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The geometry of the heat equation

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1. Introduction.

In this paper, the powerful methods of differential geometry are applied to the equivalence problem for n-dimensional generalized heat equations. The general line of attack is the same as that of a previous paper [1]) in which the geometry of the generalized Laplace equation was investigated. Here, however, the situation is slightly more complicated due to the presence of the extra variable time.

Why geometry in the subject of partial differential equations? The answer is, that in the equivalence problem, we are confronted with an array of dissimilar looking differential equations of the same general type, and we ask whether there exists a transformation of the group of transformations preserving this general type which will transform one of two given equations into the other. The criterion for this expresses itself naturally in terms of the invariants of the equations, those expressions which are formally unchanged throughout the group of transformations of these equations. And finally, the theory of the invariants of any group is the province of geometry.

In Section 2 the general theory is worked out, culminating in the equivalence theorem under the restricted heat equation transformation group. The theorem solves the equivalence problem for this group, i.e., gives a finite criterion (that is, in terms of finite equations) for this equivalence. It is remarkable here that there is no corresponding theorem for the full heat equation transformation group. A useful corollary gives the criterion that an equation be reducible to an ordinary heat equation. In Section 3, the theory is applied to a very simple heat equation arising in the theory of

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[1]) "The Geometry of the Linear Partial Differential Equation of the Second Order", Am. Journ. Math., 75, 691 (1953).

random walks with variable but isotropic step. The class of all such random walks which can be reduced to ordinary (constant step) random walks by a transformation of space is characterized by the solutions of a certain set of non-linear second order partial differential equations for the function of position defining the length of the (isotropic) step at that position. These equations are solved, the solutions exhibiting a simple algebraic form, of degree at most quadratic, for $n \neq 2$, and a more complicated form, defined by any harmonic function, for $n = 2$.

2. The heat equation geometry and the equivalence theorem.

We write the generalized heat equation in the form

$$g^{rs}\partial^2_{rs}\varphi + d^r\partial_r\varphi + \partial_t\varphi = 0 \tag{2.1}$$

$$\left[\begin{array}{c}\text{Det } g^{rs} \neq 0;\ r, s = 1, \ldots, n \\ \partial_r \equiv \partial/\partial x^r,\ \partial_t = \partial/\partial_t\end{array}\right]$$

in which it has been normalized by dividing the equation through by the (non-zero) coefficient of the time-derivative term. The summation convention is observed. g^{rs} and d^r are functions of x^r and t. The classical heat equation has the form (2.1) in which

$$g^{rs} = k\delta^{rs}(\delta^{rr} = +1;\ \delta^{rs} = 0,\ r \neq s),\ d^r = 0,$$

k a non-zero constant.

The application of differential geometric methods to these equations depends on the extraction of certain invariants formed from their coefficients. To this end it is convenient to write the first two terms of (2.1) as a covariant divergence:

$$\nabla_r(g^{rs}\partial_s\varphi) + \partial_t\varphi = 0 \tag{2.1'}$$

in which the covariant divergence $\nabla_r V^r$ of any vector V^r (here $g^{rs}\partial_s\varphi$) is defined

$$\nabla_r V^r = \partial_r V^r + V^s \Gamma^r_{rs}$$

with the linear connection Γ^p_{rs} given in this case by

$$\Gamma^p_{rs} = \left\{ {p \atop rs} \right\} - 1/2(F_r\delta^p_s + F_s\delta^p_r - F^p g_{rs}) \tag{2.2}$$

and

$$F^r \equiv -2/n(d^r + g^{pq}\left\{ {r \atop pq} \right\}),\ F_r \equiv g_{rs}F^s \tag{2.3}$$

In these formulas, δ^p_s is the Kronecker delta and $\left\{ {p \atop rs} \right\}$ the Christoffel symbol of the normalized cofactors g_{rs} of g^{rs}:

$$\left\{ {p \atop rs} \right\} \equiv 1/2 g^{pq}(\partial_r g_{qs} + \partial_s g_{qr} - \partial_q g_{rs})$$

That (2.1′) is identical with (2.1) in virtue of the definition of the covariant divergence and the definitions (2.2) and (2.3) can be verified by direct substitution. A connection Γ^p_{rs} of the form (2.2) is called a Weyl linear connection, and we have shown incidentally that a Weyl linear connection is always a general enough connection to write the heat equation in the covariant form (2.1′).

The equation (2.1) maintains its form under several groups of transformations. First, under the direct product of the two groups of transformations of variables

$$\begin{aligned} &\text{a.}\quad x^{m'} = f^{m'}(x^p, t) \\ &\text{b.}\quad t' = T(t) \end{aligned} \tag{2.4}$$

Note that although the new space variables $x^{m'}$ are allowed to be functions of the old time t, the new time t' cannot involve the old space variables, or we would get second derivatives in the new time. Second, as to the group of gauge transformations — multiplication of the whole equation by a non zero factor $\lambda(x^m, t)$ — this group has been eliminated by the normalization to the canonical form in which the coefficient of $\partial_t \varphi$ is unity.

After a transformation (2.4), eq. (2.1′) goes into another of the same form in $\bar{g}^{r's'}$ and $\bar{F}_{r'}$, where

$$\begin{aligned} \bar{g}_{r's'} &= F g_{rs} A^r_{r'} A^s_{s'}; \qquad A^r_{r'} = \partial_{r'} x^r,\ F = \frac{dt'}{dt} \\ \bar{F}_{r'} &= A^r_{r'}(F_r - 2/n A_r); \qquad A_r = g_{rs} A^s_{s'} \partial_t x^{s'} \end{aligned} \tag{2.5}$$

The „tensors" g_{rs} and F_r with the transformation rules (2.5) under the group (2.4) characterize completely the equation (vide (2.1)). Defined over an n-dimensional manifold of space coordinates x^m and a 1-dimensional manifold of time values t, they serve to define the *n-dimensional heat equation geometry.* The equivalence problem for equations (2.1) is then reduced to the equivalence problem for n-dimensional heat equation geometries.

As it now stands, the equivalence problem is a problem in analysis: Given the two sets of functions $\bar{g}_{r's'}$, $\bar{F}_{r'}$, functions of $x^{m'}$ and t' and g_{rs}, F_r, functions of x^m and t, we ask whether there exists a transformation (2.4) such that their differential coefficients satisfy (2.5). By expressing the integrability conditions of (2.5), we can reduce the problem to a question of the existence of solutions of *finite* (e.g., often algebraic) equations — in the case that a complete set of invariants of the heat equation geometry exists. By complete set of invariants is meant an array of tensors in terms of which the equivalence problem can be completely stated in this finite form.

It turns out that no complete set of invariants exists for this geometry. However, if we restrict attention to the subgroup

$$\begin{aligned} &\text{a.} \quad x^{m'} = f^{m'}(x^p) \\ &\text{b.} \quad t' = T(t) \end{aligned} \tag{2.6}$$

of (2.4), in which space and time transform separately, the resulting geometry, the restricted heat equation geometry, admits a complete set of invariants. Under (2.6) the basic invariants transform:

$$\begin{aligned} &\text{a.} \quad \bar{g}_{r's'} = F g_{rs} A^r_{r'} A^s_{s'} && \text{c.} \quad A^r_{r'} = \partial_{r'} x^r \\ &\text{b.} \quad \bar{F}_{r'} = A^r_{r'} F_r && \text{d.} \quad F = dt'/dt \end{aligned} \tag{2.7}$$

We proceed to solve the equivalence problem for this group, that is, to express the integrability conditions of the set of partial differential equations (2.7).

Eq. (2.7)a. differentiated with respect to $x^{p'}$ and rearranged, using the inverses $\bar{g}^{r's'}$ and g^{rs} to eliminate F, gives

$$\partial_{p'} A^r_{r'} + A^s_{r'} A^p_{p'} \begin{Bmatrix} r \\ ps \end{Bmatrix} - A^r_{s'} \overline{\begin{Bmatrix} s' \\ p'r' \end{Bmatrix}} = 0 \tag{2.8}$$

where $\begin{Bmatrix} r \\ ps \end{Bmatrix}$ is the Christoffel symbol of g_{pq} and $\overline{\begin{Bmatrix} s' \\ p'r' \end{Bmatrix}}$, that of $\bar{g}_{p'q'}$. (2.7)a. differentiated with respect to t gives the equivalent set

$$\begin{aligned} &\text{a.} \quad d/dt \log F - 2/n(F\bar{Q} - Q) = 0 \\ &\text{b.} \quad \bar{Q}^{r'}_{s'} A^r_{r'} = F^{-1} Q^r_s A^s_{s'} \end{aligned} \tag{2.8'}$$

where

$$\begin{aligned} &\text{c.} \quad Q = 1/2 \partial_t g_{pq} g^{qp} \\ &\text{d.} \quad Q^r_s = 1/2 \partial_t g_{sp} g^{pr} - 1/n \delta^r_s Q \end{aligned}$$

and the corresponding definitions hold for the primed barred quantities [2]). Note the identity $Q^s_s = 0$, so that (2.8′) comprises only $1n/2(n+1)$, not $1n/2(n+1)+1$, independent equations.

The integrability conditions of (2.7)c. are satisfied identically in virtue of (2.8) (by the symmetry of the Christoffel symbols) and of (2.8′)a. and b. (for the latter can be re-arranged to give $dA^r_{r'}/dt = 0$). The integrability condition of (2.7)d. is satisfied in virtue of (2.8), for this equation implies $\partial_{p'}F = 0$.

The integrability conditions of (2.8) are

$$\text{a.}\quad \overline{R}_{q'p'r'}{}^{s'}A^s_{s'} = R_{qpr}{}^{s}A^q_{q'}A^p_{p'}A^r_{r'} \tag{2.9}$$

where $R_{qpr}{}^{s}$ (the curvature tensor of $\left\{ {r \atop ps} \right\}$) is short for [3])

$$\text{b.}\quad R_{qpr}{}^{s} = -2\partial_{[q}\left\{ {s \atop p]r} \right\} - 2\left\{ {s \atop l[q} \right\}\left\{ {l \atop p]r} \right\},$$

and

$$A^s_{r'}A^p_{p'}\partial_t\left\{ {r \atop ps} \right\} = FA^r_{s'}\partial_{t'}\overline{\left\{ {s' \atop p'r'} \right\}} \tag{2.9′}$$

The integrability condition of (2.8′)a. is

$$F\partial_{p'}\overline{Q} = A^p_{p'}\partial_p Q \tag{2.10}$$

In addition, all the equations which follow from the equations (2.7)b., (2.8′)b., (2.9)a., (2.9′), and (2.10) by differentiating with respect to space and time and using the equations (2.7)c. and d., (2.8), and (2.8′)a., to eliminate the derivatives of the unknowns, must be satisfied. We shall illustrate this process on Eq. (2.7)b.; the procedure is typical.

Differentiating (2.7)b. with respect to $x^{p'}$, rearranging, and using (2.7)c. and (2.8) we get

$$\text{(2.7)b:}\qquad [S_1, T_0]\quad \overset{g}{\nabla}_{p'}\overline{F}_{r'} = A^p_{p'}A^r_{r'}\overset{g}{\nabla}_p F_r$$

where $\overset{g}{\nabla}_p$ means covariant derivative with respect to the Christoffel symbols $\left\{ {r \atop pq} \right\}$, and $\overset{g}{\nabla}_{p'}$, with respect to $\overline{\left\{ {r' \atop p'q'} \right\}}$. The notation $[S_1, T_0]$ means that we have differentiated (2.7)b., *once* with

[2]) Hereafter usually only the definitions of the invariants of the one equation will be given, the primed barred invariants being the corresponding expressions in the primed barred quantities.

[3]) The square brackets [ ] around any set of indices indicates the alternating part: e.g., $T_{[pq]} \equiv 1/2(T_{pq} - T_{qp})$.

respect to space and *zero* times with respect to time. Continuing this process, after j space derivations we get

$$(2.7)\text{b: } [S_j, T_0] \overset{g}{\nabla}{}^j_{p'_1 \dots p'_j} \overline{F}_{r'} = A^{p_1}_{p'_1} A^{p_2}_{p'_2} \dots A^{p_j}_{p'_j} A^r_{r'} \overset{g}{\nabla}{}^j_{p_1 \dots p_j} F_r$$

Differentiating (2.7)b. k times with respect to t', rearranging, and using (2.7)d. and (2.8′)a. to eliminate derivatives of the unknowns, we get, similarly

$$(2.7)\text{b.: } \qquad [S_0, T_k] \; \nabla^k_{t' \dots t'} \overline{F}_{r'} = F^{-k} A^r_{r'} \nabla^k_{t \dots t} F_r$$

where the differential operator $\nabla^k_{t \dots t}$ (k-fold covariant differentiation with respect to time) applied to any tensor is defined recursively

$$\nabla^k_{t \dots t} = (\partial_t + (k-1)2/nQ)\nabla^{k-l}_{t \dots t}, \; \nabla^0 \equiv \times 1 \; (k = 1, 2, \dots)$$

where Q is given by (2.8′)c.

Combining these two operations, we get

(2.7)b: $[S_j, T_k]$

$$\nabla^j_{p'_1 \dots p'_j}(\nabla^k_{t' \dots t'} \overline{F}_{r'}) = F^{-k} A^{p_1}_{p'_1} A^{p_2}_{p'_2} \dots A^{p_j}_{p'_j} A^r_{r'} \overset{g}{\nabla}_{p_1 \dots p_j}(\nabla^k_{t \dots t} F_r)$$

Of course the operations of covariant space and time derivation do *not* commute. However, we remark that for each j, k it is sufficient to impose only one equation with the derivatives taken in some arbitrary order, for the difference between such an equation and the equation with the differentiations performed in any other order vanishes in virtue of a previous equation. E.g., consider the difference between (2.7)b.: $[T_1, S_1]$ and: $[S_1, T_1]$ (where the symbol S_1 to the right indicates that one space differentiation is to be performed first, etc.). We get

$$\overline{F}_{s'} \partial_{t'} \overline{\left\{ \begin{matrix} s' \\ p'r' \end{matrix} \right\}} = F^{-1} F_s \partial_t \left\{ \begin{matrix} s \\ pr \end{matrix} \right\} A^p_{p'} A^r_{r'}$$

But this is satisfied in virtue of (2.7)b. and (2.9′).

We can now state the equivalence theorem:

THEOREM 1. Two equations (2.1), characterized by the invariants g_{rs} and F_s, functions of x^m and t, and $\overline{g}_{r's'}$, $\overline{F}_{s'}$, functions of $x^{m'}$ and t' respectively, are equivalent under the restricted heat equation group (2.6) if and only if the sets of equations (2.7)a and b, (2.8′)b, (2.9), (2.9′), (2.10), and the infinite sets: $[S_j, T_k]$ ($j, k = 1, 2, \dots, \infty$) derived from each of these are satisfied by some set of functions x^p, $A^r_{r'}$ of $x^{m'}$ and t, F of t'.

Of course, the equations of this infinite set are usually incompatible — the theorem asserts that in the special case that they are compatible, the equivalence exsist. In proof we appeal to the well known theorem on partial differential equations [4]).

In case that the coefficients are functions only of space variables, the theorem is simplified by the dropping away in the criterion of all equations involving time derivatives, namely (2.9′) and all the equations: $[S_j, T_k\}$ for $k > 0$. From the remaining equations it follows in particular that F must be a constant, or $t' = Ft$, $F =$ const. is the only time transformation possible here.

A case of especial interest arises when one of the equations is an ordinary heat equation, whose invariants are

$$g_{rs} = C_{rs} = \text{constants}; \quad F_r = 0 \tag{2.11}$$

Note that we allow any signature of C_{rs} in the definition of an ordinary heat equation; the classical case (cf. p. 3) is the signature $(+, +, \ldots +)$ or $(-, -, \ldots -)$, depending on the sign of k, the conductivity. For this equation, all the differential invariants vanish. We get the

Corollary 1. A heat equation (2.1) is equivalent to an ordinary heat equation if and only if its intrinsic geometry is flat.

By *flat* we mean that

Def. 1. $F_r = 0,\ Q^r_s = 0,\ R_{qpr}{}^s = 0,\ \partial_t \left\{ {r \atop qp} \right\} = 0,\ \partial_p Q = 0$

Then (2.7)a. with C_{rs} substituted for g_{rs} determines $\frac{n(n+1)}{2}$ of the available constants. For an equation (2.1) to be equivalent to the classical heat equation, it is necessary in addition that g_{rs} have definite (negative or positive) signature.

Make the

Def. 2. A (restricted) heat equation geometry is called *Riemannian* if $F_r = 0$.

This is of course an invariant demand, by (2.7)b. The Riemannian (restricted heat equation) geometries form an important subclass. For them, the equivalence problem reduces to a consideration of the metric g_{rs} and its various differential invariants.

Generalization: A trivial generalization of the above theory consists in letting the equation (2.1) have an extra term $+ d\bar{\varphi}$. The

[4]) See, say, Veblen, *Invariants of Quadratic Differential Forms.* (Cambridge University Press, 1952) p. 73.

modifications this brings to the equivalence theorem are obvious. The equations to be adjoined are

$$\bar{d} = F^{-1}d \tag{2.12}$$

and the set (2.12): $[S_j, T_k]$ ($j, k = 1, 2, \ldots \infty$) derived from it by successive space and time differentiation.

Remark on solutions: The above theory answers the equivalence question completely. It will say something about solutions insofar as this can be phrased as an equivalence question. E.g., suppose we are looking for a solution φ of an equation (2.1) in the space and time variables x^m and t, and we know by the above theory that this equation is equivalent to another (in $x^{m'}$ and t') if which we know a solution $\Psi(x^{m'}, t')$. Then a solution of the first equation is given by

$$\varphi(x^p, t) \equiv \Psi(x^{m'}(x^p), t'(t))$$

where $x^{m'} = f^{m'}(x^p)$, $t' = f(t)$ is a transformation taking the one equation into the other.

3. An application of the theory.

In the theory of random walks in n-dimensional number space with a position-dependent isotropic step one encounters the equation

$$\partial_t P = 1/2 \sum_{j=1}^{n} \partial_j [v^{2-n} \partial_j (v^n P)] \tag{3.1}$$

where v is a function of $x^1, \ldots, x^n$.

Making the substitution

$$\varphi \equiv v^n P$$

and writing it in the canonical form (2.1), we find the coefficients to be

$$g^{jk} = -1/2 v^2 \delta^{jk}, \quad d^j = \frac{n-2}{4} \partial_k v^2 \delta^{kj}$$

where δ^{jk} has been defined on p. 3. The Christoffel symbols come out to be

$$\left\{ {r \atop pq} \right\} = -1/2 (\partial_p V \delta^r_q + \partial_q V \delta^r_p - \partial_s V \delta^{sr} \delta_{pq});$$

$$V \equiv \log v^2 \tag{3.2}$$

With the help of these, the basic heat equation invariants are computed to be

$$g_{jk} = -\frac{2}{v^2} \delta_{jk}; \quad F_k = 0 \tag{3.3}$$

It is remarkable that F_k vanishes identically for this equation. The intrinsic geometry is then Riemannian in the sense of definition 2 of last section.

We now ask: *For what* v *is* (3.1) *equivalent to an ordinary heat equation in* φ? By Corollary 1 of last section this is so if and only if

$$R_{pqs}{}^{r} = 0 \tag{3.4}$$

the other conditions being satisfied identically. That is, if and only if the curvature tensor of the metric (3.3) is zero. Computing $R_{pqrt} \equiv R_{pqr}{}^{s}\delta_{st}$ from (3.2), we get

$$R_{pqrt} = [\partial^2_{s[p}V - 1/4(\nabla V)^2\delta_{s[p} + 1/2\partial_s V\partial_{[p}V]\delta_{q]t} - (s \leftrightarrows t) = 0 \tag{3.5}$$

where $(\nabla V)^2 \equiv \delta^{lm}\partial_l V\partial_m V$ and we have indicated the skew part in s and t by a departure from the standard notation with the square brackets to avoid confusion with the square brackets around p and q. Eq. (3.5) can be reduced to simpler form in the following way. Forming the Ricci tensor $R_{qr} \equiv R_{pqrt}\delta^{pt}$ from (3.5), we infer

$$(n-2)\{\partial^2_{qr}V - 1/2(\nabla V)^2\delta_{qr} + 1/2\partial_q V\partial_r V\} + \nabla^2 V\delta_{qr} = 0 \tag{3.5'}$$

where $\nabla^2 V$ is the Laplacian: $\nabla^2 V \equiv \delta^{lm}\partial^2_{lm}V$. Then for $n \neq 2$ we get, rearranging

$$\partial^2_{qr}V - 1/4(\nabla V)^2\delta_{qr} + 1/2\partial_q V\partial_r V = [1/4(\nabla V)^2 - \frac{1}{n-2}\nabla^2 V]\delta_{qr} \tag{3.6}$$

The left member is the content of the brackets in (3.5). Substituting in (3.5), we infer that the coefficient of δ_{qr} in the last equation must vanish; from the last equation it then follows that

$$\partial^2_{qr}V - 1/4(\nabla V)^2\delta_{qr} + 1/2\partial_q V\partial_r V = 0. \tag{3.7}$$

But (3.7) then implies (3.5). Hence (3.7) is equivalent to (3.5).

When the substitution $V = \log v^2$ is made in (3.7), it simplifies to

$$2v\partial^2_{pq}v - (\nabla v)^2\delta_{pq} = 0 \tag{3.8}$$

These are equivalent to

$$\begin{aligned} &\text{a.} \quad \partial^2_{pq}v = 0 \quad (p \neq q) \\ &\text{b.} \quad \partial^2_{pp}v = 1/2v(\nabla v)^2 \quad (p = 1, \ldots, n) \end{aligned} \tag{3.8'}$$

These can be integrated, giving the general solution

$$\begin{aligned} &\text{a.} \quad v = c/2\delta_{pq}x^p x^q + d_p x^p + 1/2c\delta^{pq}d_p d_q \quad (c \neq 0) \\ &\text{b.} \quad v = d_p x^p + e \quad (\delta^{pq}d_p d_q) = 0 \end{aligned} \tag{3.9}$$

In the first solution c and d_p are arbitrary constants such that

$c \neq 0$. In the second, d_p and e are constants, arbitrary up to the null length condition on the d_p.

Thus, for $n \neq 2$, *the necessary and sufficient condition that* (3.1) *be reducible to the classical heat equation in* $\varphi \equiv v^n P$.

$$- \nabla^2 \varphi + \partial_t \varphi = 0 \tag{3.10}$$

is that v *have the form* (3.9)a. *or* b. We remark that the trivial case $v =$ const. is included in the solutions b. with $d_p = 0$.

For $n = 2$, all the non-vanishing components of R_{pqrt} are the same as R_{1212} up to sign. Hence (3.5) is equivalent to (3.5′) which for $n = 2$ reduces simply to

$$\nabla^2 V = 0 \tag{3.11}$$

Hence the general solution for v is

$$v = e^w \quad (\nabla^2 w = 0);$$

i.e., v is Exp {a harmonic function}.

Therefore, *for* $n = 2$, *the necessary and sufficient condition that* (3.1) *be reducible to the classical heat equation* (3.10) *in* $\varphi \equiv v^2 P$ *is that* v *equal* e^w *where* w *is some harmonic function.*

Remark: Since the invariants here are no functions of t, by a remark of last section the time transformation can be at most

$$t' = Ft, \quad F = \text{const.}$$

But since the only equation determining the integration constants is (2.7)a., F can be taken positive, and then absorbed into the transformation of the space variables. Hence „reducible" in the above two theorems about v can be taken to mean reducible by a transformation of space variables alone.