ p \leqq q\},$$

has a least element (and therefore, *a fortiori*, an infimum). Relatively complete subalgebras are the objects whose relation to quantifiers is the same as the relation of proper ideals to homomorphisms.

Theorem 4. *If $\exists$ is a quantifier on a Boolean algebra* **A** *and if* **B** *is the range of $\exists$, then* **B** *is a relatively complete subalgebra of* **A**, *and, moreover, if $\mathbf{B}(p) = \{q \in \mathbf{B}\colon\ p \leqq q\}$, then $\exists p = \bigwedge \mathbf{B}(p)$ for every p in* **A**.

Proof. The fact that **B** is a Boolean subalgebra of **A** is already known from Lemma 6. If $q \in \mathbf{B}(p)$, then, of course, $p \leqq q$, and therefore $\exists p \leqq \exists q = q$. Since $\exists p \in \mathbf{B}(p)$, it follows that $\mathbf{B}(p)$ does indeed have a least element and that, moreover, that least element is equal to $\exists p$.

Theorem 5. *If* **B** *is a relatively complete subalgebra of a Boolean algebra* **A**, *then there exists a unique quantifier on* **A** *with range* **B**.

Proof. Write, for each p in **A**, $\exists p = \bigwedge \mathbf{B}(p)$; it is to be proved that $\exists$ is a quantifier on **A** and that $\exists(\mathbf{A}) = \mathbf{B}$.

(i) If $p = 0$, then $\mathbf{B}(p) = \mathbf{B}$ and therefore $\exists 0 = 0$.

(ii) Since $p \leqq q$ whenever $q \in \mathbf{B}(p)$, it follows that $p \leqq \bigwedge \mathbf{B}(p) = \exists p$.

(iii) If $p \in \mathbf{B}$, then $p \in \mathbf{B}(p)$ and therefore $\exists p = \bigwedge \mathbf{B}(p) \leqq p$. It follows from (ii) that (iiia) $\exists p = p$ whenever $p \in \mathbf{B}$. Since $\exists p \in \mathbf{B}$ for all p in **A**, this result in turn implies that (iiib) $\exists \exists p = \exists p$ whenever $p \in \mathbf{A}$.

(iv) If p_1 and p_2 are in **A**, then $\exists p_1$ and $\exists p_2$ are in **B** and therefore, since **B** is a Boolean algebra, $\exists p_1 \vee \exists p_2 \in \mathbf{B}$. Since, by (ii), $p_1 \vee p_2 \leqq \exists p_1 \vee \exists p_2$, so that $\exists p_1 \vee \exists p_2 \in \mathbf{B}(p_1 \vee p_2)$, it follows that $\exists(p_1 \vee p_2) = \bigwedge \mathbf{B}(p_1 \vee p_2) \leqq \exists p_1 \vee \exists p_2$. On the other hand, since $\exists(p_1 \vee p_2) \in \mathbf{B}(p_1 \vee p_2)$, it follows that $p_1 \vee p_2 \leqq \exists(p_1 \vee p_2)$ and hence that $p_1 \leqq \exists(p_1 \vee p_2)$ and $p_2 \leqq \exists(p_1 \vee p_2)$. The definition of $\exists$ implies that $\exists p_1 \leqq \exists(p_1 \vee p_2)$ and $\exists p_2 \leqq \exists(p_1 \vee p_2)$, and hence that $\exists p_1 \vee \exists p_2 \leqq \exists(p_1 \vee p_2)$.

(v) The range of $\exists$ is included in **B** by the definition of $\exists$. The result (iiia) implies the reverse inclusion, so that $\exists(\mathbf{A}) = \mathbf{B}$.

In (i)—(iv) we saw that $\exists$ is a closure operator. Since, by (v), the range of $\exists$ is a Boolean algebra, it follows from Theorem 3 that $\exists$ is a quantifier. The existence proof is complete; uniqueness is an immediate consequence of Theorem 4.

5. Monadic algebra. A *monadic algebra* is a Boolean algebra **A** together with a quantifier $\exists$ on **A**. The elementary algebraic theory of monadic algebras is similar to that of every other algebraic system, and, consequently, it is rather a routine matter. Thus, for example, a subset **B** of a monadic algebra **A** is a *monadic subalgebra* of **A** if it is a Boolean subalgebra of **A** and if it is a monadic algebra with respect to the quantifier on **A**. In other words, a Boolean subalgebra **B** of **A** is a monadic subalgebra of **A** if and only if $\exists p \in \mathbf{B}$ whenever $p \in \mathbf{B}$. The central concept is, as usual, that of a homomorphism; a *monadic homomorphism* is a mapping f from one monadic algebra into another, such that f is a Boolean homomorphism and $f \exists p = \exists f p$ for all p. Associated with every homomorphism f is its kernel $\{p: fp = 0\}$. The kernel of a monadic homomorphism is a *monadic ideal*; i.e., it is a Boolean ideal $\mathbf{I}$ in $\mathbf{A}$ such that $\exists p \in \mathbf{I}$ whenever $p \in \mathbf{I}$. The adjective "monadic" will be used with "subalgebra," "homomorphism," etc., whenever it is advisable to emphasize the distinction from other kinds of subalgebras, homomorphisms, etc. — e.g., from the plain Boolean kind. Usually, however, the adjective will be omitted and the context will unambiguously indicate what is meant.

The homomorphism theorem (every proper ideal is a kernel) and the consequent definition of monadic quotient algebras work as usual. If $\mathbf{A}$ is a monadic algebra and $\mathbf{I}$ is a monadic ideal in $\mathbf{A}$, form the Boolean quotient algebra $\mathbf{B} = \mathbf{A}/\mathbf{I}$, and consider the natural Boolean homomorphism f from $\mathbf{A}$ onto $\mathbf{B}$. There is a unique, natural way of converting $\mathbf{B}$ into a monadic algebra so that f becomes a monadic homomorphism (with kernel $\mathbf{I}$, of course). Indeed, if p_1 and p_2 are in $\mathbf{A}$, and if $fp_1 = fp_2$, then $f(p_1 + p_2) = 0$, or, equivalently, $p_1 + p_2 \in \mathbf{I}$. Since $\mathbf{I}$ is a monadic ideal, it follows that $\exists (p_1 + p_2) \in \mathbf{I}$. By Lemma 7, $\exists p_1 + \exists p_2 \in \mathbf{I}$, or, equivalently, $f\exists p_1 = f\exists p_2$. This conclusion justifies the following procedure: given q in $\mathbf{B}$, find p in $\mathbf{A}$ so that $fp = q$, and define $\exists$ on q by $\exists q = f\exists p$. The preceding argument shows that the definition is unambiguous; a straightforward verification shows that $\exists$ is a quantifier on $\mathbf{B}$.

A monadic algebra is *simple* if $\{0\}$ is the only proper ideal in it. A monadic ideal is *maximal* if it is a proper ideal that is not a proper subset of any other proper ideal. The connection between maximal ideals and simple algebras is an elementary part of universal algebra: the kernel of a homomorphism is a maximal ideal if and only if its range is a simple algebra.

Lemma 8. *A monadic algebra is simple if and only if its quantifier is simple.*

Proof. If $\mathbf{A}$ is simple and if $p \in \mathbf{A}$, $p \neq 0$, write $\mathbf{I} = \{q: q \leqq \exists p\}$. Since, clearly, $\mathbf{I}$ is a non-trivial monadic ideal, it follows that $\mathbf{I} = \mathbf{A}$, and hence, in particular, that $1 \in \mathbf{I}$. This implies that $\exists p = 1$ whenever $p \neq 0$. Suppose, conversely, that $\exists p = 1$ whenever $p \neq 0$, and suppose that $\mathbf{I}$ is a monadic ideal in $\mathbf{A}$. If $p \in \mathbf{I}$, then $\exists p \in \mathbf{I}$; if, moreover, $p \neq 0$, this implies that $1 \in \mathbf{I}$ and hence that $\mathbf{I} = \mathbf{A}$. In other words, every non-trivial ideal in $\mathbf{A}$ is improper; this proves that $\mathbf{A}$ is simple.

The only simple *Boolean* algebra is the two-element algebra, to be designated throughout the sequel as $\mathbf{O}$. This Boolean algebra is a subalgebra, and, what is more, a relatively complete subalgebra, of every Boolean algebra. Lemma 8 asserts that a monadic algebra is simple if and only if the relatively complete subalgebra associated with its quantifier is equal to **O**, or, in other words, if and only if the range of its quantifier is a simple Boolean algebra.

The connection between simple Boolean algebras and simple monadic algebras is even closer than that indicated in the preceding paragraph; it turns out that the simplest examples of monadic algebras (in both the popular and the technical sense of "simple") are the **O**-valued functional algebras.

Theorem 6. *A monadic algebra is simple if and only if it is (isomorphic to) an* **O***-valued functional monadic algebra.*

PROOF. If **A** is an **O**-valued functional monadic algebra with domain X, and if p is a non-zero element of A, then $p(x_0) = 1$ for some point x_0 in X. It follows that $1 \in \mathbf{R}(p)$ and hence that $\bigvee \mathbf{R}(p) = 1$. The definition of functional quantification implies that $\exists p = 1$. Since this proves that $\exists p = 1$ whenever $p \neq 0$, i.e., that $\exists$ is simple, the desired result follows from Lemma 8.

The converse is just as easy to prove, but the proof makes use of a relatively deep fact, namely Stone's theorem on the representation of Boolean algebras (7).

If **A** is a simple monadic algebra, then **A** is, in particular, a Boolean algebra, to which Stone's theorem is applicable. It follows that there exist (i) a set X, (ii) a Boolean subalgebra **B** of $\mathbf{O}^X$, and (iii) a Boolean isomorphism f from **A** onto **B**. Since, by Lemma 8, the quantifier of **A** is simple, and since, by Lemma 8 and the first part of this proof, the quantifier of **B** is simple, it follows that f preserves quantification, i.e., that f is automatically a monadic isomorphism between the monadic algebras **A** and **B**.

6. Monadic logics. In the usual logical treatment of Boolean algebras and their generalizations, certain elements of the appropriate Boolean algebra are singled out and called "provable". From the algebraic point of view, the definition of provability in any particular case is irrelevant; what is important is the algebraic structure of the set of all provable elements. It is convenient, in the examination of that structure, to dualize, i.e., to consider not provability but refutability. There is an obvious relation between the two concepts; clearly p should be called refutable if and only if p' is provable.

Suppose accordingly that $\mathbf{A}$ is a monadic algebra whose elements, for heuristic purposes, are thought of as propositions, or, rather, as propositional functions. What properties does it seem reasonable to demand of a subset $\mathbf{I}$ of $\mathbf{A}$ in order that its elements deserve to be called refutable? Clearly if p and q are refutable, then $p \vee q$ should also be refutable, and if p is refutable, then $p \wedge q$ should be refutable no matter what q may be. In other words, $\mathbf{I}$ should be, at least, a Boolean ideal in $\mathbf{A}$. That is not enough, however; $\mathbf{I}$ should also bear the proper relation to quantification. If, in other words, p is refutable (and here it is essential that p be thought of as a propositional function, and not merely as a proposition), then $\exists p$ should also be refutable. The requirement (satisfied by the set of refutable elements of the usual logical algebras) converts $\mathbf{I}$ into a monadic ideal.

The following definition is now adequately motivated: a *monadic logic* is a pair $(\mathbf{A}, \mathbf{I})$, where $\mathbf{A}$ is a monadic algebra and $\mathbf{I}$ is a monadic ideal in $\mathbf{A}$. The elements p of $\mathbf{I}$ are the *refutable* elements of the logic; if $p' \in \mathbf{I}$, then p is called *provable*.

For monadic logics, as for most other mathematical systems, representation theory plays an important role. Representation theory proceeds, as always, by selecting a class of particularly simple and "concrete" monadic logics, and asking to what extent every monadic logic is representable by means of logics of that class. For intuitively obvious reasons there is universal agreement on which logics should be called "concrete". The technical term for a concrete monadic logic is *model*; a model is, by definition, a monadic logic $(\mathbf{A}, \mathbf{I})$, where $\mathbf{A}$ is an $\mathbf{O}$-valued functional monadic algebra and $\mathbf{I}$ is the trivial ideal $\{0\}$. Note that since an $\mathbf{O}$-valued functional monadic algebra is simple (Theorem 6), $\mathbf{I}$ could only be $\{0\}$ or $\mathbf{A}$, and the latter choice is obviously uninteresting.

An *interpretation* of a monadic logic $(\mathbf{A}, \mathbf{I})$ in a model $(\mathbf{B}, \{0\})$ is a monadic homomorphism f from $\mathbf{A}$ into $\mathbf{B}$ such that $fp = 0$ whenever $p \in \mathbf{I}$. A convenient way of expressing the condition on the homomorphism is to say that every refutable element is *false* in the interpretation. If, in other words, an element p of $\mathbf{A}$ is called *universally invalid* whenever it is false in every interpretation, then, by definition, every refutable element is universally invalid. There could conceivably be elements in $\mathbf{A}$ that are not refutable but that are nevertheless universally invalid. If there are no such elements, i.e., if every universally invalid element is refutable, the logic is said to be *semantically complete*. This definition sounds a little more palatable in its dual form: a logic is semantically complete if every universally valid element is provable. (The element p is *universally valid* if $fp = 1$ for every interpretation f, i.e., if p is *true* in every interpretation.) Elliptically but suggestively, semantic completeness can be described by saying that everything true is provable.

7. Semisimplicity. Semantic completeness demands of a logic (**A**, **I**) that the ideal **I** be relatively large. If, in particular, **I** is very large, i.e., **I** = **A**, then the logic is semantically complete, simply because every element of **A** is refutable. (The fact that in this case there are no interpretations is immaterial.) If **I** $\neq$ **A**, then the quotient algebra **A**/**I** may be formed, and the problem of deciding whether or not the *logic* (**A**, **I**) is semantically complete reduces to a question about the *algebra* **A**/**I**.

Since every interpretation of (**A**, **I**) in a model (**B**, $\{0\}$) induces in a natural way a homomorphism from **A**/**I** into **B**, and since (by Theorem 6) the only restriction on **B** is that it be simple, the question becomes the following one. Under what conditions on a monadic algebra **A** is it true that whenever an element p of **A** is mapped on 0 by every homomorphism from **A** into a simple algebra, then $p = 0$? Since every monadic subalgebra of an **O**-valued functional monadic algebra is an algebra of the same kind, it follows from Theorem 6 that every monadic subalgebra of a simple monadic algebra is also simple, and, consequently, that the difference between "into" and "onto" is not essential here. Because of the correspondence between homomorphisms with simple ranges and maximal ideals, the question could also be put this way: under what conditions on a monadic algebra **A** is it true that whenever an element p of **A** belongs to all maximal ideals, then $p = 0$? In analogy with other parts of algebra, it is natural to say that a monadic algebra **A** is *semisimple* if the intersection of all maximal ideals in **A** is $\{0\}$. The question now becomes: which monadic algebras are semisimple? The answer is quite satisfying.

Theorem 7. *Every monadic algebra is semisimple.*

REMARK. Since monadic algebras constitute a generalization of Boolean algebras, Theorem 7 asserts, in particular, that every Boolean algebra is semisimple. This consequence of Theorem 7 is well known: it is an immediate consequence of Stone's representation theorem, and it is often presented as the most important step in the proof of that theorem. The proof of the present generalization can be carried out by a monadic imitation of any one of the usual proofs of its Boolean special case. The proof below adopts the alternative procedure of deducing the generalization from the special case.

Proof. It is to be proved that if $\mathbf{A}$ is a monadic algebra and if p_0 is a non-zero element of $\mathbf{A}$, then there exists a monadic maximal ideal $\mathbf{I}$ in $\mathbf{A}$ such that $p_0 \epsilon' \mathbf{I}$. It follows from the known Boolean version of the theorem that there exists a Boolean maximal ideal $\mathbf{I}_0$ in $\mathbf{A}$ such that $p_0 \epsilon' \mathbf{I}_0$. If $\mathbf{I}$ is the set of all those elements p in $\mathbf{A}$ for which $\exists p \in \mathbf{I}_0$, then it is trivial to verify that $\mathbf{I}$ is a monadic ideal and that $p_0 \epsilon' \mathbf{I}$. The proof of semisimplicity can be completed by showing that $\mathbf{I}$ is maximal. Suppose therefore that $\mathbf{J}$ is a monadic ideal properly including $\mathbf{I}$. It follows that $\mathbf{J}$ contains an element p such that $\exists p \epsilon' \mathbf{I}_0$. Since $\mathbf{J}$ is a monadic ideal, $\exists p \in \mathbf{J}$. Since $\exists p \epsilon' \mathbf{I}_0$, and since $\mathbf{I}_0$ is a Boolean maximal ideal, $(\exists p)' \in \mathbf{I} \subset \mathbf{J}$. The last two sentences together imply that $1 \in \mathbf{J}$, so that $\mathbf{J} = \mathbf{A}$; the proof is complete.

It should be remarked that there are natural examples of Boolean algebras with operators (4), easy generalizations of monadic algebras, that are not semisimple. Thus, for instance, the semisimplicity of the closure algebra of a topological space appears to depend on which separation axioms the space satisfies.

PART 2

Topology

8. Hemimorphisms and Boolean relations. The algebraic theory of Boolean algebras is better understood if their topological theory is taken into account; the same is true of monadic algebras. In what follows we shall therefore make use of the topological version of Stone's theorem, in the following form: there is a one-to-one correspondence between Boolean algebras $\mathbf{A}$ and Boolean spaces X such that each algebra $\mathbf{A}$ is isomorphic to the algebra of all clopen subsets of the corresponding space X (8). Explanation of terms: a *clopen* set in a topological space is a set that is simultaneously closed and open, and a *Boolean space* is a totally disconnected compact Hausdorff space, i.e., a compact Hausdorff space in which the clopen sets form a base. The one-to-one nature of the correspondence must of course be interpreted with the usual algebraic-topological grain of salt: X uniquely determines $\mathbf{A}$ to within an isomorphism, and $\mathbf{A}$ uniquely determines X to within a homeomorphism. The algebra $\mathbf{A}$ corresponding to a space X will be called the *dual algebra* of X, and the space X corresponding to an algebra **A** will be called the *dual space* of **A**.

There is a natural isomorphism between the dual algebra **A** of a Boolean space X and the set of all continuous functions from X into **O**. In order to interpret this assertion, we must, of course, endow **O** with a topology; this is done, once and for all, by declaring open every one of the four subsets of **O**, so that **O** itself becomes a (discrete) Boolean space. The isomorphism mentioned just above assigns to each clopen subset of X its characteristic function. It is algebraically convenient to identify **A** with the algebra of all continuous functions from X into **O**. In view of this identification, every Boolean algebra that occurs below is to be regarded as identical with the algebra of all **O**-valued continuous functions on its dual space. Thus, for example, if **A** is a Boolean algebra with dual space X, and if $p \in \mathbf{A}$ and $x \in X$, then $p(x)$ makes sense; it has the value 1 or 0 according as x belongs or does not belong to that clopen subset of X which corresponds to the element p of **A**.

The duality theory of monadic algebras is conveniently studied at a slightly more general level than might appear relevant at first sight. The point is that it is possible to treat simultaneously both the old theory of homomorphisms and the newer theory of quantifiers. The appropriate general concept is that of a *hemimorphism*, defined as a mapping f from a Boolean algebra **A** into a Boolean algebra **B**, such that $f0 = 0$ and $f(p \vee q) = fp \vee fq$ for all p and q in **A**. The reason for the name is that, roughly speaking, a hemimorphism preserves half the structure of a Boolean algebra. Hemimorphisms occur, under the name of "normal and additive functions", in the work of Jónsson and Tarski (4). Two elementary consequences of the definition are that a hemimorphism is *monotone* ($fp \leqq fq$ whenever $p \leqq q$) and *submultiplicative* ($f(p \wedge q) \leqq fp \wedge fq$). It is clear that every homomorphism (from a Boolean algebra **A** to a Boolean algebra **B**) and every quantifier (from a Boolean algebra **A** to itself) is a hemimorphism.

The proper topological concept is that of a Boolean relation. A *relation* between two sets Y and X (or, more accurately, from Y to X) is, as always, a subset φ of the Cartesian product $Y \times X$; the assertion $(y, x) \in \varphi$ is conveniently abbreviated as $y\varphi x$. If Q is a subset of Y, the *direct image* of Q under φ, in symbols φQ, is the set of all those points x in X for which there exists a point y in Q such that $y\varphi x$. If φ^{-1} denotes the *inverse* of the relation φ, i.e., φ^{-1} is the set of all those pairs (x, y) in $X \times Y$ for which $y\varphi x$, then the *inverse image* of a subset P of X under φ, in symbols

$\varphi^{-1}P$, is the direct image of P under φ^{-1}, or, equivalently, it is the set of all those points y in Y for which there exists a point x in P such that $y\varphi x$. If $y \in Y$ and $\{y\}$ is the set whose only element is y, then $\varphi\{y\}$ is also denoted by φy; similarly if $x \in X$, then $\varphi^{-1}\{x\}$ is also denoted by $\varphi^{-1}x$. These concepts are well known; they are explicitly mentioned here only in order to establish the notation. Several other related concepts (e.g., *equivalence relation* and *relation product*) will be used below without explicit definition.

A *Boolean relation* is a relation φ from a Boolean space Y to a Boolean space X, such that the inverse image of every clopen set in X is a clopen set in Y and such that the direct image of every point in Y is a closed set in X. It is easy to verify that a function from a Boolean space Y to a Boolean space X is a Boolean relation if and only if it is continuous. It is also pertinent to remark that the two conditions in the definition of a Boolean relation are independent of each other. Indeed, if X is a Boolean space with a single cluster point $\bar{x}$, if $Y = X$, and if $\varphi = (Y \times X) - \{(\bar{x}, \bar{x})\}$, then $\varphi^{-1}P = Y$ for every non-empty clopen subset P of X, so that the inverse image of every clopen set is clopen, but $\varphi\bar{x} = X - \{\bar{x}\}$, so that the direct image of a point is not always closed. Any discontinuous function from a Boolean space Y to a Boolean space X furnishes an example of a relation for which the direct image of every point is closed, but the inverse image of a clopen set is not always clopen.

9. Duality. Suppose that **A** and **B** are Boolean algebras, with respective dual spaces X and Y. If f is a hemimorphism from **A** into **B**, its dual, denoted by f^*, is the relation from Y to X defined by

$$f^* = \cap_{p \in \mathbf{A}}\{(y, x)\colon\ p(x) \leqq fp(y)\};$$

in other words, yf^*x if and only if $p(x) \leqq fp(y)$ for all p in **A**. (Symbols such as $fp(y)$ will be used frequently in what follows. They are to be interpreted, in every case, by first performing all the indicated functional operations and then evaluating the resulting function at the indicated point. Thus, explicitly, $fp(y) = (fp)(y)$.)

If φ is a Boolean relation from Y to X, its dual, denoted by φ^*, is the mapping that assigns to every element p of **A** a function φ^*p from Y to **O**, defined by

$$\varphi^*p(y) = \vee\{p(x)\colon\ y\varphi x\}.$$

The following theorem is the principal result of the theory of Boolean duality.

Theorem 8. *If f is a hemimorphism, then f^* is a Boolean relation, and $f^{**} = f$. If φ is a Boolean relation, then φ^* is a hemimorphism and $\varphi^{**} = \varphi$. If f and φ are each other's duals, then*

$$(*) \qquad \{y\colon fp(y) = 1\} = \varphi^{-1}\{x\colon p(x) = 1\}$$

for every p in **A**.

REMARK. A similar but weaker theorem has been published by Jónsson and Tarski (4), and the same comment is true for some of the results obtained in sections 10 and 11. The essential difference between the present approach and that of Jónsson and Tarski can be described as follows: since they do not have the concept of a Boolean relation, they are unable to state which relations between Boolean spaces can occur as the duals of hemimorphisms.

PROOF. Since

$$f^*y = \cap_{p \in \mathbf{A}}\{x\colon p(x) \leqq fp(y)\},$$

so that f^*y is obviously closed, in order to prove that f^* is a Boolean relation, it is sufficient to prove that (*) holds with f^* in place of φ. Given p_0 in **A**, write

$$P_0 = \{x\colon p_0(x) = 1\} \text{ and } Q_0 = \{y\colon fp_0(y) = 1\};$$

it is to be proved that $f^{*-1}P_0 = Q_0$. If $p_0 = 0$, then P_0 is empty; since, by the definition of a hemimorphism, $f0 = 0$, it follows that Q_0 is empty. In what follows, it is therefore permissible to assume that $p_0 \neq 0$ and hence that P_0 is not empty.

If $y \in f^{*-1}P_0$, then there exists a point x in P_0 such that yf^*x, i.e., such that $p(x) \leqq fp(y)$ for all p in **A**. It follows in particular, with $p = p_0$, that $1 = p_0(x) \leqq fp_0(y)$, so that $y \in Q_0$. This proves that $f^{*-1}P_0 \subset Q_0$; the reverse inclusion lies slightly deeper.

It is to be proved that if $y \in' f^{*-1}P_0$, then $fp_0(y) = 0$. To say that $y \in' f^{*-1}P_0$ means that the assertion yf^*x is false for every x in P_0. This, in turn, means that to every x in P_0 there corresponds an element p_x of **A** such that the assertion $p_x(x) \leqq fp_x(y)$ is false, i.e., such that $p_x(x) = 1$ and $fp_x(y) = 0$. Since

$$P_0 \subset \cup_{x \in P_0}\{z\colon p_x(z) = 1\},$$

the fact that P_0 is closed (and therefore compact), together with the fact that each set $\{z\colon p_x(z) = 1\}$ is open, implies that there exists a finite subset $\{x_1, \ldots, x_n\}$ of P_0 such that

$$P_0 \subset \cup_{j=1}^{n}\{z\colon p_{x_j}(z) = 1\}.$$

The assumption that P_0 is not empty is reflected here in the fact that $n \neq 0$, i.e., that the finite set is not empty.

Write $\bar{p} = \bigvee_{j=1}^{n} p_{x_j}$. If $z \in P_0$, then $p_{x_j}(z) = 1$ for at least one j, and, therefore, $\bar{p}(z) = 1$; it follows that $p_0 \leqq \bar{p}$. Since f is a hemimorphism, $f\bar{p} = \bigvee_{j=1}^{n} fp_{x_j}$; since $fp_x(y) = 0$ for all x in P_0, it follows that $f\bar{p}(y) = 0$. Since a hemimorphism is monotone, it follows, finally, that $f\bar{p}_0(y) = 0$; this completes the proof that f^* satisfies (*) and is, consequently, a Boolean relation.

It still remains to be shown that $f^{**} = f$. Since yf^*x implies that $p(x) \leqq fp(y)$ for all p, the inequality $\bigvee \{p(x)\colon\ yf^*x\} \leqq fp(y)$ is obvious. The reverse inequality amounts to the assertion that if $p(x) = 0$ whenever yf^*x, then $fp(y) = 0$. But to say that $p(x) = 0$ whenever yf^*x is equivalent to saying that y does not belong to $f^{*-1}\{x\colon\ p(x) = 1\}$. It follows from (*) that $\bigvee \{p(x)\colon\ yf^*x\} = fp(y)$, and hence, by the definition of the dual of a Boolean relation, that $f^{**} = f$.

Suppose now that φ is a Boolean relation from Y to X. It follows from the definition of φ^* that $\varphi^* p(y) = 1$ if and only if there is a point x such that $p(x) = 1$ and such that $y\varphi x$; in other words, (*) holds with φ^* in place of f. Since the inverse image under φ of a clopen set is clopen, it follows that φ^* maps **A** into **B**. The verification that φ^* is a hemimorphism is a matter of trivial routine.

It remains only to verify that $\varphi^{**} = \varphi$. If $y\varphi x$, then, for every p in **A**, $p(x)$ is one of the terms whose supremum is $\varphi^* p(y)$, and, consequently, $p(x) \leqq \varphi^* p(y)$. On the other hand, if the assertion $y\varphi x$ is false, then (since φy is closed) the Boolean nature of X implies that there exists an element p_0 in **A** such that $p_0(x) = 1$ and such that $p_0(z) = 0$ whenever $y\varphi z$. It follows that $\varphi^* p_0(y) = 0$ and hence that the assertion that $p(x) \leqq \varphi^* p(y)$ for all p is also false. Conclusion: $y\varphi x$ if and only if $p(x) \leqq \varphi^* p(y)$ for all p, and hence, by the definition of the dual of a hemimorphism, $\varphi^{**} = \varphi$. The proof of Theorem 8 is complete.

Corollary. *Under a Boolean relation, the inverse image of every point is closed.*

Proof. If φ is a Boolean relation and $f = \varphi^*$, then

$$\varphi^{-1}x = \bigcap_{p \in \mathbf{A}} \{y\colon\ p(x) \leqq fp(y)\}$$

for every x in X.

10. The dual of a homomorphism.

In view of the duality theorem, it is possible, in principle, to translate every algebraic property of hemimorphisms into a topological property of Boolean relations, and vice versa. The purpose of this section is to carry out the translation for some of the properties of importance in the theory of monadic algebras.

LEMMA 9. *If X and Y are Boolean spaces and if φ and ψ are Boolean relations from Y to X such that $\varphi^{-1}P = \psi^{-1}P$ for every clopen subset P of X, then $\varphi = \psi$.*

PROOF. If x and y are such that $y\varphi x$ and if P is a clopen subset of X such that $x \in P$, then $y \in \varphi^{-1}P$ and therefore, by assumption, $y \in \psi^{-1}P$. This implies that $P \cap \psi y$ is not empty. In other words: every neighborhood of x meets the set ψy. Since ψy is closed, it follows that $x \in \psi y$, i.e., that $y\psi x$. This proves that $\varphi \subset \psi$; the reverse inclusion follows by symmetry.

Theorem 9. *If* **A**, **B**, *and* **C** *are Boolean algebras, with respective dual spaces X, Y, and Z, and if f and g are hemimorphisms from* **A** *into* **B** *and from* **B** *into* **C** *respectively, then $(gf)^* = f^*g^*$.*

PROOF. An application of (*) (section 9) to gf shows that

$$(gf)^{*-1}\{x\colon\ p(x) = 1\} = \{z\colon\ gfp(z) = 1\}$$

for all p in **A**. An application of (*) to g shows that

$$g^{*-1}\{y\colon\ q(y) = 1\} = \{z\colon\ gq(z) = 1\}$$

for all q in **B**; replacing q by fp and applying (*) to f, we deduce that

$$g^{*-1}f^{*-1}\{x\colon\ p(x) = 1\} = \{z\colon\ gfp(z) = 1\}$$

for all p in **A**. Together the two equations involving gfp imply that

$$(gf)^{*-1}P = g^{*-1}f^{*-1}P$$

for every clopen subset P of X. Since it is an elementary fact about relations that $g^{*-1}f^{*-1} = (f^*g^*)^{-1}$, the desired result follows from Lemma 9.

LEMMA 10. *If a hemimorphism f (from* **A** *to* **B***) and a Boolean relation φ (from Y to X) are each other's duals, then a necessary and sufficient condition that f be multiplicative ($f(p \wedge q) = fp \wedge fq$) is that φ be a function.*

PROOF. If φ is a function, then $y\varphi x$ means that $\varphi y = x$ and therefore

$$fp(y) = \bigvee \{p(x)\colon\ y\varphi x\} = p(\varphi y).$$

The multiplicativity of f is the result of a straightforward computation. If φ is not a function, then there exists a point y in Y and there exist distinct points x_0 and x_1 in X so that $y\varphi x_0$ and

$y\varphi x_1$. If p is an element of **A** such that $p(x_0) = 0$ and $p(x_1) = 1$, then (since $p'(x_0) = 1$) $fp'(y) = fp(y) = 1$ and therefore $fp' \wedge fp \neq 0$. Since $f(p \wedge p') = f0 = 0$, it follows that f is not multiplicative.

In the following two lemmas, as in Lemma 10, f is a hemimorphism from **A** to **B** and φ is the corresponding Boolean relation from Y to X.

Lemma 11. *A necessary and sufficient condition that $f1 = 1$ is that φ have Y for its domain.*

Proof. It follows from (*), with $p = 1$, that

$$\{y\colon f1(y) = 1\} = \varphi^{-1}X,$$

and hence that $f1 = 1$ if and only if $\varphi^{-1}X = Y$.

Lemma 12. *A necessary and sufficient condition that f be a homomorphism ($fp' = (fp)'$) is that φ be a function with domain Y.*

Proof. It is a well-known and easily proved fact about Boolean homomorphisms that a hemimorphism f is a homomorphism if and only if it is multiplicative and satisfies the equation $f1 = 1$. The desired conclusion now follows from Lemmas 10 and 11.

At this point the generalized version of Boolean duality makes contact with the standard version. The duality between homomorphisms f (from **A** to **B**) and continuous functions φ (from Y to X) is known. It is known also that f is a monomorphism (i.e., an isomorphism into) if and only if φ maps Y onto X, and f is an epimorphism (i.e., a homomorphism onto) if and only if φ is one-to-one (8); these facts will be used below.

11. The dual of a quantifier. Suppose, throughout this section, that **A** is a Boolean algebra, with dual X, and that f is a hemimorphism from **A** into itself, with dual φ. A useful auxiliary concept in this case is the relation $\hat{\varphi}$ in X; by definition

$$\hat{\varphi} = \cap_{p \in \mathbf{A}}\{(y, x)\colon fp(x) = fp(y)\}.$$

In other words, $y\hat{\varphi}x$ if and only if $fp(x) = fp(y)$ for all p in **A**. It is obvious that the relation $\hat{\varphi}$ is an equivalence with domain X (and, therefore, with range X).

Lemma 13. *A necessary and sufficient condition that f be a quantifier is that $\varphi = \hat{\varphi}$.*

Proof. If f is a quantifier, then, of course, f is increasing. It follows that if $y\hat{\varphi}x$, then $p(x) \leqq fp(x) = fp(y)$ for all p in **A**, so that $y\varphi x$; in other words, $\hat{\varphi} \subset \varphi$. To prove the reverse inclusion, suppose that $y\varphi x$, i.e., that $p(x) \leqq fp(y)$ for all p in **A**. This inequality, applied first to fp and then to $(fp)'$ in place of p, yields

(i) $$fp(x) \leqq ffp(y) = fp(y)$$

(cf. Theorem 1), and

(ii) $$(fp)'(x) \leqq f(fp)'(y) = (fp)'(y)$$

(cf. Lemma 5). Since (ii) is equivalent to the reverse of (i), it follows that $fp(x) = fp(y)$ for all p in $\mathbf{A}$, and hence that $y\hat{\varphi}x$. This means that $\varphi \subset \hat{\varphi}$ and therefore completes the proof of the necessity of the condition.

For sufficiency, it is to be proved that if $\varphi = \hat{\varphi}$, then f is increasing and quasi-multiplicative. The assertion that f is increasing means that $p(x) \leqq fp(x)$ whenever $p \in \mathbf{A}$ and $x \in X$, and, consequently, it is equivalent to the assertion that φ is reflexive. If $\varphi = \hat{\varphi}$, or, more generally, if φ is known to be an equivalence with domain X, then φ is reflexive and therefore f is increasing. (This comment will be used again in the proof of Lemma 14 below.) To prove that f is quasi-multiplicative, note first that

$$f(p \wedge fq)(y) = \bigvee\{(p \wedge fq)(x)\colon\ y\varphi x\} = \bigvee\{p(x) \wedge fq(x)\colon\ y\varphi x\}.$$

Since $\varphi = \hat{\varphi}$, so that, in particular, $\varphi \subset \hat{\varphi}$, the condition $y\varphi x$ implies that $fq(x) = fq(y)$. It follows that

$$f(p \wedge fq)(y) = \bigvee\{p(x) \wedge fq(y)\colon\ y\varphi x\} = \bigvee\{p(x)\colon\ y\varphi x\} \wedge fq(y) = \\ = fp(y) \wedge fq(y);$$

this completes the proof of sufficiency.

Lemma 14. *A necessary and sufficient condition that $\varphi = \hat{\varphi}$ is that φ be an equivalence with domain X.*

Proof. Necessity is trivial. To prove sufficiency, assume that φ is an equivalence with domain X. It was already remarked that this implies that f is increasing.

It follows, exactly as in the corresponding part of the proof of Lemma 13, that $\hat{\varphi} \subset \varphi$. To prove the reverse inclusion, assume that $y\,\varphi\,x$. Since

$$fp(x) = \bigvee\{p(z)\colon\ x\varphi z\} \text{ and } fp(y) = \bigvee\{p(z)\colon\ y\varphi z\},$$

and since $x\varphi z$ is equivalent to $y\varphi z$, it follows that $fp(x) = fp(y)$ for all p, i.e., that $\varphi \subset \hat{\varphi}$.

Theorem 10. *A necessary and sufficient condition that f be a quantifier is that φ be an equivalence with domain X.*

PROOF. Obvious from Lemmas 13 and 14.

The two extreme quantifiers are sometimes of interest.

LEMMA 15. *If* $\exists$ *is the discrete quantifier on* **A**, *then* $\exists^*$ *is the identity (i.e.,* $y \exists^* x$ *if and only if* $y = x$*); if* $\exists$ *is the simple quantifier on* **A**, *then* $\exists^* = X \times X$ *(i.e.,* $y \exists^* x$ *for all* x *and* y*).*

PROOF. If $\exists$ is discrete, then, by definition, $\exists p = p$ for all p. It follows that $y \exists^* x$ if and only if $p(x) = p(y)$ for all p (cf. Lemma 13), and hence if and only if $x = y$. (Observe also that the discrete quantifier on **A** coincides with the identity homomorphism from **A** onto itself; it follows from the duality theory of homomorphisms that its dual is the identity mapping from X onto itself.) If $\exists$ is simple, then $\exists p = 1$ whenever $p \neq 0$; it follows *a fortiori* that $p(x) \leqq \exists p(y)$ whenever $p \neq 0$ (for all x and y). Since the inequality is still true when $p = 0$, the proof is complete.

PART 3

Representation

12. Boolean mappings. Just as the ordinary duality theory shows that the study of Boolean algebras is equivalent to the study of Boolean spaces, the duality theory of quantifiers shows that the study of monadic algebras is equivalent to the study of Boolean spaces equipped with a Boolean equivalence relation. An equivalence relation φ in a topological space X determines a quotient space X/φ; the first purpose of the present section is to use this fact in order to describe a topological version of monadic algebra in terms of objects that are more familiar than Boolean equivalence relations. In the approach it is convenient not to use the theory of quotient spaces but to use, instead, methods more directly relevant to Boolean theory. The appropriate concept is that of a *Boolean mapping*, defined as a continuous and open mapping from one Boolean space onto another. The principal result is that the study of quantifiers is equivalent to the study of Boolean mappings.

LEMMA 16. *If* $\exists$ *is a quantifier on a Boolean algebra* **A**, *then there exists a Boolean mapping* π, *from the dual space* X *of* **A** *to a Boolean space* Y, *such that* $x_1 \exists^* x_2$ *is equivalent to* $\pi x_1 = \pi x_2$.

PROOF. Let **B** be the range of $\exists$, so that **B** is a Boolean subalgebra of **A**, and let Y be the dual space of **B**. Despite the identification convention of section 8, it is not permissible to identify **B** with the algebra of all continuous functions from Y to **O**;

the trouble is that, in view of that identification convention, **B** is already concretely given as an algebra of continuous functions from X to **O**. All that it is possible to say is that there is an isomorphism, say f, from **B** onto the algebra of all continuous functions from Y to **O**, and, fortunately, this is all that is needed.

The identity mapping from **B** into **A** is a homomorphism; its dual is a continuous function π from X onto Y. The assertion that π is the dual of the embedding of **B** into **A** means that the result of evaluating a function q in **B** at a point x in X is the same as the value of the image function fq on Y at the point πx of Y; in other words,

$$q(x) = fq(\pi x)$$

for all q and all x.

In view of Lemma 13, the condition $x_1 \exists^* x_2$ is equivalent to the validity of $\exists p(x_1) = \exists p(x_2)$ for all p in **A**; since $\exists$ maps **A** onto **B**, this, in turn, is equivalent to the validity of $q(x_1) = q(x_2)$ for all q in **B**. It follows from the definition of π that $x_1 \exists^* x_2$ if and only if $fq(\pi x_1) = fq(\pi x_2)$ for all q in **B**; since, finally, f is an isomorphism, the condition is equivalent to $\pi x_1 = \pi x_2$.

One consequence of the preceding paragraph is that $\exists^* Q = \pi^{-1}\pi Q$ for every subset Q of X. Indeed, if $x_0 \in \exists^* Q$, then $x_0 \exists^* x$ for some x in Q, and therefore $\pi x_0 = \pi x$ for some x in Q. The last assertion means that $\pi x_0 \in \pi Q$, or, equivalently, that $x_0 \in \pi^{-1}\pi Q$. This proves that $\exists^* Q \subset \pi^{-1}\pi Q$; the reverse inclusion follows by retracing the steps of the argument in the reverse order.

Since $\exists^*$ is a Boolean relation, the equation $\exists^* Q = \pi^{-1}\pi Q$ implies that $\pi^{-1}\pi Q$ is clopen whenever Q is clopen. Since it is an elementary fact about mappings between compact Hausdorff spaces, i.e., mappings such as π, that if $\pi^{-1}P$ is open (or closed), then P is open (or closed), it follows that πQ is clopen whenever Q is clopen. The Boolean nature of X implies now that π is open; the proof is complete.

LEMMA 17. *If π is a Boolean mapping from a Boolean space X to a Boolean space Y, then there exists a quantifier $\exists$ on the dual algebra* **A** *of X such that $x_1 \exists^* x_2$ is equivalent to $\pi x_1 = \pi x_2$.*

PROOF. Define a relation φ in X by writing $x_1 \varphi x_2$ if and only if $\pi x_1 = \pi x_2$. Clearly φ is an equivalence relation with domain X, and $\varphi Q = \pi^{-1}\pi Q$ for every subset Q of X. The Boolean nature of π implies that the (inverse) image under φ of a clopen set is clopen and that the (direct) image under φ of a point is closed. (Since φ is an equivalence, the distinction between direct and inverse images is purely verbal.) In other words, φ is a Boolean equivalence with domain X; the desired result follows from Theorem 10.

It is not difficult to verify that the correspondences described in Lemmas 16 and 17 are in an obvious sense dual to each other. If, in other words, $\exists$ is a quantifier and π is the Boolean mapping that corresponds to $\exists$ via Lemma 16, then the quantifier that corresponds to π via Lemma 17 is the same as $\exists$, to within an isomorphism, and a similar statement holds with the order of π and $\exists$ reversed. Since the proof of this assertion involves absolutely no conceptual difficulties, it may safely be omitted.

13. Constants. The first-order monadic functional calculus is often described as the modern version of Aristotelean (syllogistic) logic. Correspondingly, the algebraic formulation of Aristotelean logic is to be found in the theory of monadic algebras; the fact that a certain syllogism is valid can be described by asserting that the ideal generated by two specified elements of a monadic algebra contains a third specified element.

The point is worth a little closer examination. It turns out that the logically relevant concept is not that of an ideal (cf. section 6) but the dual concept of a filter. A (Boolean) *filter*, by definition, is a non-empty subset $\mathbf{F}$ of a Boolean algebra $\mathbf{A}$, such that if p and q are in $\mathbf{F}$, then $p \wedge q \in \mathbf{F}$, and if $p \in \mathbf{F}$, then $p \vee q \in \mathbf{F}$ for all q in $\mathbf{A}$. A *monadic filter* is a subset $\mathbf{F}$ of a monadic algebra $\mathbf{A}$, such that $\mathbf{F}$ is a Boolean filter in $\mathbf{A}$ and such that $\forall p \in \mathbf{F}$ whenever $p \in \mathbf{F}$. The validity of the syllogism *Barbara* can now be described in the following terms: if p, q, and r are elements of a monadic algebra, then the monadic filter generated by $\forall(p' \vee q)$ and $\forall(q' \vee r)$ contains $\forall(p' \vee r)$.

If the preceding example is examined with a view to specializing it to the well-known syllogism concerning the mortality of Socrates, an intuitively unsatisfactory aspect of the situation emerges. In highly informal language, the trouble is that the theory is equipped to deal with generalities only, and is unable to say anything concrete. Thus, for instance, by an appropriate choice of notation, a monadic algebra might be taught to say "all men are mortal", but difficulties are encountered in trying to teach it to say "Socrates is a man." The point is that "manhood" and "mortality" can easily be thought of as elements of a monadic algebra, since they are the obvious abstractions of the propositional functions whose value at each x of some set is, respectively, "x

is a man" and "x is mortal". Socrates, on the other hand, is a "constant", and there is no immediately apparent way of pointing to him. A classical artifice, designed to avoid this situation, is to promote Socrates to a propositional function, namely the function whose value at x is "x is Socrates". This procedure is both intuitively and algebraically artificial; Socrates is not a proposition, but an entity about which propositions may be made.

Intuitively, constants play the same role in algebraic logic as distinguished elements (e.g., the unit element of a group) play in ordinary algebra. What is desired is to single out certain points of a given domain X and to build them into the theory. Since, however, the ultimate objects of the theory are not points of X, but abstract elements suggested by functions on X, the desideratum becomes that of finding an algebraic description of what it means to replace the argument of a function by a constant.

The correct description is quickly suggested by the study of functional algebras. Suppose, to be specific and to avoid irrelevant, non-algebraic difficulties, that **B** is a complete Boolean algebra, and that **A** is the functional monadic algebra of all functions from some domain X into **B**. A convenient way to describe the act of replacing the argument x of a function p in **A** by a fixed element x_0 of X is to introduce the mapping c that associates with p the function cp, where

$$cp(x) = p(x_0)$$

for all x in X. It is clear that c is a Boolean endomorphism on **A**. Since, moreover, $\exists p$ is always a constant function, its value at x_0 is the same as its constant value, so that $c\exists p = \exists p$. Since, similarly, cp is always a constant function, an application of the quantifier leaves it unchanged, so that $\exists cp = cp$.

The following definition is now adequately motivated: a *constant* of a monadic algebra **A** is a Boolean endomorphism c on **A**, such that

$$c\exists = \exists \text{ and } \exists c = c.$$

It follows from the definition that (i) c is the identity on the range of $\exists$, and (ii) the range of c is included in the range of $\exists$. Conversely, (iii) if c is a Boolean endomorphism satisfying (i) and (ii), then c is a constant. (iv) A constant is idempotent ($c^2 = c$). (v) If c is a constant of **A**, then $cp \leqq \exists p$ for all p in **A**. If c is a constant of **A**, then (vi) $c\forall = \forall$, and (vii) $\forall c = c$, and, conversely, (viii) if c is a Boolean endomorphism satisfying (vi) and (vii), then c is a constant. In view of (vi), (vii), and (viii), all true assertions about constants have true duals, and, in particular, (ix) if c is a constant of **A**, then $\forall p \leqq cp$ for all p in **A**.

With all the machinery at hand, it is easy to formulate the algebraic version of the syllogism about the mortality of Socrates. It asserts that if p and q are elements of a monadic algebra **A**, and if c is a constant of **A**, then the filter generated by $\forall(p' \vee q)$ and cp contains cq.

The introduction of constants was motivated above by logical considerations. It turns out that constants are also of great algebraic importance, and that, in fact, they play a central role in the proof of the fundamental representation theorem for monadic algebras. A final pertinent comment is this: although the concept of a constant is a purely algebraic one, in the discussion of the existence of constants it is convenient to make use of the topological theory of duality; it is for that reason that the definition was not given before.

14. Cross sections. A *cross section* of a continuous mapping π from a topological space X onto a topological space Y is a continuous mapping σ from Y into X such that $\pi\sigma y = y$ for all y in Y. Cross sections do not always exist, not even if X and Y are Boolean spaces. Example: let X be a Boolean space with exactly two cluster points, let Y be obtained from X by identifying the two cluster points, and let π be the identification mapping. One reason this example works is that π is not a Boolean mapping. Unfortunately, however, even Boolean mappings do not always have a cross section. Examples to show this are not trivial to construct; a suitable one has been constructed by J. L. Kelley and is described in the work of Arens and Kaplansky (1).

Cross sections of Boolean mappings are intimately related to constants of monadic algebras; the relation depends, naturally, on the correspondence between Boolean mappings and quantifiers. Suppose, indeed, that **A** is a monadic algebra, with quantifier $\exists$ and dual space X, and let π be a Boolean mapping from X onto a Boolean space Y such that $x_1 \exists^* x_2$ is equivalent to $\pi x_1 = \pi x_2$ (Lemma 16).

LEMMA 18. *There is a one-to-one correspondence between all constants c of* **A** *and all cross sections σ of π, such that*

$$cp(x) = p(\sigma\pi x)$$

for all p in **A** *and all x in X.*

PROOF. Suppose that c is a constant and let γ be its dual; in other words, γ is a continuous mapping from X into X such that $cp(x) = p(\gamma x)$ for all p and all x. The cross section σ corresponding to c is defined for a point $y = \pi x$ of X by writing $\sigma y = \gamma x$. It must, of course, be proved that this definition is unambiguous; i.e., that if $\pi x_1 = \pi x_2$, than $\gamma x_1 = \gamma x_2$. Indeed: if $\pi x_1 = \pi x_2$, then $x_1 \exists^* x_2$, so that $\exists p(x_1) = \exists p(x_2)$ for all p. This relation, applied to cp in place of p, yields the conclusion that $cp(x_1) = cp(x_2)$ for all p, and hence that $p(\gamma x_1) = p(\gamma x_2)$ for all p; the equation $\gamma x_1 = \gamma x_2$ now follows immediately. Clearly σ maps Y into X, and, if $p \in \mathbf{A}$, $x \in X$, and $y = \pi x$, then

$$cp(x) = p(\gamma x) = p(\sigma y) = p(\sigma \pi x).$$

It remains to prove that σ is continuous. Suppose, for this purpose, that $p \in \mathbf{A}$. The set $\{x\colon p(x) = 1\}$ is a typical clopen set in X; it is to be proved that its inverse image under σ is open. Since $y \in \sigma^{-1}\{x\colon p(x) = 1\}$ if and only if $y = \pi x$ with $p(\gamma x) = 1$, i.e., if and only if $y \in \pi\{x\colon cp(x) = 1\}$, the desired result follows from the fact that π is open.

Suppose next that σ is a cross section of π; the constant c corresponding to σ is defined for an element p of $\mathbf{A}$ by writing $cp(x) = p(\sigma\pi x)$ for all x in X. Clearly c is a Boolean endomorphism of $\mathbf{A}$. Since $\pi\sigma y = y$ for all y, so that $\pi\sigma\pi x = \pi x$ for all x, it follows from the relation between π and $\exists$ that $\sigma\pi x \exists^* x$ for all x. This implies that

$$c\exists p(x) = \exists p(\sigma\pi x) = \exists p(x)$$

for all p and all x, and hence that $c\exists = \exists$. Finally, if $x_1 \exists^* x_2$, then $\pi x_1 = \pi x_2$, so that $p(\sigma\pi x_1) = p(\sigma\pi x_2)$, or $cp(x_1) = cp(x_2)$; this implies that

$$\exists cp(x_0) = \bigvee\{cp(x)\colon x \exists^* x_0\} = cp(x_0)$$

for all p and all x_0, and hence that $\exists c = c$. The proof of the lemma is complete.

From Lemmas 17 and 18 we can conclude that there exist monadic algebras that possess no constants. The assertion of the existence of a constant for a monadic algebra $\mathbf{A}$, whenever it is true, can be regarded as an extension theorem. Indeed, a Boolean endomorphism of $\mathbf{A}$ is a constant of $\mathbf{A}$ if and only if it is an extension (to a homomorphism from $\mathbf{A}$ to $\exists(\mathbf{A})$) of the identity mapping (from $\exists(\mathbf{A})$ to $\exists(\mathbf{A})$). These considerations make contact with a theorem of Sikorski (6). The reason Sikorski's theorem is not available to prove the existence of constants is that Sikorski needs the added assumption that the prescribed range algebra is complete.

15. Rich algebras. A constant c of a monadic algebra **A** with quantifier will be called a *witness* to an element p of **A** if $\exists p = cp$; we shall also say that c is a witness to p *with respect to* **A**, or, more simply, *in* **A**. If **A** and $\mathbf{A}^+$ are monadic algebras such that **A** is a monadic subalgebra of $\mathbf{A}^+$ and such that every element of **A** has a witness in $\mathbf{A}^+$, we shall say that $\mathbf{A}^+$ is a *rich extension* of **A**, or, more simply, that $\mathbf{A}^+$ is rich *for* **A**. A *rich algebra* is one that is rich for itself.

LEMMA 19. *If* p_0 *is an arbitrary element of a monadic algebra* **A**, *then there exists a monadic algebra* $\mathbf{A}_0^+$ *including* **A** *as a monadic subalgebra and such that* (i) *there is a witness* c_0 *to* p_0 *in* $\mathbf{A}_0^+$, *and* (ii) *every constant of* **A** *has an extension to a constant of* $\mathbf{A}_0^+$.

PROOF. Let X be the dual space of **A** and let π be a Boolean mapping from X onto a Boolean space Y, such that $x_1 \exists^* x_2$ is equivalent to $\pi x_1 = \pi x_2$ (Lemma 16). If $X^+ = X \times X$, $Y^+ = X \times Y$, and $\pi^+(x, y) = (x, \pi y)$ whenever $(x, y) \in X^+$, then π^+ is a Boolean mapping from X^+ onto Y^+. To this Boolean mapping there corresponds a quantifier $\exists^+$ on the dual algebra $\mathbf{A}^+$ of X^+ such that $(x_1, y_1) \exists^{+*} (x_2, y_2)$ is equivalent to $\pi^+(x_1, y_1) = \pi^+(x_2, y_2)$ (Lemma 17). It follows from the definition of π^+ that if $p(x, y) = q(x) \wedge r(y)$, where q and r are in **A**, then

$$\exists^+ p(x_0, y_0) = \bigvee \{p(x, y)\colon x_0 = x, y_0 \exists^* y\} = q(x_0) \wedge \bigvee \{r(y)\colon y_0 \exists^* y\}$$
$$= q(x_0) \wedge \exists r(y_0).$$

If we write $p^+(x, y) = p(y)$ for every p in **A**, it follows from what was just proved about $\exists^+$ that the mapping $p \to p^+$ (which is obviously a Boolean isomorphism from **A** into $\mathbf{A}^+$) is a monadic isomorphism. We may therefore regard **A** as a monadic subalgebra of $\mathbf{A}^+$ whenever it is convenient to do so.

We observe next that every constant c of **A** has a natural extension to a constant c^+ of $\mathbf{A}^+$. To say that c^+ is an extension of c means, of course, that $c^+p^+ = (cp)^+$ whenever $p \in \mathbf{A}$. To prove this, let σ be a cross section of π such that $cp(x) = p(\sigma\pi x)$ (Lemma 18) and write $\sigma^+(x, y) = (x, \sigma y)$ whenever $(x, y) \in Y^+$. It is easy to verify that σ^+ is a cross section of π^+ and that the constant c^+ corresponding to the cross section σ^+ via Lemma 18 is the desired extension.

The algebra $\mathbf{A}^+$ has a simple Boolean endomorphism c_0^+ that is almost a constant of $\mathbf{A}^+$; by definition, $c_0^+ p(x, y) = p(x, x)$. It is trivial to verify that c_0^+ is an idempotent Boolean endomorphism and that $\exists^+ c_0^+ = c_0^+$; the only reason c_0^+ is not a constant is that the equation $c_0^+ \exists^+ = \exists^+$ need not hold. (The equation does not hold, because, for instance, if $p \in \mathbf{A}$, then $\exists^+ p^+(x, y) = \exists p(y)$, whereas $c_0^+ \exists^+ p^+(x, y) = \exists p(x)$.) We shall force c_0^+ to become a constant by identifying $c_0^+ \exists^+ p$ with $\exists^+ p$ for every p in $\mathbf{A}^+$. Precisely speaking, we shall consider in $\mathbf{A}^+$ the monadic ideal $\mathbf{I}^+$ generated by all elements of the form $\exists^+ p - c_0^+ \exists^+ p$, and we shall form the quotient algebra $\mathbf{A}^+/\mathbf{I}^+$. We shall prove that during the reduction modulo $\mathbf{I}^+$ the subalgebra $\mathbf{A}$ is not disturbed, and that after that reduction the endomorphism c_0^+ becomes a constant.

An element of $\mathbf{A}^+$ belongs to $\mathbf{I}^+$ if and only if it is dominated by the supremum of a finite set of generators. If $q \in \mathbf{I}^+$, so that

$$q \leqq \bigvee_{i=1}^{n} (\exists^+ p_i - c_0^+ \exists^+ p_i),$$

then, applying c_0^+ to both sides, we obtain $c_0^+ q = 0$. If, in particular, $q \in \mathbf{A}$ and $q^+ \in \mathbf{I}^+$, then $c_0^+ q^+ = 0$, and, since $c_0^+ q^+(x, y) = q(x)$, it follows that $q = 0$. If f^+ is the natural homomorphism from $\mathbf{A}^+$ onto $\mathbf{A}^+/\mathbf{I}^+$, the result we just obtained implies that f^+ is one-to-one on $\mathbf{A}$, i.e., that we may regard $\mathbf{A}$ as a monadic subalgebra of $\mathbf{A}^+/\mathbf{I}^+$ whenever it is convenient to do so.

Since, as shown in the preceding paragraph, c_0^+ maps $\mathbf{I}^+$ onto 0, it follows that if p and q are in $\mathbf{A}^+$ and if $f^+ p = f^+ q$, then $c_0^+ p = c_0^+ q$. This implies that c_0^+ maps cosets of $\mathbf{I}^+$ onto cosets of $\mathbf{I}^+$ and hence that c_0^+ may be regarded as a mapping of $\mathbf{A}^+/\mathbf{I}^+$ into itself. A routine verification shows that c_0^+ is, in fact, a constant of $\mathbf{A}^+/\mathbf{I}^+$, as promised.

An argument similar to the one just given shows that every constant of $\mathbf{A}^+$ can be transferred to $\mathbf{A}^+/\mathbf{I}^+$. Suppose indeed that c^+ is a constant of $\mathbf{A}^+$. If p and q are in $\mathbf{A}^+$ and if $f^+ p = f^+ q$, then $p + q \in \mathbf{I}^+$. Since $c^+(p + q) \leqq \exists^+(p + q)$ and $\exists^+(p + q) \in \mathbf{I}^+$, it follows that $c^+ p + c^+ q \in \mathbf{I}^+$ and hence that $f^+ c^+ p = f^+ c^+ q$. This implies that c^+ maps cosets of $\mathbf{I}^+$ onto cosets of $\mathbf{I}^+$ and hence that c^+ may be regarded as a mapping of $\mathbf{A}^+/\mathbf{I}^+$ into itself. The fact that c^+ is a constant of $\mathbf{A}^+/\mathbf{I}^+$ follows from the fact that f^+ is a monadic homomorphism. From this result and from what we said earlier about extending constants from $\mathbf{A}$ to $\mathbf{A}^+$, it follows that every constant of $\mathbf{A}$ has a natural extension to a constant of $\mathbf{A}^+/\mathbf{I}^+$. The proof of the transferability of c^+ to $\mathbf{A}^+/\mathbf{I}^+$ did not use any special properties of $\mathbf{A}^+$ and $\mathbf{I}^+$; we have proved, in fact, that a constant can always be transferred to a quotient algebra. Since the final algebra $\mathbf{A}_0^+$ that we shall construct will be a quotient algebra of $\mathbf{A}^+/\mathbf{I}^+$, as soon as we prove that the natural homomorphism from $\mathbf{A}^+/\mathbf{I}^+$ onto $\mathbf{A}_0^+$ does not disturb $\mathbf{A}$ (i.e., is one-to-one on $\mathbf{A}$), it will follow automatically that $\mathbf{A}_0^+$ has property (ii).

To construct $\mathbf{A}_0^+$ out of $\mathbf{A}^+/\mathbf{I}^+$, we simply force the identification of $c_0^+p_0^+$ with $\exists^+p_0^+$. Precisely speaking, we shall consider in $\mathbf{A}^+/\mathbf{I}^+$ the monadic ideal $\mathbf{I}_0^+$ generated by $\exists^+p_0^+ - c_0^+p_0^+$ (or, rather, by the coset of $\mathbf{I}^+$ containing that element), and we shall form the quotient algebra $\mathbf{A}_0^+ = (\mathbf{A}^+/\mathbf{I}^+)/\mathbf{I}_0^+$. In accordance with what was said above all that remains to be proved is that if an element q of $\mathbf{A}$ belongs to $\mathbf{I}_0^+$, then $q = 0$. An element of $\mathbf{A}^+/\mathbf{I}^+$ belongs to $\mathbf{I}_0^+$ if and only if it is dominated by the coset of $\exists^+p_0^+ - c_0^+p_0^+$. What we must prove reduces therefore to the following implication: if $q \in \mathbf{A}$ and if

$$q^+ \leqq \exists^+p_0^+ - c_0^+p_0^+,$$

then $q^+ \in \mathbf{I}^+$ (i.e., q^+ is equal to 0 modulo $\mathbf{I}^+$). In fact we can conclude that $q^+ = 0$. If $p_0 = 0$, this is trivial, since $q^+ \leqq \exists^+p_0^+$. If $p_0 \neq 0$, let x_0 be a point of X such that $p_0(x_0) = 1$, and form the infimum of both sides of the assumed inequality with $c_0^+p_0^+$. The result is that $q^+ \wedge c_0^+p_0^+ = 0$ and hence that $q(y) \wedge p_0(x) = 0$ for all x and y; the desideratum follows by setting x equal to x_0. The proof of Lemma 19 is complete.

Theorem 11. *Every monadic algebra is a subalgebra of a rich algebra.*

Proof. Repeated applications of Lemma 19 show that if $\mathbf{A}_0$ is a monadic algebra, then there exists a monadic algebra $\mathbf{A}_1$ including $\mathbf{A}_0$ as a monadic subalgebra and such that (i) $\mathbf{A}_1$ is a rich extension of $\mathbf{A}_0$ and (ii) every constant of $\mathbf{A}_0$ has an extension to a constant of $\mathbf{A}_1$. The applications of Lemma 19 have to repeated rather often, to be sure; precisely speaking, what is involved is transfinite induction. The witnesses to the elements of $\mathbf{A}_0$ are introduced one by one. The constants obtained at each stage are carried along to all subsequent stages, and, finally, to the union of the chain of algebras so obtained. The details are automatic.

The proof of the theorem consists of repeated applications of the result of the preceding paragraph. The applications have to be repeated countably often in this case; what is involved is elementary mathematical induction. Given $\mathbf{A}_n$, we denote by $\mathbf{A}_{n+1}$ a rich extension of it with the constant-extension-property. Thus the constants obtained at each stage are carried along to all subsequent stages, and, finally, to the union of the increasing sequence of algebras so obtained. Since that union is obviously a rich algebra, the proof of Theorem 11 is complete.

Constants play the same role in the theory of monadic algebras as homomorphisms into **O** play in the theory of Boolean algebras; in a (somewhat vague) sense a constant is "locally" a homomorphism into **O**. Theorem 11 is the monadic substitute for Stone's theorem on the existence of sufficiently many maximal ideals; the representation theorem of the next section is based on Theorem 11 almost the same way as Stone's representation theorem is based on the maximal ideal theorem.

16. Representation. It is a routine application of universal algebraic techniques (2) to put together the simplicity theorem (Theorem 6) and the semisimplicity theorem (Theorem 7) to obtain a representation theorem that exhibits every monadic algebra as a subdirect union of **O**-valued functional monadic algebras. The purpose of this final section is to discuss a stronger and more useful representation theorem that asserts, in effect, that the functional algebras with which we began the theory of monadic algebras exhaust all possible cases.

Theorem 12. *If* **A** *is a monadic algebra, then there exists a set* X *and there exists a Boolean algebra* **B**, *such that* (i) **A** *is isomorphic to a* **B**-*valued functional algebra* $\tilde{\mathbf{A}}$ *with domain* X, *and* (ii) *for every element* $\tilde{p}$ *of* $\tilde{\mathbf{A}}$ *there exists a point* x *in* X *with* $\tilde{p}(x) = \exists\tilde{p}(x)$.

Proof. The conclusions of the theorem are such that if they are valid for an algebra, then they are automatically valid for all its subalgebras. It follows from this comment and from Theorem 11 that there is no loss of generality in assuming that **A** is rich. This means that for each element p of **A** there exists at least one constant $c = c_p$ of **A** such that $cp = \exists p$; let X be a set of constants containing at least one such c for each p. Let the Boolean algebra **B** be the range $\exists(\mathbf{A})$ of the quantifier $\exists$ on **A**. Define a mapping f from **A** into $\mathbf{B}^X$, i.e., associate a function $\tilde{p} = fp$, from X into **B**, with every element p of **A**, by writing $\tilde{p}(x) = cp$. Since $\exists cp = cp$, the value $\tilde{p}(c)$ is indeed in **B** for every c in X, so that $\tilde{p} \in \mathbf{B}^X$. Since each c in X is a Boolean endomorphism on **A**, and since the Boolean operations in $\mathbf{B}^X$ are defined pointwise, a routine verification shows that f is a Boolean homomorphism. If $fp = 0$, i.e., if $\tilde{p}(c) = 0$ for all c in X, then, in particular, $\exists p = c_p p = \tilde{p}(c_p) = 0$, and therefore, $p = 0$; this proves that the homomorphism f is one-to-one.

Let $\tilde{\mathbf{A}}$ be the range of f, so that $\tilde{\mathbf{A}}$ is a Boolean subalgebra of

$\mathbf{B}^X$; it is to be proved that $\tilde{\mathbf{A}}$ is a functional monadic algebra and that f is a monadic isomorphism between $\mathbf{A}$ and $\tilde{\mathbf{A}}$. If $\tilde{p} = fp \in \tilde{\mathbf{A}}$, the range $\mathbf{R}(\tilde{p})$ of the function $\tilde{p}$ contains, in particular, the element $\exists p = c_p p = \tilde{p}(c_p)$ of $\mathbf{B}$; since $cp \leqq \exists p$ for every constant c, it follows that $\mathbf{R}(\tilde{p})$ has a largest element, and therefore a supremum, namely $\exists p$. This proves that $\exists \tilde{p}$ exists and has the value $\exists p$ at each c in X. On the other hand, $f\exists p(c) = c\exists p = \exists p$ for all c, so that $\exists fp = \exists \tilde{p} = f\exists p$; the proof of the theorem is complete.

It is instructive to observe that the simplicity and semisimplicity theorems can be recaptured from this general representation theorem. The purpose of the following considerations is to show how that can be done.

Lemma 20. *If* $\mathbf{A}$ *is a* $\mathbf{B}$*-valued functional monadic algebra with domain* X*, such that for every* p *in* $\mathbf{A}$ *there exists a point* x *in* X *with* $p(x) = \exists p(x)$*, and if* f_0 *is a Boolean homomorphism from* B *into* $\mathbf{O}$*, then the mapping* f*, from* $\mathbf{A}$ *into* $\mathbf{O}^X$*, defined by* $fp(x) = f_0(p(x))$*, is a monadic homomorphism.*

Proof. The fact that f is a Boolean homomorphism is an easily verified consequence of the fact that f_0 is such, and of the fact that the Boolean operations in $\mathbf{O}^X$ are defined pointwise. The proof of the fact that f is a monadic homomorphism is also similar to the corresponding part of the proof of Theorem 12. Indeed, if $p_0 \in \mathbf{A}$, and if x_0 is a point such that $p(x_0) = \exists p(x_0)$, then $\mathbf{R}(fp)$ contains $f_0(p(x_0))$ as its largest element, and therefore $\exists fp(x) = f_0(p(x_0))$ for all x. Since, on the other hand, $\exists p$ is a constant function, so that

$$f\exists p(x) = f_0(\exists p(x)) = f_0(\exists p(x_0)) = f_0(p(x_0)),$$

it follows that $\exists fp = f\exists p$, as desired.

It is now an easy matter to give an alternative proof of the deeper half of the simplicity theorem. In view of Theorem 12, there is no loss of generality in restricting attention to a functional monadic algebra $\mathbf{A}$ that satisfies the condition of Lemma 20. If $\mathbf{A}$ is simple, then select the f_0 in Lemma 20 arbitrarily; since, by simplicity, the homomorphism f so obtained has a trivial kernel, it follows that f is an isomorphism.

Once the simplicity theorem is known, the assertion of semisimplicity takes the following form: if $p \in \mathbf{A}$ and $p \neq 0$, then there exists a monadic homomorphism f from $\mathbf{A}$ into an $\mathbf{O}$-valued functional algebra, such that $fp \neq 0$. For the proof, find x in X so that $p(x) \neq 0$, and (by the semisimplicity theorem for ordinary

Boolean algebras) find f_0 so that $f_0(p(x)) \neq 0$; the f obtained from Lemma 20 satisfies the stated condition.

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