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The boundary layer problem for certain non-linear ordinary differential equations

by

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Introduction.

The theory of buckling of circular plates leads to a boundary value problem for a pair of non-linear ordinary differential equations of second order depending upon a parameter. When this parameter approaches zero as a limit, the solution will approach a limit function which is no longer satisfied by all boundary conditions. The non-uniform convergence in the "boundary layer" can be studied by an appropriate stretching process.

These phenomena have so far been treated only for the special equations resulting from the theory of plates, in particular in [1]. It is the intention of this paper to establish similar phenomena for differential equations of a simpler type, which lend themselves more easily to generalization.

The differential equations considered are

$$\begin{aligned} p_{xx} &= \tfrac{1}{2}q^2 \\ kq_{xx} + pq &= 0 \end{aligned}$$

where p and q are functions of x defined, without loss of generality, in the interval $-1 \leqq x \leqq 1$, with the associated boundary conditions

$$p(-1) = p_1,\ p(1) = p_2,\ q_x(-1) = 0,\ q_x(1) = 0,$$

p_1, p_2 being given constants.

We shall distinguish three cases:

$$\text{I: } \begin{matrix} p_1 > 0 \\ p_2 > 0 \end{matrix} \qquad \text{II: } \begin{matrix} p_1 > 0 \\ p_2 < 0 \end{matrix} \qquad \text{III: } \begin{matrix} p_1 < 0 \\ p_2 < 0 \end{matrix}$$

We are interested in what happens to the solution $p(x)$ as the parameter k approaches zero. We might expect that as $k \to 0$, p and q would approach limit functions satisfying the same boundary conditions and the differential equations obtained by putting $k = 0$ in the original equations. This is not true. There is indeed a limit function and it does solve the so-called limit equations, but it does not always satisfy the boundary conditions. More specifically, we should expect that the boundary condition on q is lost in the limit, for q_{xx} disappears in the limit equation, and so it is reasonable that the boundary condition on q should no longer be met. This is generally so; that in our case the boundary condition for q is assumed after all is quite accidental, due to the simplicity of the particular problem. It turns out that q_x does not converge uniformly, although q does.

However, the interesting result is that the boundary condition on p may be lost, although the limit differential equation remains of the same order in p. We shall show that if the original boundary value is negative, the limit function will still assume this value; however, if the assigned value is positive, it will no longer satisfy the limit function—instead, the function will take on that value, multiplied by a constant, —.47271. From these remarks, it is clear that the limit solution is $q \equiv 0$, $p =$ a linear function, determined by these new boundary values.

In order to establish these facts, we must investigate the details of the non-uniform convergence. To do this, we introduce a stretching transformation in which the new variable depends upon the parameter, which at the same time no longer occurs explicitly in the differential equations. This follows the procedure of [1]. Considering q now as a function of the new variable, it does converge, the second order terms in the differential equations are not lost as k approaches 0, and both boundary conditions are satisfied at the fixed end point; but here the difference is that the interior region as well as the other end is pushed out to infinity. How it behaves there we settle by the methods of the calculus of variations. The result is that if $p_1 > 0$ (assuming the left end to be the unstretched one), the limit function assumes the prescribed value at the unstretched end, and $-.47271p_1$ at the stretched end; if $p_1 < 0$, the limit function is the constant p_1 (cf. Fig. 1b). It is a strange paradox that the limit value at the stretched end should be the value assumed by the limit function at the unstretched end.

The limit function in the stretched variable satisfies the same differential equations and boundary conditions as the function in [1], so that in this paper the author has been able to use many of the numerical calculations made there. However, this paper supplements that work by a proof of the convergence of the power series expansion of the limit function in the stretched variable. Use is also made here of the results in [1] of the stretched limit process to investigate the limit process in the unstretched variable, the situation again being similar to [1], § 10. However, certain complications arise here, due to the boundary layer at both ends in the problem discussed in this paper.

Specifically, our program will be as follows: with reference to Case I (both boundary values positive), we formulate the problem in terms of functionals (§ 1). This formulation enables us to apply the methods of the calculus of variations, and prove uniqueness and existence theorems (§§ 2, 3). Employing now the stretching procedure, we next investigate the asymptotic behavior of the solutions (§ 4). The limit solution in the stretched variable is now expanded in a power series which is proved convergent (§ 5). The information thus gained enables us to return to the original variables and discuss the limit state in the interior (§ 6). Here we find an explicit representation for the limit solution in terms of the unstretched variables. We conclude the paper with a dicussion of Cases II and III (§ 7).

We close these introductory remarks with several figures. illustrating the various cases.

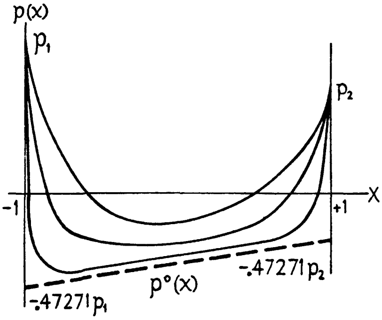

Fig. 1a.

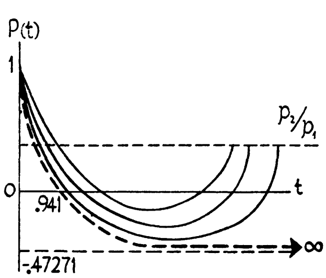

Fig. 1b.

Figure 1a illustrates Case I. Several curves $p^k(x)$ are shown, for varying values of $k \neq 0$. The limit solution $p^0(x)$ [cf. Th. 6.2, P. 162] is represented by the dotted line. We note the non-uniform convergence, and the region of rapid change moving toward the extremities of the interval as $k \to 0$.

In Figure 1b we see the corresponding situation in the stretched variable. $P^0(t)$, shown as a broken line curve, approaches the value —.47271 as $t \to \infty$; the end points of $P^k(t)$, at $t = a$, move toward the right as $k \to 0$, i.e., as $a \to \infty$. The convergence here is uniform in every finite interval.

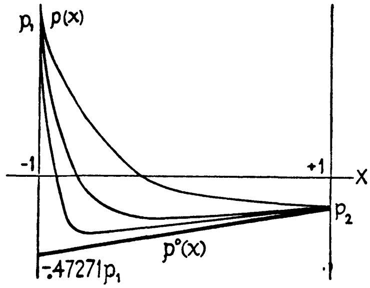

Fig. 2.

Figure 2 illustrates Case II. The limit function $p^0(x)$ [cf. Th. 7.1, P. 167] here is satisfied by the right hand boundary value, so that it is approached non-uniformly by $p^k(x)$ only on the left side, i.e., we have here a boundary layer phenomenon on one side only. A figure for the stretched variable in this case would resemble Fig. 1b, except that the right end points of the several $P^k(t)$ would be below the axis, since $P^k(a) = p_2/p_1$ is < 0 here.

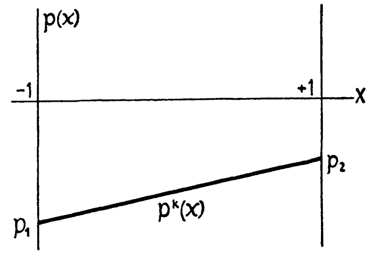

Fig. 3.

In Case III, $p^k(x)$ is the same linear function for all values of k, including $k = 0$. This is illustrated in Figure 3.

§ 1. Formulation of the problem in terms of Functionals.

In order to investigate the existence and uniqueness of solutions of our problem, it is convenient and useful to formulate the problem in a new way. Accordingly, we introduce the functionals

$$H^k(q) = \tfrac{1}{2}\int_{-1}^{1} f(x)q^2(x)dx \tag{1.01}$$

$$D^k(q) = k\int_{-1}^{1} q_x^2(x)dx \tag{1.02}$$

$$\begin{aligned} K^k(q) &= \int_{-1}^{1} p_x^2(x)dx - \tfrac{1}{2}(p_2 - p_1)^2 \\ &= \int_{-1}^{1} [p_x(x) - \tfrac{1}{2}(p_2 - p_1)]^2 dx, \end{aligned} \tag{1.03}$$

where $p_x(x)$ is a functional in q through

$$4p_x(x) = \int_{-1}^{x} q^2(\tilde{x})d\tilde{x} - \int_{x}^{1} q^2(\tilde{x})d\tilde{x} + \int_{-1}^{1} \tilde{x}q^2(\tilde{x})d\tilde{x} + 2(p_2 - p_1), \tag{1.04}$$

or, alternately,

$$p_x(x) = p_x(1) - \tfrac{1}{2}\int_{x}^{1} q^2(\tilde{x})d\tilde{x}, \tag{1.04a}$$

and where

$$f(x) = (p_2 + p_1) + x(p_2 - p_1). \tag{1.05}$$

The functional to be minimized is

$$(1.06)\qquad W^k(q) = D^k(q) - H^k(q) + K^k(q) = \int_{-1}^{1} [kq_x^2 - \tfrac{1}{2}fq^2 + p_x^2]dx - \tfrac{1}{2}(p_2 - p_1)^2$$

By *admissible functions* we mean functions $q(x)$ continuous in $-1 \leqq x \leqq 1$, with L^2-integrable drivatives in the same interval. The *minimum problem*, M^k, is that of minimizing $W^k(q)$; the problem S^k is that of making $W^k(q)$ stationary, in each case with respect to admissible functions q^k. The *boundary value problem*, B^k, requires the determination of an admissible function $q(x)$ possessing a continuous second derivative, and satisfying the differential equation

$$(1.07)\qquad kq_{xx} + pq = 0$$

and the boundary conditions

$$(1.08)\qquad q_x(-1) = 0, \quad q_x(1) = 0.$$

The function $p(x)$ in (1.07) is defined by

$$(1.09)\qquad p(x) = p_1 + \int_{-1}^{x} p_{\tilde{x}}(\tilde{x})d\tilde{x}$$

where $p_x(x)$ is given by (1.04). The function p therefore satisfies the differential equation

$$(1.10)\qquad p_{xx} = \tfrac{1}{2}q^2$$

and the boundary conditions

$$(1.11)\qquad p(-1) = p_1, \quad p(1) = p_2.$$

The first of (1.11) is immediately evident from (1.09); the second results from substituting (1.04) in (1.09) and simplifying.

The connection between the problems S^k and B^k is based on two "Green's" formulas. They refer to the first variation

$$(1.12)\qquad \delta W = 2\int_{-1}^{1} [kq_x\delta q_x - \tfrac{1}{2}fq\delta q + p_x\delta p_x]dx,$$

where δp_x is defined, in accordance with (1.04a), by

$$(1.13)\qquad \delta p_x = \delta p_x(1) - \int_{x}^{1} q(\tilde{x})\delta q(\tilde{x})d\tilde{x}.$$

Using product integration and the fact that $p_{xx} = \tfrac{1}{2}q^2$, we may write (1.12) as our first Green's formula:

$$(1.14)\qquad \delta W^k(q) = 2\int_{-1}^{1} [kq_x\delta q_x - pq\delta q]dx.$$

If q possesses a continuous second derivative, we may again employ product integration, this time to simplify the integrand $q_x \delta q_x$. We thus obtain

$$(1.15) \qquad \delta W^k(q) = -2\int_{-1}^{1} (kq_{xx} + pq)\delta q\,dx + 2k[q_x(1)\delta q(1) - q_x(-1)\delta q(-1)],$$

our second Green's formula, which holds for all admissible functions q possessing continuous second derivatives. Formula (1.15) yields immediately

THEOREM 1.1: *A solution of B^k solves S^k.*

The converse also holds, as we shall now prove.

THEOREM 1.2: *A solution of S^k ($\therefore$ also a solution of M^k) possesses a continuous second derivative and solves B^k.*

To prove Theorem 1.2, we make use of the following

LEMMA: Let $R(x)$ be an L^2-integrable function, and $S(x)$ be a continuous function such that

$$\int_{-1}^{1} RT_x\,dx = \int_{-1}^{1} ST\,dx$$

holds for all continuous functions $T(x)$ with L^2-integrable derivatives which vanish identically in the neighborhood of $x = -1$ and $x = 1$. Then $R(x)$ coincides (almost everywhere) with a function R^*which possesses the continuous derivative $-S$. We note in addition that $R^* = R$ in case R is the derivative of a continuous function.

We apply this Lemma to $R = kq_x$, $S = pq$, where q is a solution of S^k. Since $\delta W^k(q) = 0$ for the admissible variations $\delta q = T$, it follows from (1.14) that kq possesses the continuous second derivative $-pq$; thus q satisfies the differential equation (1.07). We now apply (1.15); it yields

$$(1.16) \qquad q_x(1)\delta q(1) - q_x(-1)\delta q(-1) = 0$$

Now δq is arbitrary; thus, when among possible values, $\delta q(1) = 0$, then $q_x(-1)\delta q(-1) = 0$ implies $q_x(-1) = 0$; similarly, $\delta q(-1) = 0$ leads to $q_x(1) = 0$. Thus the boundary conditions (1.11) are satisfied, and Theorem 1.2 is proved.

§ 2. Uniqueness theorems.

Continuing the notation used in § 1, we prove

THEOREM 2.1: *There is at most one solution q of the minimum problem M^k, apart from the sign of q.*

We first dispose of the case in which $q \equiv 0$ is a solution of M^k. In this case the functional W^k is non-negative; otherwise there would be a function q^* with $W^k(q^*) < 0$, and $W^k(q) = 0$ would not be the minimum. If q is *any* solution of M^k with $W(q) = D(q) - H(q) + K(q) = 0$, then for constant e, $W(eq) = e^2[D(q) - H(q)] + e^4K(q)$, as we see from the definitions of D, H, K, and p_x in (1.01—4). From the two preceding equations we have $W(eq) = (e^4 - e^2)K(q)$,, which would be negative for $e < 1$, unless $K(q) \equiv 0$. From (1.03) and (1.04), $K \equiv 0$ implies $q \equiv 0$. Hence $q \equiv 0$ is the only solution of M^k if W^k is non-negative. From now on we may thus leave aside the case in which $q \equiv 0$ solves M^k.

We have noted in Theorem 1.2 that a solution of the minimum problem M^k also solves S^k and the boundary problem B^k. Hence Theorem 2.1 (for $q \not\equiv 0$) results from the following two theorems:

THEOREM 2.2: *A solution $q(x) \not\equiv 0$ of M^k is nowhere zero in the interval $-1 \leqq x \leqq 1$.*

THEOREM 2.3: *A solution $q(x)$ of B^k which is nowhere zero in the interval $-1 \leqq x \leqq 1$ is, apart from the sign of q, the sole solution of M^k.*

An immediate consequence of Theorem 2.3 is the following

COROLLARY: *The problem M^k has only one solution, apart from the sign of q, which is nowhere zero in the interval $-1 \leqq x \leqq 1$.*

We prove Theorem 2.2 indirectly. We may assume $p \neq$ a constant, for otherwise $q \equiv 0$, a case already considered. Let us assume that the solution $q(x)$ of M^k vanishes for some value of x, say x_0. We now construct an admissible function q^e for which $W(q^e) < W(q)$, in contradiction with the minimum property of q. First, let us replace q by $|q|$. Since q occurs in all three functionals only to even powers, we observe that $W(|q|) = W(q)$. We have

$$|q|_x = q_x \quad \text{for} \quad -1 \leqq x \leqq x_0$$
$$|q|_x = -q_x \text{ for } \quad x_0 \leqq x \leqq 1$$

We next introduce a positive functional $n(x) \not\equiv 0$ with continuous derivatives, and a constant e whose propertes will be assigned later, and then define $q^e = |q| + en$. We note that

$$\int_x^1 [q^e(x)]^2 dx - \int_x^1 q^2(x)dx = e \,.\, d^e(x),$$

where

$$d^e(x) = 2\int_x^1 n(x) \,.\, |q(x)|\, dx + e\int_x^1 n^2(x)dx.$$

For future reference we note that

$$(2.01) \qquad d^e(1) = 0; \; d^e_x(x) = -2n \mid q \mid - en^2;$$

$$\int_{-1}^{1} x[2n \mid q \mid + en^2]dx = xd^e(x)|_{-1}^{1} - \int_{-1}^{1} d^e(x)dx$$

$$= d^e(-1) - 2A, \text{ where } 2A = \int_{-1}^{1} d^e(x)dx.$$

To obtain our contradiction, we make some calculations. First,

$$D(q^e) - D(q) = k\int_{-1}^{1} [| q |_x^2 + 2en_x \mid q \mid_x + e^2 n_x^2 - q_x^2]\, dx$$

$$= ke\int_{-1}^{1} [2n_x \mid q \mid_x + en_x^2]\, dx. \qquad \text{Next,}$$

$$H(q^e) - H(q) = \tfrac{1}{2}\int_{-1}^{1} f(x)\, [| q |^2 + 2en \mid q \mid + e^2 n^2 - q^2]\, dx$$

$$= ep_2 d^e(-1) - eA(p_2 - p_1).$$

To calculate $K(q^e) - K(q)$, we must first calculate $p_x(q^e) - p_x(q)$. Using (1.04), somewhat altered, and (2.01), we obtain, after straightforward calculations, the result

$$K(q^e) - K(q) = \int_{-1}^{1} ep_x q[d^e(-1) - d^e(x) - A]\, dx + e^2 B,$$

where $4B$ is a temporary abbreviation for $\int_{-1}^{1} [d^e(-1) - d^e(x) - A]^2 dx$. Since $d^e(-1)$ and A are constants, the integration of the product of these numbers by p_x can be effected. Recalling that $p(-1) = p_1$ and that $p(1) = p_2$, we have

$$K(q^e) - K(q) = e(p_2 - p_1)\, [d^e(-1) - A] - e\int_{-1}^{1} p_x d^e(x) dx + e^2 B.$$

Since $W = D - H + K$, we now have

$$W(q^e) - W(q) = ke\int_{-1}^{1} [2n_x | q |_x + en_x^2]dx - e\int_{-1}^{1} p_x d^e(x) dx$$

$$- ep_1 d^e(-1) + e^2 B.$$

Using product integration on the second integral, we may finally write the right hand side in the form

$$= e\int_{-1}^{1} [2kn_x \mid q \mid_x - 2np \mid q \mid]\, dx + e^2\int_{-1}^{1} [kn_x^2 - n^2 p]\, dx + e^2 B.$$

We shall show that for a suitably chosen n the quantity in the first bracket is negative, so that for a sufficiently small positive e, $W(q^e) - W(q)$ will be < 0, contrary to the minimum property of q. We must at the same time show that this quantity is not identically zero.

Since $q(x_0) = 0$ by hypothesis, we have $|q| = q$ for $x \leqq x_0$, and $-|q| = q$ for $x \geqq x_0$. Hence $p\,|q| = -kq_{xx}$ for $x \leqq x_0$, and $p\,|q| = kq_{xx}$ for $x \geqq x_0$. We note that here we are making use of the fact that q solves B^k and hence satisfies the differential equation $kq_{xx} + pq = 0$, also that q is initially positive. This last we may assume since the sign of q is clearly arbitrary. We now employ the fact that if a solution $q(x)$ of such a differential equation vanishes at a point $x = x_0$, then $q_x(x_0) \neq 0$ unless $q(x) \equiv 0$. Hence, in our case, $q_x(x_0) \neq 0$. Since q is a decreasing function, $q_x(x_0) < 0$.

Examining the coefficient of e, we find

$$2\int_{-1}^{1}[kn_x|q|_x - np|q|]dx = 2k\int_{-1}^{1} n_x|q|_x dx + 2k\int_{-1}^{x_0} nq_{xx}dx - 2k\int_{x_0}^{1} nq_{xx}dx,$$

which by product integration is

$$2k\int_{-1}^{x_0} n_x[|q|_x - q_x]dx + 2k\int_{x_0}^{1} n_x[|q|_x + q_x]dx + 4kn(x_0)q_x(x_0).$$

As already remarked, $|q|_x = q_x$ for $-1 \leqq x \leqq x_0$, so that the first integral $\equiv 0$; similarly, since $|q|_x = -q_x$ for $x_0 \leqq x \leqq 1$, the second integral also $\equiv 0$. Hence the coefficient of e is merely $4kn(x_0)q_x(x_0)$. n has been defined as positive throughout the range of x, and, as we have seen, $q_x(x_0) < 0$. Hence the coefficient of e is negative, so that our constructed function q^e contradicts the minimum property of q. This contradiction establishes Theorem 2.2.

We now turn to the proof of Theorem 2.3. This theorem is equivalent to the statement: If q^* is any solution of B^k which does not vanish for $-1 \leqq x \leqq 1$, then $W(q) \geqq W(q^*)$ for every admissible function q, and the equality holds only for $q = \pm q^*$.

Let p and p^* be the functions corresponding to q and q^*, respectively. We derive by product integration the identity

$$\int_{-1}^{1}(p^* - p_1)q^2dx = 2(p_2 - p_1)p_x(1) - 2\int_{-1}^{1} p_x^* p_x dx.$$

We now introduce the quadratic functional

$$T(q) = \int_{-1}^{1}(kq_x^2 - p^*q^2)dx.$$

Both the identity and the functional clearly exist for admissible q. From (1.04), simplified,

$$p_x(1) = \tfrac{1}{4}\int_{-1}^{1}(1 + x)q^2dx + \tfrac{1}{2}(p_2 - p_1).$$

This, used in conjunction with (1.05), gives us

$$H(q) = p_1 \int_{-1}^{1} q^2 dx + 2(p_2 - p_1)p_x(1) - (p_2 - p_1)^2.$$

Employing this form of H, we have

$$\begin{aligned} W(q) &= \int_{-1}^{1} (kq_x^2 - p^* q^2 + [p^* - p_1] q^2 + p_x^2) dx \\ &\qquad - 2(p_2 - p_1)p_x(1) + \tfrac{1}{2}(p_2 - p_1)^2 \\ &= T(q) + \int_{-1}^{1} p_x^2 dx - 2\int_{-1}^{1} p_x^* p_x dx + \tfrac{1}{2}(p_2 - p_1)^2. \end{aligned}$$

$$W(q^*) = T(q^*) - \int_{-1}^{1} (p_x^*)^2 dx + \tfrac{1}{2}(p_2 - p_1)^2$$

Subtraction yields the identity

$$(2.02) \qquad W(q) - W(q^*) = T(q) - T(q^*) + \int_{-1}^{1} (p_x - p_x^*)^2 dx$$

Theorem 2.3 is a consequence of (2.02) and

Lemma 2.1: *For admissible* q

$$(2.03) \qquad T(q^*) \geqq 0,$$

where the equality holds only for $q = cq^*$, c *a constant.*

Lemma 2.1 implies

$$(2.04) \qquad T(q^*) = 0.$$

If Lemma 2.1 holds, (2.02) yields $W(q) \geqq W(q^*)$, the equality holding only for $q = cq^*$, $p_x = p_x^*$. This last statement gives

$$-\tfrac{1}{2}\int_{x}^{1} q^2 d\tilde{x} + \tfrac{1}{4}\int_{-1}^{1} (1+\tilde{x}) q^2 d\tilde{x} = -\tfrac{1}{2}\int_{x}^{1} (q^*)^2 d\tilde{x} + \tfrac{1}{4}\int_{-1}^{1} (1+\tilde{x})(q^*)^2 d\tilde{x};$$

i.e., $$-\tfrac{1}{2}\int_{x}^{1} (q^2 - q^{*2}) d\tilde{x} = \tfrac{1}{4}\int_{-1}^{1} (1+\tilde{x})(q^{*2} - q^2) d\tilde{x}.$$

Since the right side is clearly a constant, while the left is a function of x, the left integrand must be zero, i.e.,

$$q = \pm q^*$$

Hence Theorem 2.3 is proved once the inequality (2.03) is established. This inequality states that q^* minimizes the quadratic functional $T(q)$. We proceed to prove Lemma 2.1.

Since $q^* > 0$, we may introduce the function

$$(2.05) \qquad G = q/q^*$$

With this function G, we shall prove the identity

$$(2.06) \qquad T(q) = k\int_{-1}^{1} G_x^2 q^{*2} dx$$

holds. This identity implies that $T(q) \geqq 0$, and that the equality holds only if $G_x = 0$ (since $q^* \neq 0$), i.e., if $G \equiv c$, a constant, or if $q = cq^*$. Thus Lemma 2.1 follows from (2.06). To prove (2.06) we apply Jacobi's identity

$$\int_a^b [f_x^2 + w^{-1}w_{xx}f^2]dx = \int_a^b w^2[w^{-1}f]_x^2 dx + w^{-1}w_x f^2 \Big|_a^b$$

to $w = q^*$ and $f = q$, and obtain

$$\begin{aligned}\int_{-1}^{1} [q_x^2 + q^{*-1}q_{xx}^* q^2]dx &= \int_{-1}^{1} q^{*2}[q^{*-1}q]_x^2 dx + q^{*-1}q_x^* q^2 \Big|_{-1}^{1} \\ &= \int_{-1}^{1} G_x^2 q^{*2} dx + [q_x^*/q^*]q^2 \Big|_{-1}^{1}\end{aligned}$$

However, q^2 is finite, $q^* \neq 0$, and $q_x^*(-1) = q_x^*(1) = 0$.

$$\therefore k\int_{-1}^{1} [q_x^2 + q^{*-1}q_{xx}^* q^2]\, dx = k\int_{-1}^{1} G_x^2 q^{*2} dx$$

But $kq_{xx}^* = -p^*q^*$, so that we have

$$\begin{aligned}k\int_{-1}^{1} G_x^2 q^{*2} dx &= \int_{-1}^{1} [kq_x^2 + q^{*-1}(kq_{xx}^*)q^2]\, dx \\ &= \int_{-1}^{1} [kq_x^2 + q^{*-1}(-p^*q^*)q^2]\, dx \\ &= \int_{-1}^{1} [kq_x^2 - p^*q^2]\, dx = T(q)\end{aligned}$$

Hence Lemma 2.1 is proved, since (2.06) holds, and, with it, Theorem 2.3.

§ 3. Existence theorems.

In this section we prove the existence of the solutions of the minimum problem M^k. We apply direct methods similar to those used for linear boundary value problems (cf. [1], Vol. II, Chap. VII). We use the same formulation of the minimum problem M^k as in § 1, except that we find another form of the functional $K(q)$ more convenient. Functions q admissible with respect to the problem M^k in the sense of § 1 (p. 123) are here referred to as *k-admissible functions*.

For the new form of $K^k(q)$ we introduce the function $y(x)$ by means of the relation

$$(3.01)\qquad \begin{aligned} p(x) &= \tfrac{1}{2}[f(x) - y(x)] \\ &= -\tfrac{1}{2}y(x) + \tfrac{1}{2}x(p_2 - p_1) + \tfrac{1}{2}(p_2 + p_1) \end{aligned}$$

whence $\quad p_x(x) = -\tfrac{1}{2}y_x(x) + \tfrac{1}{2}(p_2 - p_1).$

From (1.03) we have at once our new form of K:

$$(3.02)\qquad K^k(q) = \tfrac{1}{4}\int_{-1}^{1} y_x^2\, dx$$

If we express p as an integral of the equation $p_{xx} = \frac{1}{2}q^2$ and calculate the constants of integration by means of the boundary conditions, we obtain

$$(3.03)\qquad \begin{aligned} p(x) = \tfrac{1}{4}\int_{-1}^{x} (x - \tilde{x})q^2\, d\tilde{x} - \tfrac{1}{4}\int_{x}^{1} (x - \tilde{x})q^2\, d\tilde{x} \\ + \tfrac{1}{4}\int_{-1}^{1} (x\tilde{x} - 1)\, q^2\, d\tilde{x} + \tfrac{1}{2}f(x). \end{aligned}$$

Hence

$$(3.04)\qquad \begin{aligned} y(x) = -\tfrac{1}{2}\int_{-1}^{x} (x - \tilde{x})q^2\, d\tilde{x} + \tfrac{1}{2}\int_{x}^{1} (x - \tilde{x})q^2\, d\tilde{x} \\ - \tfrac{1}{2}\int_{-1}^{1} (x\tilde{x} - 1)\, q^2\, d\tilde{x} \end{aligned}$$

We have then for future reference that

$$(3.05)\qquad y_x(x) = -\tfrac{1}{2}\int_{-1}^{x} q^2\, d\tilde{x} + \tfrac{1}{2}\int_{x}^{1} q^2\, d\tilde{x} - \tfrac{1}{2}\int_{-1}^{1} \tilde{x}q^2\, d\tilde{x}$$

$$(3.06)\qquad y_{xx}(x) = -q^2(x)$$

$$(3.07)\qquad y(-1) = y(1) = 0$$

Our theorem here is

THEOREM 3.1: *To every $k > 0$ there exists at least one k-admissible function $q(x)$ for which $W^k(q)$ attains its minimum.*

Such a minimizing function will be denoted henceforth by $q^k(x)$; it is uniquely determined (Th. 2.1) once the condition $q^k(0) \geqq 0$ has been imposed.

The proof of this theorem, as well as of those in § 4, is based on a number of preliminary lemmas and inequalities, which we now proceed to establish.

$$(3.08)\qquad \int_{a}^{b} y\, dx \leqq \sqrt{b - a}\, .\, \sqrt{\int_{a}^{b} y^2\, dx} \qquad \text{(Schwarz)}$$

A. Now $y(x) = y(-1) + \int_{-1}^{x} y_{\tilde{x}}(\tilde{x})d\tilde{x}$

$$\therefore |y(x)| \leqq |y(-1)| + \left| \int_{-1}^{x} y_{\tilde{x}}(\tilde{x})d\tilde{x} \right|$$

$$\therefore |y(x)|^2 \leqq 2|y(-1)|^2 + 2\left| \int_{-1}^{x} y_{\tilde{x}}d\tilde{x} \right|^2$$

$$\leqq 2\left| \int_{-1}^{x} y_{\tilde{x}}d\tilde{x} \right|^2 \text{ since } y(-1) = 0.$$

$$\leqq 2\,|x+1|\,.\left| \int_{-1}^{x} y_{\tilde{x}}^2 d\tilde{x} \right| \leqq 4\int_{-1}^{1} y_x^2 dx \text{ from (3.08)}$$

(3.09) $\therefore \int_{-1}^{1} y^2(x)dx \leqq 8\int_{-1}^{1} y_x^2(x)dx$; hence, from (3.02),

$$\int_{-1}^{1} y^2(x)dx \leqq 32K^k(q) \tag{3.10}$$

B. We next proceed to establish the inequality

$$\int_{-1+c}^{1-c} q^2 dx \leqq 2\sqrt{\frac{2K^k(q)}{c}}. \tag{3.11}$$

We start with $\int_{-1+c}^{1-c} q^2 dx \leqq \int_{v}^{u} q^2 dx$, where $1-c \leqq u \leqq 1$, $-1 \leqq v \leqq -1+c$, and c is a positive constant yet to be determined, but necessarily < 1. From (3.06) we then have

$$\int_{-1+c}^{1-c} q^2 dx \leqq -\int_{v}^{u} y_{xx}dx \leqq -[y_x(u) - y_x(v)]$$

$$\therefore \left(\int_{-1+c}^{1-c} q^2 dx\right)^2 \leqq y_x^2(u) - 2y_x(u)\,.\,y_x(v) + y_x^2(v).$$

Integrate with respect to u and then v. We have

$$c^2\left(\int_{-1+c}^{1-c} q^2 dx\right)^2 \leqq \int_{-1}^{-1+c}\int_{1-c}^{1} [y_x^2(u) + 2y_x(u)\,.\,y_x(v) + y_x^2(v)]du\,dv$$

$$\leqq c\int_{1-c}^{1} y_x^2(u)du + c\int_{-1}^{-1+c} y_x^2(v)dv - 2\int_{1-c}^{1} y_x(u)du\int_{-1}^{-1+c} y_x(v)dv$$

$$\leqq c\int_{1-c}^{1} y_x^2 du + c\int_{-1}^{-1+c} y_x^2 dv + 2c\sqrt{\int_{1-c}^{1} y_x^2 du}\sqrt{\int_{-1}^{-1+c} y_x^2 dv},$$

the last from (3.08). Since $a^2 + b^2 \geqq 2ab$,

$$c^2\left(\int_{-1+c}^{1-c} q^2 dx\right)^2 \leqq c\int_{1-c}^{1} y_x^2 du + c\int_{-1}^{-1+c} y_x^2 dv + c\left[\int_{1-c}^{1} y_x^2 du + \int_{-1}^{-1+c} y_x^2 dv\right]$$

$$\leqq 2c\int_{1-c}^{1} y_x^2 du + 2c\int_{-1}^{-1+c} y_x^2 dv \leqq 2c\int_{-1}^{1} y_x^2(x)dx \leqq 8cK^k(q), \text{ from (3.02).}$$

Now dividing both sides of the inequality by c^2 and extracting the square root, we have the desired result, (3.11).

C. Next, we shall show the existence of a constant $g > 0$ such that $H \leqq 2g\sqrt{K} + \frac{1}{2}D/k$ for any k-admissible q and all $k > 0$. We start with

$$q(x') = q(x'') + \int_{x''}^{x'} q_x dx, \quad -1 \leqq x' \leqq 1, \quad -1 \leqq x'' \leqq 1$$

$$\therefore |q(x')|^2 \leqq 2|q(x'')|^2 + 2\left|\int_{x''}^{x'} q_x dx\right|^2$$

$$\leqq 2|q(x'')|^2 + 2|x' - x''| \cdot \left|\int_{x''}^{x'} q_x^2 dx\right| \quad \text{[by (3.08)]}$$

$$\therefore |q(x')|^2 \leqq 2|q(x'')|^2 + 2|x' - x''| \cdot \int_{-1}^{1} q_x^2 dx \tag{3.12}$$

Recall that $D^k(q) = k\int_{-1}^{1} q_x^2 dx$; also, let us integrate (3.12) with respect to x' in the interval -1 to $-1 + c$, and then with respect to x'' over the interval $-1 + c$ to $-1 + 2c$, where c is the constant referred to in section B, P. 131.

$$\therefore \int_{-1+c}^{-1+2c}\int_{-1}^{-1+c} q^2\, dx'\, dx'' \leqq 2c \int_{-1+c}^{-1+2c} |q(x'')|^2 dx''$$

$$+ \frac{D(q)}{k}\int_{-1+c}^{-1+2c}\int_{-1}^{-1+c} |x' - x''|\, dx'\, dx''$$

$$\therefore c\int_{-1}^{-1+c} q^2 dx \leqq 2c\int_{-1+c}^{-1+2c} q^2 dx + \frac{2c^3 D(q)}{k}$$

$$\therefore \int_{-1}^{-1+c} q^2 dx \leqq 2\int_{-1+c}^{-1+2c} q^2 dx + \frac{2c^2 D}{k}. \tag{3.13}$$

We return to (3.12) and integrate twice again, first with respect to x' over the interval $1 - c$ to 1, and then with respect to x'' over the interval $1 - 2c$ to $1 - c$. We have

$$\int_{1-c}^{1} q^2 dx \leqq 2\int_{1-2c}^{1-c} q^2 dx + \frac{2c^2 D}{k} \tag{3.14}$$

Add (3.13) and (3.14); then add $\int_{-1+c}^{1-c} q^2 dx$ to both sides. Then

$$\int_{-1}^{-1+c} q^2 dx + \int_{-1+c}^{1-c} q^2 dx + \int_{1-c}^{1} q^2 dx \leqq 2\left[\int_{-1+c}^{-1+2c} q^2 dx + \int_{1-2c}^{1-c} q^2 dx\right]$$

$$+ \int_{-1+c}^{1-c} q^2 dx + 4c^2 D/k, \qquad \text{or}$$

$$\int_{-1}^{1} q^2 dx \leqq 3 \int_{-1+c}^{1-c} q^2 dx + 4c^2 D/k \tag{3.15}$$

Now from (1.05), if $p_2 > p_1$, the maximum value, $2p_2$, of $f(x)$ in the interval $-1 \leqq x \leqq 1$ is attained when $x = 1$; if $p_2 < p_1$, the maximum value is $2p_1$, and occurs when $x = -1$. From (1.01), $H^k(q) = \frac{1}{2}\int_{-1}^{1} f(x) q^2(x) dx$, so that $H \leqq \frac{1}{2} \max_{-1 \leqq x \leqq 1} f(x) \int_{-1}^{1} q^2 dx$. Denoting by N the greatest of p_1, p_2, and 1, we may now write

$$H^k(q) \leqq N \int_{-1}^{1} q^2 dx \tag{3.16}$$

Using (3.15) and then (3.11), we obtain

$$H \leqq 3N \int_{-1+c}^{1-c} q^2 dx + \frac{4c^2 ND}{k} \leqq 6N \sqrt{\frac{2K}{c}} + \frac{4c^2 ND}{k} \tag{3.17}$$

Now we choose $c = (8N)^{-1/3}$. Therefore, finally,

$$H \leqq 2g\sqrt{K} + D/2k, \tag{3.18}$$

where $g = 3(2N)^{5/4}$ and is therefore a positive constant, greater than 1, and depending only upon the constants involved in the boundary conditions.

From (3.18) and (1.06) we deduce

$$\begin{aligned} W^k(q) &\geqq \frac{D^k(q)}{2q} + K^k(q) - 2g\sqrt{K^k(q)} \\ &\geqq \frac{D^k(q)}{2q} + (\sqrt{K^k(q)} - g)^2 - g^2 \end{aligned} \tag{3.19}$$

which implies

Lemma 3.1: *For k-admissible functions q, $W^k(q)$ has a lower bound, $-g^2$, independent of k.*

From (3.19) we obtain the following inequalities:

$$\begin{aligned} \frac{D^k(q)}{k} &\leqq 2W^k(q) + 2g^2 \\ K^k(q) &\leqq 2W^k(q) + 4g^2 \\ H^k(q) &\leqq 3W^k(q) + 6g^2 \end{aligned} \tag{3.20}$$

Consider now a set of k-admissible functions q (with k not necessarily fixed) for which W^k has an upper bound M. We conclude from (3.20)

Lemma 3.2: *An upper bound M for W^k implies upper bounds for D^k/k, H^k, K^k which depend upon M but not upon k.*

We now prove the basic

LEMMA 3.3: *For a fixed k, let q_m be a sequence of k-admissible functions for which $W^k(q_m)$ is bounded. Then there exists a subsequence q_b converging uniformly to a k-admissible function q such that*

$$\underline{\lim}\, D^k(q_b) \geqq D^k(q) \tag{3.21}$$

$$\underline{\lim}\, H^k(q_b) = H^k(q) \tag{3.22}$$

$$\underline{\lim}\, K^k(q_b) = K^k(q) \tag{3.23}$$

$$\underline{\lim}\, W^k(q_b) \geqq W^k(q) \tag{3.24}$$

A bound for W implies, by Lemma 3.2, bounds for D_m, H_m, K_m; i.e., the sequences $\int_{-1}^{1}(q_m)_x^2 dx$ and $\frac{1}{2}\int_{-1}^{1} f q_m^2 dx$ are bounded. In the interval considered, $f(x)$ is always > 0 whenever p_1 and p_2 are (as they are in the present Case I), and is clearly bounded. Hence we may explicitly assume

$$\int_{-1}^{1}(q_m)^2 dx \leqq A, \qquad \int_{-1}^{1}(q_m)_x^2 dx \leqq B, \tag{3.25}$$

A and B representing positive constants. Now

$$\begin{aligned} | q_m(x') - q_m(x'') | &= \left| \int_{x'}^{x''} (q_m)_x dx \right| \\ &\leqq \sqrt{x'' - x'} \cdot \sqrt{\int_{-1}^{1}(q_m)_x^2 dx} \quad \text{[by (3.08)]} \\ &\leqq \sqrt{2B} \quad \text{[from the range of } x \text{ \& (3.25)]} \end{aligned}$$

Hence the sequence $q_m(x)$ is equicontinuous. Moreover, all q_m are bounded, for, as above,

$$q_m(x) - q_m(\tilde{x}) = \int_{\tilde{x}}^{x}(q_m)_x dx \leqq \sqrt{x+1} \cdot \sqrt{\int_{-1}^{1}(q_m)_x^2 dx} \leqq \sqrt{2B}$$

$\therefore q_m(x) \leqq q_m(\tilde{x}) + \sqrt{2B}$, whence $| q_m(x) |^2 \leqq 2 | q_m(\tilde{x}) |^2 + 4B$.

Integrate with respect to $\tilde{x}$ between -1 and 1. We have

$$2 | q_m(x) |^2 \leqq 2 \int_{-1}^{1} (q_m)^2 d\tilde{x} + 8B \leqq 2A + 8B, \text{ from (3.25).}$$

$\therefore | q_m(x) |^2 \leqq A + 4B$, whence q_m is uniformly bounded. Thus the sequence q_m satisfies the conditions of Ascoli's Theorem ([3], P. 336). Thus we have the existence of a continuous limit function. To establish our inequalities, we must also show that this limit function possesses a quadratically integrable derivative. To this end, we assume for $(q_m)_x$ the Fourier expansion $\sum_1^\infty a_r^{(m)} u_r(x)$. Then

$$a_r^{(m)} = \int_{-1}^{1} (q_m)_x u_r(x)\,dx \text{ (aside from constant factors)}$$

$$= -\int_{-1}^{1} (q_m)[u_r(x)]_x dx + q_m(x) \,.\, u_r(x)]\Big|_{-1}^{1} \text{ by product integration.}$$

Ascoli's Theorem showed that there is a subsequence of $q_m(x)$ converging uniformly to the continuous limit function $q(x)$; but if in an interval a sequence of functions F_n tends uniformly to the limit function $F(x)$, then $\int_a^b F(x)dx = \lim_{n\to\infty} \int_a^b F_n(x)dx$. Hence $a_r^{(m)}$ approaches a limit, which we denote by a_r. Then clearly

$$a_r = -\int_{-1}^{1} q[u_r(x)]_x\, dx + q(x) \,.\, u_r(x)\Big|_{-1}^{1}$$

From (3.25) $\int_{-1}^{1} (q_m)_x^2 dx \leqq B$; then Bessel's inequality ([4], V. I, P. 451) gives us $\sum_{r=1}^{\infty} [a_r^{(m)}]^2 \leqq B$. Then all the more $\sum_1^R (a_r^{(m)}]^2 \leqq B$, where R is arbitrary; whence $\sum_1^R [a_r]^2 \leqq B$, the convergence certainly being true for a finite number. Since R is arbitrary, however, we have $\sum_1^\infty [a_r]^2 \leqq B$. This inequality permits us to apply the Riesz-Fischer Theorem ([5], V. II, P. 577). Thus there exists a unique function $h(x)$ for which a_r are the Fourier constants and which is quadratically integrable. We now proceed to demonstrate that $h(x) = q_x(x)$. Let us consider an arbitrary function $g(x) = \sum_1^\infty b_r u_r(x)$, with $\sum_1^\infty b_r^2 < \infty$. Again from the Riesz-Fischer Theorem, $g(x)$ is also L^2-integrable. Since both $h(x)$ and $(q_m)_x$ are L^2-integrable, we may apply Parseval's Theorem to obtain

$$\int_{-1}^{1} g(x)\, q_{m_x}(x)dx = \sum_1^\infty b_r a_r^{(m)}; \quad \int_{-1}^{1} g(x)\, h(x)\, dx = \sum_1^\infty b_r a_r.$$

We claim that as $m \to \infty$ the first of these expressions approaches the second. We have

$$\int_{-1}^{1} g(x)[q_{m_x}(x) - h(x)]\, dx = \sum_1^\infty b_r[a_r^{(m)} - a_r]$$

$$= \sum_1^R b_r[a_r^{(m)} - a_r] + \sum_{R+1}^\infty b_r[a_r^{(m)} - a_r]$$

The absolute value of this last expression, by Schwarz-Cauchy, is

$$\leqq \sum_{1}^{R} |b_r| \cdot |a_r^{(m)} - a| + \sqrt{\sum_{R+1}^{\infty} b_r^2} \left\{ \sqrt{\sum_{R+1}^{\infty} a_r^2} + \sqrt{\sum_{R+1}^{\infty} [a_r^{(m)}]^2} \right\}$$

Since $\sum\limits_{1}^{\infty} b_r^2 < \infty$, we can choose R so large that $\sum\limits_{R+1}^{\infty} b_r^2 \leqq e$; and because of the convergence of the $a_r^{(m)}$ to the a_r, we can then choose m so large that $| a_r^{(m)} - a_r | \leqq e$. Hence our previous expression is $\leqq \sum\limits_{1}^{R} | b_r | . e + \sqrt{e}\,[\sqrt{B} + \sqrt{B}]$, which clearly can be made arbitrarily small.

Now let $g(x) = 1$ for $-1 \leqq x \leqq z$
$ = 0$ for $ z \leqq x \leqq 1$.

Then from the result just obtained,

$$\int_{-1}^{z} [q_m(x)]_x dx \to \int_{-1}^{z} h(x)dx; \text{ i.e., } q_m(z) - q_m(-1) \to \int_{-1}^{z} h(x)dx.$$

But we already know that $q_m(x) \to q(x)$; hence $q(z) - q(-1) = \int_{-1}^{z} h(x)dx$; that is, $h(x) = q_x(x)$.

The inequalities of our Lemma now follow almost immediately. Indeed, the relation $\sum\limits_{1}^{\infty} a_r^2 \leqq B$ is essentially (3.21). For (3.22) we write

$$H(q_b) - H(q) = \tfrac{1}{2}\int_{-1}^{1} f(x)[q_b^2 - q^2]\,dx.$$

Since, as already remarked, $f(x)$ remains bounded, and q_b tends to q uniformly, the right side $\to 0$ as $b \to \infty$; i.e.,

$$\underline{\lim}\, H^k(q_b) = H(q).$$

To prove (3.23), we derive the identity

$$2(p_2 - p_1)[p_{b_x} - p_x] \equiv H_b - H + (p_2 - p_1)\int_{-1}^{x} [q_b^2 - q^2]\,dx$$
$$- p_2\int_{-1}^{1} [q_b^2 - q^2]dx$$

from the definitions of H, p_x, and f in § 1. Hence, as before, the right side $\to 0$, so that $p_{b_x} \to p_x$ uniformly as $b \to \infty$. Therefore, $p_{b_x}^2 \to p_x^2$. But $K(q_b) - K(q) = \int_{-1}^{1} [p_{b_x}^2 - p_x^2]dx$. Hence, we have (3.23).

In view of (3.21—3), q is k-admissible, and, moreover, (3.24) holds. Thus Lemma 3.3 is proved.

We are now in a position to prove Theorem 3.1. We turn, therefore, to the problem of minimizing $W^k(q)$ by a k-admissible function q. From Lemma 3.1 we know that the g.l.b. w^k of $W^k(q)$ is finite; hence there exists a minimizing sequence, i.e., a sequence of k-admissible functions q_m for which $W^k(q_m)$ has as limit w^k. We now apply Lemma 3.3; it yields the existence of a subsequence q_b and a k-admissible function $q = \tilde{q}$ for which [cf. (3.24)]

$$W^k(q) \leqq \underline{\lim}\, W^k(q_b).$$

Since the right member here is the g.l.b. w^k of W^k, the equality must hold. Hence q solves the minimum problem M^k. This proves Theorem 3.1.

§ 4. Asymptotic solutions: uniqueness, existence, and convergence.

In the preceding sections we have discussed the existence and uniqueness of solutions of our problem for a fixed value, not zero, of the parameter k. In order to determine the asymptotic behavior of the solutions, it is necessary to formulate a limit boundary value problem. A simple and rather natural procedure would be by a passage to the limit in the original differential equations and boundary conditions. If we let $k \to 0$ in the equations, they take the form

$$p_{xx} = \tfrac{1}{2}q^2, \quad pq = 0.$$

The only solution of these equations satisfying the boundary conditions is $q \equiv 0$, $p =$ a linear function, with the constants in this function fixed by the values of p at the boundaries. However, the results of numerical calculations in [1] which are applicable here indicate that wrong results are obtained by this procedure. In the interior of the circular plate, the study of which gives rise to our problem, the above procedure seems valid, but the constants cannot be determined by using the values of p at the edge. The constants can be fixed only by an investigation of the transition phenomena from tension in the interior to the prescribed compression at the edges—phenomena which occur in a narrow strip, the breadth of which decreases as $k \to 0$. These boundary layer phenomena are related to the fact that the order of the system of differential equations has been reduced in q, although remaining the same in p. The above discussion indicates that the lost boundary conditions are at the edge.

A treatment of such an edge effect requires that the scale be stretched with decreasing k in such a manner that the width of the edge strip, or boundary layer, as measured in the new scale, does not shrink to zero. This will be accomplished by introducing new variables, first one that stretches the right hand edge off to infinity, and then one that does the same for the left hand edge. Because of the symmetry of the problem, only one such stretching need be studied in detail.

Accordingly, we make the transformation of independent and dependent variables

$$t = (x + 1)\sqrt{p_1/k}, \; P = p/p_1, \; Q = \sqrt{k} \,.\, q/p_1. \tag{4.01}$$

These transform our original equations into

$$P_{tt} = \tfrac{1}{2}Q^2, \; Q_{tt} + PQ = 0, \tag{4.02}$$

with the corresponding new boundary conditions

$$\begin{aligned} P(0) = 1, \; P(a) = p_2/p_1, \\ Q_t(0) = 0, \; Q_t(a) = 0, \end{aligned} \tag{4.03}$$

with t defined in the interval $0 \leqq t \leqq a$, and where, as a convenient abbreviation, we have set

$$a = \sqrt{2p_1/k}. \tag{4.04}$$

We note that $k \to 0$ implies $a \to \infty$, and conversely; we shall use these statements interchangeably. We observe also that the new equations do not contain the parameter k explicitly, but that it is involved in the right end point, as well as the interval of variation.

These new equations have the trivial solution

$$P = [(p_2 - p_1)/ap_1]t + 1, \; Q \equiv 0;$$

also, if there is another solution (P, Q), with $Q \not\equiv 0$, then $(P, -Q)$ is also a solution, so that the sign of Q is arbitrary. We shall therefore assume Q as positive whenever it is not identically zero.

In this section we wish to prove the existence of the solutions of the minimum problem M^k as restated for the new variable t, including the asymptotic case M^0, and to establish the convergence of the solutions for $k > 0$ to the asymptotic solution $(k = 0)$ as k tends to 0. As before, we apply direct methods similar to those used for linear boundary value problems.

We shall now formulate simultaneously the stretched problems for $k > 0$ and for $k = 0$, the asymptotic case.

We require first functionals similar to those in § 1:

$$(4.05)\quad H^k[Q^k] = \int_0^a F(t)Q^2(t)dt \qquad (4.05^0)\quad H^0[Q^0] = \int_0^\infty Q^2(t)dt$$

$$(4.06)\quad D^k[Q^k] = \int_0^a Q_t^2(t)dt \qquad (4.06^0)\quad D^0[Q^0] = \int_0^\infty Q_t^2(t)dt$$

$$(4.07)\quad K^k[Q^k] = \int_0^a P_t^2 dt - \frac{1}{a}\left(\frac{p_2 - p_1}{p_1}\right)^2 \qquad (4.07^0)\quad K^0[Q^0] = \int_0^\infty P_t^2(t)dt$$

$$= \int_0^a \left[P_t(t) - \frac{1}{a}\left(\frac{p_2 - p_1}{p_1}\right)\right]^2 dt$$

where $P_t(t)$ is a functional in Q through

$$(4.08)\quad P_t^k[Q^k] = -\tfrac{1}{2}\int_t^a Q^2(\tilde{t})d\tilde{t} \qquad (4.08^0)\quad P_t^0[Q^0] = -\tfrac{1}{2}\int_t^\infty Q^2(\tilde{t})\,d\tilde{t}$$

$$+ \frac{1}{2a}\int_0^a \tilde{t}Q^2(\tilde{t})dt + \frac{p_2 - p_1}{ap_1} = P_t^k(a) - \tfrac{1}{2}\int_t^a Q^2(\tilde{t})d\tilde{t}, \qquad \text{and}$$

$$(4.09)\qquad F(t) = 1 + (p_2 - p_1)t/ap_1.$$

The functional to be minimized is

$$(4.10)\qquad W^k[Q^k] = D^k[Q^k] - H^k[Q^k] + K^k[Q^k]$$

$$(4.10^0)\qquad W^0[Q^0] = D^0[Q^0] - H^0[Q^0] + K^0[Q^0]$$

By *admissible functions* we mean functions $Q(t)$ continuous in $\begin{matrix} 0 \leqq t \leqq a \\ 0 \leqq t \leqq \infty \end{matrix}$ with L^2-integrable derivatives in $\begin{matrix} 0 \leqq t \leqq a < \infty \\ 0 \leqq t < \infty \end{matrix}$ and for which the integrals in $\begin{matrix} (4.05\text{—}7) \\ (4.05^0\text{—}7^0) \end{matrix}$ are $\begin{matrix} \text{finite for all } k > 0, \\ \text{finite,} \end{matrix}$ $\begin{matrix} P_t^k \\ P_t^0 \end{matrix}$ being defined by $\begin{matrix} (4.08) \\ (4.08^0) \end{matrix}$. The *minimum problem* M^k (M^0) is that of minimizing $W^k[Q^k]$ ($W^0[Q^0]$); the problem S^k (S^0) is that of making W^k (W^0) stationary, in each case with respect to admissible functions Q. The *boundary value problem* B^k (B^0) requires the determination of an admissible function Q possessing a continuous second derivative and satisfying the equation

$$(4.11)\qquad Q_{tt} + PQ = 0$$

and the boundary condition(s)

$$(4.12)\qquad Q_t(0) = Q_t(a) = 0 \qquad\qquad (4.12^0)\quad Q_t(0) = 0.$$

The function $P(t)$ in (4.11) is defined by

$$(4.13)\qquad P(t) = 1 + \int_0^t P_{\tilde{t}}(\tilde{t})d\tilde{t},$$

where P_t^k (P_t^0) is given by (4.08) ((4.08⁰)). The function P therefore satisfies the differential equation

$$P_{tt} = \tfrac{1}{2}Q^2 \tag{4.14}$$

and the boundary condition(s)

(4.15) $P(0) = 1,\ P(a) = p_2/p_1$ (4.15⁰) $P(0) = 1.$

The condition at $t = 0$ is obvious from (4.13); that at $t = a$ follows from substituting (4.08) in (4.13).

The asymptotic problem as here formulated is identical with the asymptotic problem treated in [1]; accordingly there is no need to repeat that work here. We consequently take as proven that the asymptotic problem has a solution and that, apart from the sign of Q, the solution is unique.

Moreover, the theorems of §§ 1, 2, establishing the connection between the minimum problem and the solution of the differential equations, with the accompanying boundary conditions, and proving the uniqueness of the solution, are equally true for the functions P and Q in the stretched variable t and the corresponding new functionals in (4.05—6—7); their proofs require merely the appropriate change in notation. Theorem 3.1 is similarly true in terms of the stretched variable and functions for the case $k > 0$; for $k = 0$, we have its truth from [1], as mentioned in the preceding paragraph. However, for convenience we restate the theorem in these new terms:

THEOREM 4.1: *For every $k \geqq 0$ there exists at least one k-admissible function $Q(t)$ for which $W^k[Q]$ attains its minimum.*

Here, as before, functions Q admissible with respect to the problem M^k in the sense given above (P. 139) are referred to as k-admissible functions. Such a minimizing function as mentioned in the theorem will be denoted henceforth by $Q^k(t)$; it is uniquely determined once the condition $Q^k(0) \geqq 0$ has been imposed (Th. 2.1 and preceding remarks here).

Our real concern in this section, then, is the existence of solutions for k tending to zero, and the convergence of these solutions to the asymptotic solution. Since much preparation is necessary in order to prove our principal theorem (4.2) in this section, its statement will be deferred until we are ready for it.

Our subsequent results are based on a number of preliminary inequalities and lemmas, which we now give. Since most of these relations are identical with or analogous to those derived in § 3, we merely list the results here.

$$\int_0^a Y^2(t)dt \leqq 8a^2K^k[Q] \tag{4.16}$$

$$\int_c^{a-c} Q^2(t)dt \leqq 2\sqrt{2K/c} \tag{4.17}$$

$$H^k[Q] \leqq 2g\sqrt{K} + \tfrac{1}{2}D \tag{4.18}$$

In these

$$Y(t) = -2P(t) + 2t(p_2 - p_1)/ap_1 + 2; \tag{4.19}$$

whence

$$Y(t) = \int_0^t \bar{t}[1 - t/a]Q^2(\bar{t})d\bar{t} + \int_t^a t[1 - \bar{t}/a]Q^2(\bar{t})d\bar{t}; \tag{4.20}$$

$$Y_t(t) = -\int_0^a [\bar{t}/a]Q^2(\bar{t})d\bar{t} + \int_t^a Q^2(\bar{t})d\bar{t}; \tag{4.21}$$

$$Y_{tt}(t) = -Q^2(t). \tag{4.22}$$

Relation (4.20) results from (4.19) and the integral

$$P^k(t) = -\tfrac{1}{2}\int_0^t \bar{t}[1-t/a]Q^2(\bar{t})d\bar{t} - \tfrac{1}{2}\int_t^a t[1-\bar{t}/a]Q^2(\bar{t})d\bar{t} + F(t) \tag{4.23}$$

of the equation $P_{tt} = \frac{1}{2}Q^2$, with the constants of integration determined by the boundary conditions (4.03).

We note for future reference that $Y(0) = Y(a) = 0$; and that, as before, $K^k[Q] = \frac{1}{4}\int_0^a Y_t^2(t)dt$. Also, here $c = (8N)^{-1/3}$, where N is the greater of 1 and the ratio p_2/p_1, and so is positive; and $g = 3(2N)^{5/4}$, i.e., again a positive constant > 1 and depending only upon the constants involved in the boundary conditions.

As before, from (4.18) and (4.10) we deduce

$$\begin{aligned} W^k[Q] &\geqq \tfrac{1}{2}D^k[Q] + K^k[Q] - 2g\sqrt{K^k[Q]} \\ &\geqq \tfrac{1}{2}D[Q] + (\sqrt{K^k[Q]} - g)^2 - g^2 \end{aligned} \tag{4.24}$$

which implies

LEMMA 4.1: *For k-admissible functions Q, $W^k[Q]$ has a lower bound, $-g^2$, independent of k.*

From this Lemma, we obtain inequalities corresponding to (3.20), except that the divisor k of $D^k(q)$ is dropped here. These lead as before to

LEMMA 4.2: *An upper bound M for $W^k[Q]$ implies upper bounds for D^k, H^k, K^k which depend upon M but not upon k.*

In what follows we shall make frequent use of the following well-known lemmas:

LEMMA A: *If $f_s(t)$ is a sequence of non-negative continuous functions defined for $0 \leqq t < \infty$ which converge uniformly in every* *finite interval to a limit function $f_0(t)$, then $f_0(t)$ is continuous, non-negative, and*

$$\int_0^\infty f_0(t)dt \leqq \varliminf_{a\to\infty} \int_0^a f_s(t)dt.$$

LEMMA B: *If, in addition, for every $e > 0$, there exists a quantity $T = T(e)$ such that $\int_T^a f_s(t)dt \leqq e$ for all a, then*

$$\int_0^a f_s(t)dt \to \int_0^\infty f_0(t)dt \text{ as } a \to \infty.$$

Now in the integrals of the functions P and Q taken from 0 to ∞, the contribution of the boundary layer at the stretched end has disappeared as the curve was smoothed out. Hence we cannot expect that the contribution of such integrals over the left boundary layer and of those over the right boundary layer can be obtained from the integral taken over the entire present infinite domain. Therefore it is impossible further to investigate the problem without an additional transformation. We require one which splits our interval in two, bringing the boundary layer of the stretched end back to the finite portion of the interval; although at the same time, the central region in the vicinity of the split will then be carried off to infinity, its resulting inaccessability is not troublesome, since this region is of no interest to us, and can be studied, if desired, before this splitting. Accordingly, to apply these lemmas and inequalities and continue our treatment we now make the further transformation

$$\begin{aligned} R &= t \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2}a \\ a - S &= t \quad \text{for} \quad \tfrac{1}{2}a \leqq t \leqq a. \end{aligned} \tag{4.25}$$

The functionals in (4.05—6—7) then become

$$\begin{aligned} H^k[Q] &= \int_0^{a/2} F'(R)Q'^2(R)dR + \int_0^{a/2} F''(S)Q''^2(a-S)dS \\ &= H^k[Q'] + H^k[Q''], \end{aligned} \tag{4.26}$$

$$D^k[Q] = \int_0^{a/2} Q_R'^2 dR + \int_0^{a/2} Q_S''^2 dS = D^k[Q'] + D^k[Q''], \tag{4.27}$$

$$K^k[Q] = \tfrac{1}{4}\int_0^{a/2} Y_R'^2 dR + \tfrac{1}{4}\int_0^{a/2} Y_S''^2 dS = K^k[Q'] + K^k[Q''], \tag{4.28}$$

respectively, where $Q'(R) \equiv Q(t)$ for $0 \leqq t \leqq \frac{1}{2}a$, $\equiv 0$ for $\frac{1}{2}a < t \leqq a$;
$Q''(a-S) \equiv 0$ for $0 \leqq t < \frac{1}{2}a$, $\equiv Q(t)$ for $\frac{1}{2}a \leqq t \leqq a$;
$Y'(R)$ and $Y''(a-S)$ are similarly defined;

$F'(R) = 1 + R(p_2 - p_1)/ap_1;\ \ F''(S) = 1 + (a - S)(p_2 - p_1)/ap_1.$ We note that $Q'(\frac{1}{2}a) = Q''(\frac{1}{2}a)$ and that $Y'(\frac{1}{2}a) = Y''(\frac{1}{2}a)$. Also, in (4.28)

$$(4.29)\quad Y'_R[Q] = -\int_0^{a/2} \frac{\tilde{R}}{a} Q'^2 d\tilde{R} + \int_0^{a/2} \frac{\tilde{S}}{a} Q''^2 d\tilde{S} + \int_R^{a/2} Q'^2 d\tilde{R}$$

for $t \leqq \frac{1}{2}a$; while for $t \geqq \frac{1}{2}a$, we have

$$(4.30)\quad Y''_S[Q] = \int_0^{a/2} \frac{\tilde{R}}{a} Q'^2 d\tilde{R} - \int_0^{a/2} \frac{\tilde{S}}{a} Q''^2 d\tilde{S} + \int_S^{a/2} Q''^2 d\tilde{S}.$$

We now require the basic

LEMMA 4.3: *Let k_m be a sequence of values of k tending to 0, and let Q_m be a sequence of k_m-admissible functions for which $W^{k_m}[Q_m]$ is bounded. Then there exists a subsequence $Q'_b(R)$ of $Q'_m(R)$ and a subsequence $Q''_b(a - S)$ of $Q''_m(a - S)$ converging uniformly in every finite interval $0 \leqq R \leqq R_0 < \infty$ and $0 \leqq S \leqq S_0 < \infty$, respectively, to an L^2-integrable limit function $Q'_0(R)$ and $Q''_0(S)$, respectively, for which*

$$(4.31)\quad \varliminf_{k \to 0} W^{k_b}[Q^k_b(t)] \geqq W^0[Q'_0] + W^0[Q''_0].$$

By Lemma 4.2, a bound for W^{k_m} implies bounds for the sequences $D[Q_m]$, $H[Q_m]$, $K[Q_m]$. In particular,

$$H^{k_m}[Q_m] = \int_0^{a/2} F'Q'^2_m dR + \int_0^{a/2} F''Q''^2_m dS \leqq A, \text{ say, and}$$

$$D^{k_m}[Q_m] = \int_0^{a/2} [Q'_m]^2_R dR + \int_0^{a/2} [Q''_m]^2_S dS \leqq B, \text{ say, where } A$$

and B are positive constants independent of k. From the second inequality we have at once that

$$(4.32)\quad \int_0^{a/2} [Q'_m]^2_R dR \leqq B, \quad \int_0^{a/2} [Q''_m]^2_S dS \leqq B.$$

In Case I, with $p_1 > 0$, $p_2 > 0$, both F' and F'' are also > 0, so that the first inequality above yields immediately

$$(4.33)\quad \int_0^{a/2} F'Q'^2_m dR \leqq A, \quad \int_0^{a/2} F''Q''^2_m dS \leqq A.$$

If $p_2 \geqq p_1$, both F' and F'' are $\geqq 1$, so that (4.33) remains true when the F^i are deleted; if, however, $p_2 < p_1$, we have somewhat weaker forms. We have then the final results

$$(4.34)\quad \int_0^{a/2} Q'^2_m dR \leqq 2A, \quad \int_0^{a/2} Q''^2_m dS \leqq \theta A,$$

where $\theta = 1$ when $p_2 \geqq p_1$, and $= p_1/p_2$ otherwise.

Since $D[Q'_m]$, $H[Q'_m]$, $K[Q'_m]$ are bounded, we may apply the results of Lemma 3.3 for fixed k. Therefore there exists a subsequence $Q'_b(R)$ converging uniformly in every finite interval $0 \leqq R \leqq R_0 < \infty$ to an L^2-integrable limit function $Q'_0(R)$ for which [cf. (3.21)]

$$\varliminf_{k \to 0} \int_0^{R_0} [Q'_b]^2_R \, dR \geqq \int_0^{R_0} [Q'_0]^2_R \, dR$$

Clearly we can choose a k so small that $\frac{1}{2}a > R_0$.

$$\therefore \lim_{k \to 0} \int_0^{a/2} [Q'_b]^2_R \, dR \geqq \int_0^{R_0} [Q'_0]^2_R \, dR$$

The left side is independent of R_0, which is arbitrary; hence we have finally that $\varliminf_{k \to 0} \int_0^{a/2} [Q'_b]^2_R \, dR \geqq \int_0^{\infty} [Q'_0]^2_R dR$. The treatment of the functional involving Q'' is exactly the same, so that we have

$$(4.35) \qquad \varliminf_{k \to 0} D^k[Q'_b] \geqq D^0[Q'_0]; \quad \varliminf_{k \to 0} D^k[Q''_b] \geqq D^0[Q''_0].$$

For the H's, we begin with $\int_c^{a/2} F' Q'^2_b dR \leqq N \int_c^{a/2} Q'^2_b dR$, with N as defined on P. 141, and c necessarily $\leqq \frac{1}{2}a$. Then $a - c \geqq \frac{1}{2}a$, so that $\int_c^{a/2} Q'^2_b dR \leqq \int_c^{a-c} Q'^2_b dR \leqq 2\sqrt{2K/c}$, the latter from (4.17). Thus $\int_c^{a/2} F' Q'^2_b dR \leqq 2N\sqrt{2K/c}$. Since $K^k[Q'_m]$ is bounded by the hypothesis on $W^k[Q_m]$, then for a given e and c_0, both > 0, we can choose a $c > c_0$ such that $2N\sqrt{2K/c} \leqq e$, and a k so small that $c < \frac{1}{2}a$. We further choose k such that $\left| \int_0^c F' Q'^2_b dR - \int_0^c Q'^2_0 dR \right| \leqq e$. This is possible because we have already proven the existence of subsequences of Q'_m converging uniformly in every finite interval to a continuous limit function Q'_0.

From the last two inequalities we have $\left| \int_0^{a/2} F' Q'^2_b dR - \int_0^c Q'^2_0 dR \right| \leqq 2e$. This result, in conjunction with (4.33), gives us $\int_0^c Q'^2_0 dR \leqq A + 2e$, so that all the more $\int_0^{c_0} Q'^2_0 dR \leqq A + 2e$. However, c_0 is as yet arbitrary. Hence, $\int_0^{\infty} Q'^2_0 dR < \infty$, so that a fortiori $\int_{c_0}^{\infty} Q'^2_0 dR < \infty$. We now choose c_0 such that $\int_{c_0}^{\infty} Q'^2_0 dR \leqq e$, whence $\int_c^{\infty} Q'^2_0 dR \leqq e$.

Combining this with the preceding inequality, we have

$$\left|\int_0^{a/2} F'Q_b'^2 dR - \int_0^\infty Q_0'^2 dR\right| \leqq 3e, \text{ or } \lim_{\overline{k\to 0}} \int_0^{a/2} F'Q_b'^2 dR = \int_0^\infty Q_0'^2 dR.$$

The treatment of Q'' is virtually identical with that of Q' above, so that we have then

$$(4.36) \qquad \lim_{\overline{k\to 0}} H^k[Q_b'] = H^0[Q_0']; \quad \lim_{\overline{k\to 0}} H^k[Q_b''] = H^0[Q_0''].$$

To establish the desired relations for the K's, we employ the identity (for $t \leqq \frac{1}{2}a$)

$$\frac{p_2 - p_1}{p_1}(Y_R^k[Q_b'] - Y_R^0[Q_0']) = H^0[Q_0'] - H^k[Q_b'] + H^0[Q_0'']$$
$$-H^k[Q_b''] + \frac{p_2}{p_1}\left[\int_0^{a/2} Q_b'^2 dR - \int_0^\infty Q_0'^2 dR + \int_0^{a/2} Q_b''^2 dS - \int_0^\infty Q_0''^2 dS\right]$$
$$-[(p_2 - p_1)/p_1]\int_0^R [Q_b'^2 - Q_0'^2]\, d\bar{R}.$$

Since we have already proven that Q_b' converges uniformly to Q_0' in every finite interval, the integral with limits 0 to R clearly tends to zero as $k \to 0$. On the preceding page we showed that $\int_c^{a/2} Q_b'^2 dR \leqq e$, so that $Q_b'^2$ satisfies the requirements of Lemma B; similarly, so does $Q_b''^2$. Hence this Lemma and (4.36) used in the identity above yield the uniform convergence of $Y_R^k[Q_b']$ to $Y_R^0[Q_0']$ as $k \to 0$. Now applying Lemma A to $K^k[Q_b'] = \frac{1}{4}\int_0^{a/2} [Y_b']_R^2 dR$, we have $\lim_{\overline{k\to 0}} K^k[Q_b'] \geqq K^0[Q_0']$. Again, the treatment of the right side ($t \geqq \frac{1}{2}a$), involving $Y[Q'']$, is essentially the same as the case detailed above. Thus we have the result

$$(4.37) \qquad \lim_{\overline{k\to 0}} K^k[Q_b'] \geqq K^0[Q_0']; \quad \lim_{\overline{k\to 0}} K^k[Q_b''] \geqq K^0[Q_0''].$$

From its definition,

$$W^k[Q_b(t)] = D^k[Q_b] - H^k[Q_b] + K^k[Q_b]$$
$$= D^k[Q_b'] + D^k[Q_b''] - H^k[Q_b'] - H^k[Q_b''] + K^k[Q_b'] + K^k[Q_b''].$$

From this identity and (4.35—6—7), (4.31) now follows. Also, from the same three relations, we see that Q_0' and Q_0'' are 0-admissible.

Before stating our next Lemma, it will be simpler to develop the notation required in its statement. We recall that $Q'(R) = Q(t)$

for $0 \leqq t \leqq \frac{1}{2}a$, while $Q''(a-S) = Q(t)$ for $\frac{1}{2}a \leqq t \leqq a$. From Theorem 4.1, we have the existence of the function $Q^0(t)$, a solution of the asymptotic minimum problem. Then by $Q^{0\prime}(R)$ and $Q^{0\prime\prime}(S)$ we shall designate the solutions of the similar asymptotic minimum problems set up for each half of the split interval. We now construct the k-admissible function $Q^*(t)$ as follows:

First we define $Q^{*\prime}(R) = Q^{0\prime}(R)$ for $0 \leqq R \leqq a/4$
$= L'(R)$ for $a/4 \leqq R \leqq \frac{1}{2}a$,

where $L'(R)$ is the linear function joining the points $[\frac{1}{2}a, 0]$ and $\left[\frac{a}{4}, Q^{0\prime}\left(\frac{a}{4}\right)\right]$; similarly, we define

$Q^{*\prime\prime}(S) = Q^{0\prime\prime}(S)$ for $0 \leqq S \leqq a/4$
$= L''(S)$ for $a/4 \leqq S \leqq \frac{1}{2}a$, where again $L''(S)$ is the linear function joining the points $[\frac{1}{2}a, 0]$ and $\left[\frac{a}{4}, Q^{0\prime\prime}\left(\frac{a}{4}\right)\right]$. We note for later use that the absolute values of the slopes L' and L'' are $\frac{4}{a}Q^{0\prime}\left(\frac{a}{4}\right)$ and $\frac{4}{a}Q^{0\prime\prime}\left(\frac{a}{4}\right)$, respectively. Finally, we now define

$$\begin{aligned} Q^*(t) &= Q^{*\prime}(R) \text{ for } 0 \leqq t \leqq \tfrac{1}{2}a, \\ &= Q^{*\prime\prime}(S) \text{ for } \tfrac{1}{2}a \leqq t \leqq a. \end{aligned}$$

We are now ready for

LEMMA 4.4:

$$\lim_{\overline{k\to 0}} W^k[Q^*(t)] = W^0[Q^{0\prime}] + W^0[Q^{0\prime\prime}] \tag{4.38}$$

We have $D^k[Q^*(t)] = \int_0^a Q_t^{*2}(t)dt$

$$= \int_0^{a/2} Q_R^{*\prime 2}(R)dR + \int_0^{a/2} Q_S^{*\prime\prime 2}(S)dS$$

$$= \int_0^{a/4} Q_R^{0\prime 2}dR + \int_{a/4}^{a/2} L_R'^2\,dR + \int_0^{a/4} Q_S^{0\prime\prime 2}\,dS + \int_{a/4}^{a/2} L_S''^2 dS$$

$$= \int_0^{a/4} Q_R^{0\prime 2}dR + \int_0^{a/4} Q_S^{0\prime\prime 2}dS + \frac{4}{a}\cdot\left[Q^{0\prime 2}\left(\frac{a}{4}\right) + Q^{0\prime\prime 2}\left(\frac{a}{4}\right)\right].$$

Since $Q^{0\prime}$ and $Q^{0\prime\prime}$ are 0-admissible, $\int_0^\infty Q_R^{0\prime 2}dR < \infty$, $\int_0^\infty Q_S^{0\prime\prime 2}dS < \infty$; also $Q^{0\prime}$ and $Q^{0\prime\prime}$ must remain bounded as $k \to 0$. Hence, as k does $\to 0$, the last term $\to 0$, while the first two approach $\int_0^\infty Q_R^{0\prime 2}dR$ and $\int_0^\infty Q_S^{0\prime\prime 2}dS$, respectively. Hence

$$\underline{\lim}_{k\to 0} D^k[Q^*(t)] = D^0[Q^{0\prime}] + D^0[Q^{0\prime\prime}].$$

For H, we have $H^k(Q^*(t)] = \int_0^a F(t)\,Q^{*2}(t)dt$

$$= \int_0^{a/2} F'(R)Q^{*\prime 2}(R)dR + \int_0^{a/2} F''(S)Q^{*\prime\prime 2}(a-S)dS$$

$$= \int_0^{a/4} F'Q^{0\prime 2}dR + \int_{a/4}^{a/2} F'L'^2dR + \int_0^{a/4} F''Q^{0\prime\prime 2}dS + \int_{a/4}^{a/2} F''L''^2dS.$$

Now since $\int_0^\infty Q^{0\prime 2}dR < \infty$, $\int_0^\infty \frac{RQ^{0\prime 2}}{R}dR < \infty$. Hence $RQ^{0\prime 2}$ cannot remain above a positive bound. Therefore, there exists a subsequence of R's, with $R \to \infty$, such that $RQ^{0\prime 2} \to 0$; in particular, then, for this subsequence, $aQ^{0\prime 2}\left(\frac{a}{4}\right) \to 0$ as $a \to \infty$. Then

$$0 \leqq \lim \int_{a/4}^{a/2} F'L'^2dR \leqq \lim \underset{a/4 \leqq R \leqq a/2}{\mathrm{Max}} L'^2 \int_{a/4}^{a/2} F'(R)dR$$
$$\leqq \lim aQ^{0\prime 2}\left(\frac{a}{4}\right) \cdot \left[\frac{1}{4} + 3(p_2 - p_1)/32p_1\right] = 0$$

(where we have used the definition of $F'(R)$ on P. 143), i.e., $\lim \int_{a/4}^{a/2} F'L'^2dR = 0$. Similarly, $\lim \int_{a/4}^{a/2} F''L''^2dS = 0$. As shown in the derivation of (4.36), $\lim \int_0^{a/2} F'Q^{0\prime 2}dR = \int_0^\infty Q^{0\prime 2}dR$; also, $\lim \int_0^{a/2} F''Q^{0\prime\prime 2}dS = \frac{p_2}{p_1}\int_0^\infty Q^{0\prime\prime}dS$. Therefore

$$\underline{\lim}\, H^k[Q^*(t)] = \int_0^\infty Q^{0\prime 2}dR + \frac{p_2}{p_1}\int_0^\infty Q^{0\prime\prime 2}dS$$
$$= H^0[Q^{0\prime}] + H^0[Q^{0\prime\prime}].$$

Lastly, for K we have $K^k[Q^*(t)] = \int_0^a Y_t^2[Q^*(t)]dt = \int_0^{a/2} Y_R'^2\,dR + \int_0^{a/2} Y_S''^2dS$, where $Y_t^k[Q^*(t)]$, Y_R', and Y_S'' are as given in (4.21), (4.29), and (4.30), respectively.

For each half of the split interval we must consider two cases, according as the variable R (or S) is $\leqq$ or $\geqq \frac{a}{4}$. We give the details for the left side only. Thus, if $R \leqq \frac{a}{4}$,

$$Y'^k_R = -\int_0^{a/4}\frac{\tilde{R}}{a}Q^{0'2}d\tilde{R} - \int_{a/4}^{a/2}\frac{\tilde{R}}{a}L'^2d\tilde{R} + \int_0^{a/4}\frac{\tilde{S}}{a}Q^{0''2}d\tilde{S}$$
$$+\int_{a/4}^{a/2}\frac{\tilde{S}}{a}L''^2d\tilde{S} + \int_R^{a/4}Q^{0'2}d\tilde{R} + \int_{a/4}^{a/2}L'^2d\tilde{R},$$

while for this case, $Y'^0_R = \int_R^\infty Q^{0'2}d\tilde{R}$. Now (omitting the various indices on R and Q for simplicity) we may write $\frac{1}{a}\int_0^{a/4}RQ^2dR \leqq \frac{b}{a}\int_0^b Q^2dR + \frac{1}{4}\int_b^{a/4}Q^2dR$, ($b$ arbitrary), which in turn $\leqq \frac{b}{a}\int_0^\infty Q^2dR + \frac{1}{4}\int_b^\infty Q^2dR$. Since Q is here a 0-admissible function, $\int_0^\infty Q^2dR < \infty$. Hence, given an $e > 0$, we may first choose b such that $\frac{1}{4}\int_b^\infty Q^2dR < \frac{1}{2}e$, and then choose an a large enough so that $(b/a)\int_0^\infty Q^2dR < \frac{1}{2}e$. Thus we have that as $a \to \infty$, $\int_0^{a/4}\frac{\tilde{R}}{a}Q^{0'2}d\tilde{R} \to 0$. Hence, as $k \to 0$, the first and third integrals in Y'^k_R above clearly tend to zero; while from the argument on P. 33, the second, fourth, and sixth integrals likewise tend to zero. Hence $Y'^k_R \to Y'^0_R$ as $k \to 0$. When $R \geqq a/4$, the expression for Y'^k_R is the same as formerly, except that the single term $\int_R^{a/2}L'^2d\tilde{R}$ replaces the last two integrals of the first case, while Y'^0_R becomes $\int_R^\infty L'^2d\tilde{R}$. It is at once clear, then, that again $Y'^k_R \to Y'^0_R$ as $k \to 0$. Having the convergence of the $Y'^k_{R,S}$ to the $Y'^0_{R,S}$, we can apply Lemma A to K as before; this gives us that

$$\underline{\lim}\, K^k[Q^*(t)] = K^0[Q^{0'}] + K^0[Q^{0''}].$$

Combining the results for D, H, and K, we have (4.38).

We now prove our principal theorem,

THEOREM 4.2: *Given the minimizing function $Q^k(t)$ and its associated functions $Q'^k(R)$ and $Q''^k(a-S)$, with $Q'^k(0) \geqq 0$, $Q''^k(0) \geqq 0$. As $k \to 0$, $Q'^k(R)$ and $Q''^k(a-S)$ tend uniformly in every finite interval $0 \leqq R \leqq R_0 < \infty$, $0 \leqq S \leqq S_0 < \infty$, respectively, to the minimizing functions $Q^{0'}(R)$ and $Q^{0''}(S)$, respectively, with $Q^{0'}(0) \geqq 0$ and $Q^{0''}(0) \geqq 0$. These limit functions are unique, and, moreover, $W^k[Q'^k(R)] \to W^0[Q^{0'}]$ and $W^k[Q''^k(a-S)] \to W^0[Q^{0''}]$.*

We now take *any* sequence of positive values of k tending to zero, and solutions $Q^k(t)$ of the corresponding minimum problems M^k — solutions which exist according to Theorem 4.1. The values of the minima, w^k, of $W^k[Q^k]$ have the common upper bound zero, i.e., $w^k \leqq 0$. This follows immediately from $W^k[0] = 0$, since $Q \equiv 0$ is an admissible function. We can therefore apply Lemma 4.3 to the sequence Q^k with $k \to 0$. This Lemma assures the existence of the subsequences $Q'(R)$ and $Q''(a - S)$ converging in the sense of the Lemma to 0-admissible limit functions $Q_0'(R)$ and $Q_0''(S)$, respectively. From now on, Q^k refers to such a sequence. From (4.31),

$$\underline{\lim}\, W^k[Q^k(t)] \geqq W^0[Q_0'] + W^0[Q_0'']. \tag{4.39}$$

We proceed to show that Q_0' and Q_0'' *solve* the corresponding minimum problems M^0.

The minimum problem M^0, according to Theorem 4.1, has a solution $Q^*(t)$ for which, by Lemma 4.4,

$$\underline{\lim}\, W^k[Q^*(t)] = W^0[Q^{0\prime}] + W^0[Q^{0\prime\prime}]. \tag{4.40}$$

As a consequence of the minimum properties of $Q^*(t)$ and $Q^k(t)$,

$$w^0 = W^0[Q^{0\prime}] + W^0[Q^{0\prime\prime}] \leqq W^0[Q_0'] + W^0[Q_0''], \tag{4.41}$$

$$w^k = W^k[Q^k] \leqq W^k[Q^*]. \tag{4.42}$$

This last gives

$$\overline{\lim}\, W^k[Q^k] \leqq \overline{\lim}\, W^k[Q^*]. \tag{4.43}$$

Successive consideration of (4.43), (4.40), (4.41), and (4.39) yields

$$\begin{aligned} \overline{\lim}\, W^k[Q^k] &\leqq \overline{\lim}\, W^k[Q^*] = W^0[Q^{0\prime}] + W^0[Q^{0\prime\prime}] \\ &\leqq W^0[Q_0'] + W^0[Q_0''] \leqq \underline{\lim}\, W^k[Q^k]. \end{aligned} \tag{4.44}$$

Since $\underline{\lim}\, W^k[Q^k] = \overline{\lim}\, W^k[Q^k]$, this implies the equality

$$W^0[Q^{0\prime}] + W^0[Q^{0\prime\prime}] = W^0[Q_0'] + W^0[Q_0'']. \tag{4.45}$$

This, in turn, implies

$$W^0[Q^{0\prime}] = W^0[Q_0']; \quad W^0[Q^{0\prime\prime}] = W^0[Q_0'']; \tag{4.46}$$

for otherwise either $W^0[Q^{0\prime}] > W^0[Q_0']$,

or $$W^0[Q^{0\prime\prime}] > W^0[Q_0''].$$

However, each of these alternatives is impossible, since $Q^{0\prime}$ an($Q^{0\prime\prime}$ are solutions of the minimum problems. Hence (4.46) hold Since $W^0[Q^{0\prime}]$ is the g.l.b. of W^0 for the left hand stretched asymp totic problem, and $W^0[Q^{0\prime\prime}]$ the same for the right, it follows tha the functions $Q = Q_0'(R)$ and $Q = Q_0''(S)$ are the solutions of the two minimum problems M^k for $k = 0$, with $Q_0'(0) \geqq 0$ and $Q_0''(0) \geqq 0$.

From Theorem 4.1 and the results of [1] (Th. 8.1) for the asymptotic problem, we know that each minimum problem M^0 has at most one solution Q with $Q(0) \geqq 0$. Therefore $Q^{0\prime} = Q_0'$ and $Q^{0\prime\prime} = Q_0''$; i.e., all convergent sequences $Q'^k(R)$ and $Q''^k(a - S)$ converge to the same limit functions $Q^{0\prime}(R)$ and $Q^{0\prime\prime}(S)$, respectively. If a sequence has the property that every subsequence contains a convergent subsequence with limit L, and if L is the same for all such convergent subsequences, then the original sequence itself converges to L. Therefore we can conclude in our case that the solutions $Q'^k(R)$ and $Q''^k(a - S)$, with $Q'^k(0) \geqq 0$ and $Q''^k(0) \geqq 0$, of the minimum problems M^k converge, as $k \to 0$, to the unique solutions $Q^{0\prime}(R)$ and $Q^{0\prime\prime}(S)$, respectively, [with $Q^{0\prime}(0) \geqq 0$ and $Q^{0\prime\prime}(0) \geqq 0$] of the minimum problems M^0. This completes the proof of Theorem 4.2.

We conclude this section with

THEOREM 4.3: *For a fixed t, $P^k(t) \to P^{0\prime}(R)$ if $0 \leqq t \leqq \frac{1}{2}a$ and $\to P^{0\prime\prime}(S)$ if $\frac{1}{2}a \leqq t \leqq a$, as $k \to 0$.*

In the proof of Lemma 4.3, we showed the uniform convergence of $Y_R'^k$ to $Y_R^{0\prime}$ and $Y_S''^k$ to $Y_S^{0\prime\prime}$ as $k \to 0$ [P. 145]. Since both $Y'^k(0) = Y^{0\prime}(0) = 0$ and $Y''^k(0) = Y^{0\prime\prime}(0) = 0$, the foregoing gives us the uniform convergence of $Y'^k(R)$ to $Y^{0\prime}(R)$ and of $Y''^k(a - S)$ to $Y^{0\prime\prime}(S)$. From the definition of $Y(t)$ in (4.19) this implies that

$$\begin{aligned} P^k(t) - t(p_2 - p_1)/ap_1 &\to P^{0\prime}(R) \text{ if } 0 \leqq t \leqq \tfrac{1}{2}a \\ &\to P^{0\prime\prime}(S) \text{ if } \tfrac{1}{2}a \leqq t \leqq a, \text{ uniformly.} \end{aligned}$$

Since $a \to \infty$ as $k \to 0$, for a fixed t, $t/a \to 0$ with k. Hence, for a fixed t, $P^k(t) \to P^{0\prime}(R)$ if $0 \leqq t \leqq \frac{1}{2}a$, and $\to P^{0\prime\prime}(S)$ if $\frac{1}{2}a \leqq t \leqq a$ as $k \to 0$.

§ 5. Expansion in series.

Before we can discuss the limit procedure in the interior, it is necessary to have some numerical details concerning the asymptotic solution of the stretched problem. The reader will recall that this asymptotic problem was precisely formulated at the beginning of § 4 [cf. Pp. 138—9).

Accordingly, we introduce new variables x, y, z (not to be confused with the space variables used earlier) as follows:

$$(5.01) \qquad x = je^{-wt},\ y = -w^{-2}P(t),\ z = \tfrac{1}{2}\sqrt{2}w^{-2}Q(t),$$

where j and w are numbers to be determined. The interval for x is $j_a \leqq x \leqq j$, where $j_a = je^{-wa}$, and tends to zero as a becomes infinite. In these new variables, the differential equations (4.02) become

$$x(xy_x)_x + z^2 = 0 \qquad (5.02)$$
$$x(xz_x)_x - yz = 0.$$

The introduction of the new variable x has the effect that the resulting differential equations (5.02) possess solutions expressible as power series in x:

$$y = \sum_{k=0}^{\infty} (-1)^k y_k x^{2k}; \; z = \sum_{m=0}^{\infty} (-1)^m z_m x^{2m+1} \qquad (5.03)$$

Substituting these series into (5.02), we find the following formulas for y_k and z_m:

$$(2k)^2 y_k = \sum_{m+n=k-1} z_m z_n \qquad k = 1, 2, \ldots \qquad (5.04)$$

$$(2m+1)^2 z_m = \sum_{n+k=m} z_n y_k \qquad m = 0, 1, \ldots \qquad (5.05a)$$

From the second equation, $m = 0$ yields $z_0 = y_0 z_0$. Assuming for the moment that $z_0 \neq 0$, this coefficient is then arbitrary; we assign to it the numerical value $z_0 = 4$, for the reason given below. Obviously then $y_0 = 1$. We may now rewrite (5.05a) as a proper recursion formula:

$$4m(m+1)z_m = \sum_{n=0}^{m-1} z_n y_{m-n} \qquad (5.05)$$

It is found amply sufficient to calculate ten terms in each series.

We turn now to consideration of the boundary condition associated with (5.02). The right hand end values, now taken off to infinity, are automatically satisfied in view of (5.02) and the assumed development into the power series (5.03). The boundary conditions (4.03) for $t = 0$ become

$$y(j) = -w^{-2}, \; z_x(j) = 0. \qquad (5.06)$$

The second is a transcendental equation in j, to be solved for its lowest root, which is found to be $j = .98618$. (The reason for assuming $z_0 = 4$ was to make $j \sim 1$). This value inserted in the first equation determines w, which is found to be $w = .68754$.

Once j and w are determined, the limit boundary value problem is solved in principle. The function $P(t)$ begins with the prescribed value $P(0) = 1$, decreases monotonically, assumes the value zero at $t = .941$, and approaches the value $P(\infty) = -w^2 = -.47271$

as $t \to \infty$, the latter value resulting from (5.01). The function $Q(t)$ decreases monotonically and approaches zero as t tends to infinity. [1])

The results just obtained were predicated on the assumption that $z_0 \neq 0$. We must now consider the alternate possibility. If z_0 does equal zero, then it is y_0 that is arbitrary. In this case it is not difficult to show that if y_0 is not chosen in the form $(2m + 1)^2$, all remaining coefficients in both series are zero. Thus this choice of y_0 leaves us with the limit solution of the trivial case, $y = y_0$, $z \equiv 0$ (cf. § 4, P. 138). If on the other hand, y_0 *is* chosen $= (2m+1)^2$, m any integer, then the only non-vanishing coefficients are y_0, y_{2m+1}, $y_{2(2m+1)}$, ... and z_m, z_{3m+1}, z_{5m+1}, ... This gives us, then, an infinitude of solutions, depending upon the value of m. However, each of these solutions may be reduced to (5.03) by a transformation which simultaneously carries the original differential equations into formally identical equations in the new variables. Hence this choice of y_0 does not lead to an essentially new solution. Consequently, taking $z_0 = 0$ leads to results which are either not new or of no interest.

We turn now to a proof of the convergence of the solutions (5.03). For this purpose we first establish the inequality

$$\text{(5.08)} \qquad \sum_{i=1}^{k+1} \frac{1}{i^2} \frac{1}{(k - i + 2)^2} < (k + 1)^{-1}$$

We remark at once that the pairs of terms of the sum equally distant from either end are identical. Next, we assert that

$$\frac{1}{i} \cdot \frac{1}{k - i + 2} > \frac{1}{i + 1} \cdot \frac{1}{k - i + 1} \quad \text{for certain } i\text{'s.}$$

This is true if $\qquad (i + 1)(k - i + 1) > i(k - i + 2),$

i.e., if $\qquad \frac{1}{2}(k + 1) > i.$

Hence the terms of our sum decrease after the first, $(k + 1)^{-2}$, until we reach either the two equal — and minimum — central terms (when $k + 1$ is even), or else the single minimum middle term (when $k + 1$ is odd), and then begin increasing until we reach the last term, again $(k + 1)^{-2}$. This follows from the inequality just demonstrated and the fact that, for positive integers,

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[1]) The numerical results in the foregoing paragraph are taken directly, with only partial verification, from the Friedrichs-Stoker paper referred to previously. This is another point where the two papers are identical in form.

$a > b$ implies $a^2 > b^2$. Hence we have

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \cdot \frac{1}{(k-i+2)^2} < \frac{1}{(k+1)^2} \cdot (k+1)$$
$$< (k+1)^{-1},$$

our desired result.

Referring now to the coefficients (5.04) and (5.05) of the expansion in series, we shall prove by induction

$$z_{n-1} \leqq 2^3/n^2 \quad \text{and} \tag{5.09}$$

$$y_{n-1} \leqq 2^4/n^2 \tag{5.10}$$

Using the recursion formulas (5.04—5) we calculate

$$y_0 = 1,\ y_1 = 4,\ y_2 = 1,\ y_3 = 2/9, \quad \text{and}$$
$$z_0 = 4,\ z_1 = 2,\ z_2 = \tfrac{1}{2},\ z_3 = 11/108,$$

except that z_0, being arbitrary, was assumed as 4, as discussed earlier in this section. We observe that these values verify (5.09) and (5.10) for $n = 1$, 2, 3, and 4.

For our double induction, we assume that (5.09—10) hold for all n from $n = 4$ to $n = k$, the earlier cases having been verified directly. Then (5.04) gives

$$\begin{aligned}
(2k)^2 y_k &= \Sigma z_m z_n = \Sigma z_m z_{k-m-1} \\
&\leqq \Sigma [2^3/(m+1)^2][2^3/(k-m)^2] && \text{[using (5.09)]} \\
&\leqq 2^6 \sum_{i=1}^{k} \frac{1}{i^2} \cdot \frac{1}{(k-i+1)^2} && [\text{with } m+1=i] \\
&\leqq 2^6/k && [\text{from (5.08), with } k+1=k]
\end{aligned}$$

$$\therefore y_k \leqq 2^4/k^3, \text{ which is } \leqq 2^4/(k+1)^2 \text{ for } k \geqq 3.$$

Since our induction started with $k = 4$, we have that when (5.09) and (5.10) are true for $n = k$, (5.10) is true for $n = k + 1$. What about (5.09) then? We have

$$\begin{aligned}
4k(k+1)z_k &= \Sigma z_m y_{k-m} && \text{which, by hypothesis,} \\
&\leqq \sum_{0}^{k-1} [2^3/(m+1)^2][2^4/(k-m+1)^2] \\
&\leqq 2^7 \sum_{1}^{k} \frac{1}{i^2} \cdot \frac{1}{(k-i+2)^2} && [\text{with } m+1=i] \\
&\leqq 2^7 \sum_{1}^{k+1} \frac{1}{i^2} \cdot \frac{1}{(k-i+2)^2} && \text{[adding an extra term on the right]}
\end{aligned}$$

$$\therefore 4k(k+1)z_k \leqq 2^7/(k+1) \qquad \text{[from (5.08)]}$$

$\therefore z_k \leqq 2^5/k(k+1)^2$, which is $\leqq 2^3/(k+1)^2$ whenever $k \geqq 4$. Again, since the induction started with $k = 4$, we have shown that when (5.09) and (5.10) are true for $n = k$, (5.09) is also true for $n = k + 1$. With this result, (5.09) and (5.10) are established for all n.

We are now able to prove the convergence of (5.03). For the first series, $y_k x^{2k} \leqq 2^4 x^{2k}/(k+1)^2 = u_k$, say. Using the Cauchy Ratio Test for the u-series, we have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x^2 \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = x^2 \,.\, 1$$

Thus the u-series converges at least for $-1 < x < 1$. However, the range of x is $0 \leqq j_a \leqq x \leqq j = .98618 < 1$ [cf. Pp. 150—1]. Hence the u-series is convergent for all values of x under consideration, and therefore, by comparison, so is the series for y. The solution z is seen to be convergent in exactly the same manner. Thus both power series expansions in (5.03) are convergent. Moreover, these series converge uniformly ([4], V. I, P. 392).

The information just obtained about the asymptotic solutions P^0, Q^0 of the stretched problem enables us to derive some inequalities involving p, q, our original unstretched functions. These inequalities will be of great use in our study of the limit procedure in the interior, to be discussed in the next section. We also desire some numerical properties of the limit functions as expanded in the power series, as well as two theorems identifying these functions with the limit functions of § 4.

Since the alternating series $z = \sum_{m=0}^{\infty} (-1)^m z_m x^{2m+1}$, with $z_{n-1} \leqq 8/n^2$, is uniformly convergent for all x's considered, we may write $z < z_0 x - z_1 x^3$, or, using the calculated values of z_i, $z < 4x - 2x^3$. Transforming back to our stretched variables by (5.01), we have

$$Q(t) < \sqrt{2}w^2(4je^{-wt} - 2j^3e^{-3wt})$$

where $j = .98618$ and $w = .68754$. Hence we have

$$Q^2(t) < 2w^4(16j^2e^{-2wt} - 16j^4e^{-4wt}).$$

Moreover, because of the uniform convergence, we may differentiate term by term, and the result is also uniformly convergent Similarly, Q^2 and Q_t^2 may be integrated term by term. Hence we also have

$$Q_t(t) < \sqrt{2}w^3(-4je^{-wt} + 6j^3e^{-3wt}) \quad \text{and}$$
$$Q_t^2(t) < 2w^6(16j^2e^{-2wt} - 48j^4e^{-4wt}).$$

Since the lim $e^{-rt} = 0$ as $t \to \infty$, with r any positive constant, and

and $\int e^{-rt}dt$ is essentially e^{-rt}, we have

$$(5.11) \qquad Q(t) \to 0 \text{ as } t \to \infty;$$

$$(5.12) \qquad D^0[Q] = \int_0^\infty Q_t^2 dt \quad \text{is finite, i.e., exists;}$$

$$(5.13) \qquad H^0[Q] = \int_0^\infty Q^2 dt \quad \text{is finite, i.e., exists.}$$

Using the series for y, we find

$$P(t) = -w^2(1 - 4j^2e^{-2wt} + j^4e^{-4wt}\ldots);$$
$$P_t(t) = -w^3(8j^2e^{-2wt} - 4j^4e^{-4wt}\ldots);$$
$$[P_t(t)]^2 < w^6(64j^4e^{-4wt}). \qquad \text{Hence}$$

$$(5.14) \qquad K^0[Q] = \int_0^\infty P_t^2 dt \quad \text{is finite, i.e., exists.}$$

We are now prepared for our theorems. First,

THEOREM 5.1: *The function $Q(t)$ defined by the infinite series (5.03) is identical, for $0 \leqq t \leqq \frac{1}{2}a$, with the unique limit function $Q^{0\prime}(R)$ referred to in Theorem 4.2.*

From (5.12—3—4), we see that the admissibility conditions are all satisfied by the power series function $Q(t)$. It is also a solution of our differential equations and satisfies the boundary conditions at the finite end — these were used, we recall, in the calculation of the constants j and w. From Theorem 4.2, the limit solution $Q^{0\prime}(R)$, for $0 \leqq t \leqq \frac{1}{2}a$, is unique, apart from sign and once the restriction $Q^{0\prime}(0) \geqq 0$ is imposed. Hence $Q(t)$ is identical, for $0 \leqq t \leqq \frac{1}{2}a$, with $Q^{0\prime}(R)$.

THEOREM 5.2: *The function $P(t)$ defined by the infinite series (5.03) is identical, for $0 \leqq t \leqq \frac{1}{2}a$, with the unique limit function $P^{0\prime}(R)$ referred to in Theorem 4.3.*

Writing the series for z in terms of Q and t, we have

$$\tfrac{1}{2}\sqrt{2}w^{-2}Q(t) = \sum_0^\infty (-1)^m z_m (je^{-wt})^{2m+1}, \quad \text{whence}$$

$$Q^2(t) = 2w^4 \sum_{k=1}^\infty \left(\sum_{m=0}^{k-1} (-1)^{k+1} z_m z_n \right) j^{2k} e^{-2kwt},$$

while the expressions for $P(t)$ and $P_t(t)$ were given above. Elementary calculations show that for the power series functions, $P_t(t) = -\frac{1}{2}\int_t^\infty Q^2(\tilde{t})d\tilde{t}$. This is the same relation [cf. 4.08⁰), P. 139] satisfied by P_t^0 of the minimum problem. Hence the derivatives of the two functions are identical, so that the functions themselves can differ at most by a constant. However, both functions satisfy the boundary conditions at the finite end. Consequently, they are identical.

We claim here that the restriction on the values of t to the interval $0 \leqq t \leqq \frac{1}{2}a$ is not a serious one, for, given any numerical value of t, however large, we shall ultimately come to a k so small that t is $\leqq \frac{1}{2}a$. The inner significance of this will be made clear very shortly, when we shall also answer a question which naturally arises here — what roles do $Q^{0\prime\prime}(S)$ and $P^{0\prime\prime}(S)$ play?

We conclude this section with the development of the inequalities referred to on P. 154, and the interpretation just promised.

We return to the discussion of P. 151. Since the limit function $P^0(t)$ is a monotonically decreasing function with its zero at $t = .941$, we can select a value of t, say $t = l$, $> .941$, such that $P^0(l) < 0$; moreover, because of the uniform convergence of $P^k(t)$ to $P^{0\prime}(R)$ — now established identical with $P^0(t)$ for $0 \leqq t \leqq \frac{1}{2}a$ — with t fixed but $\leqq \frac{1}{2}a$, (cf. Th. 4.3), we can find a value of k, $k = k'$, such that for all $k \leqq k'$ all t's under consideration will be $\leqq \frac{1}{2}a$ and $P^k(l) < 0$. Specifically, we shall choose our k' so that

$$(5.15) \qquad 2P^0(l) \leqq P^k(l) \leqq \tfrac{1}{2}P^0(l).$$

Since P^0 is a known function and l a definite quantity, $P^0(l)$ is a definite quantity independent of k.

Now $P_{tt} = \frac{1}{2}Q^2$, so that P is everywhere concave upward. For $k \neq 0$, $P^k(0) = 1$, $P^k(a) = p_2/p_1$, where the latter value is positive whenever p_1, p_2 have the same sign. Consequently, for $0 < k \leqq k'$, P^k starts out at $+ 1$, crosses the axis somewhere to the left of l, remains negative for a while, and then recrosses the axis to become positive again before $t = a$. Hence our k' can be determined such that P^k remains negative for $l \leqq t \leqq \frac{1}{2}a$ and for $k \leqq k'$. Returning now to the original variables and functions of §§ 1, 2, and 3 by means of the inverse of (4.01),

$$(5.16) \qquad x = t\sqrt{k/p_1} - 1 = 2t/a - 1$$

we have from (5.15) that

$$(5.17) \qquad p^k(x) < 0 \qquad \text{for } x_l \leqq x \leqq 0,$$

where x_l is the transform of l; i.e., $x_l = 2l/a - 1$. We note that $x_l \to - 1$ as $k \to 0$.

It was remarked in the introduction to § 4 that because of the symmetry, a detailed study of the stretching procedure was necessary for one side only. However, at this point we find it required to examine briefly the stretching off to infinity of the other side; this will provide us with another inequality analogous to (5.17), as well as a full insight into the dual limit situation.

Accordingly, we subject our original variables and functions to the transformation

$$(5.18) \qquad T = (1 - x)\sqrt{p_2/k}; \; P = p/p_2; \; Q = \sqrt{k}q/p_2.$$

This gives us the new equations

$$(5.19) \qquad P_{TT} = \tfrac{1}{2}Q^2, \; Q_{TT} + PQ = 0,$$

and the new boundary conditions

$$(5.20) \qquad \begin{aligned} P(0) &= 1, \quad P(A) = p_1/p_2, \\ Q_T(0) &= 0, \quad Q_T(A) = 0, \end{aligned}$$

with $0 \leqq T \leqq A$, where $A = 2\sqrt{p_2/k}$.

We note that this variable T is essentially the S of § 4. There $S = a - t = a - \frac{1}{2}a(x + 1) = \frac{1}{2}a(1 - x)$, while here directly $T = \frac{1}{2}A(1 - x)$. The only difference is in the constant factors p_1, p_2 involved in a and A. Actually $a = \sqrt{p_1/p_2}A$, so that precisely $S = \sqrt{p_1/p_2}T$. This relationship affords the insight promised on P. 156.

For x's on the left side of the original interval, i.e., for $-1 \leqq x \leqq 0$, (4.01) carries us into the stretched variable t in the range $0 \leqq t \leqq \frac{1}{2}a$. Theorems 4.2 and 4.3 demonstrated that for this range $Q^k(t)$ and $P^k(t)$ approached unique limit functions (apart from sign) $Q^{0\prime}(R)$ and $P^{0\prime}(R)$, respectively, which the theorems of this section in turn identified with the power series expansions of Q and P with the right side carried off to infinity. In this discussion, if t is given in terms of a, it must be held $\leqq \frac{1}{2}a$, to be a transform of an x from the left side; but if t is given merely as an arithmetical quantity, it may always be regarded as a transform of such an x, for we can always consider k's sufficiently close to 0 that $\frac{1}{2}a$ is larger than any preassigned numerical value. Hence, as remarked, the restriction on t in Theorems 5.1 and 5.2 is not a serious one.

When we are concerned with k's on the right or positive side of the original range, we employ the transformation (5.18) which carries us over into the stretched variable T, now held in $0 \leqq T \leqq \frac{1}{2}A$. We observe that the new equations in T are identical with those in t, as are the boundary conditions, except for those at A and a, respectively. However, in the expansion giving $P^0(t)$ it is precisely the boundary condition at the non-zero end which can no longer be satisfied. Hence the power series expansion of the function $P^0(T)$ is formally identical with that for $P^0(t)$. Because of this formal identity with the expressions in the variable t, § 4 demonstrates also the existence of unique limit functions analogous to $Q^{0\prime}(R)$ and $Q^{0\prime\prime}(S)$ — let us temporarily refer to them as Q_1 and Q_2, respectively. As in Theorem 5.1, we should find the $Q(T)$ of the power series expansion identical, for $0 \leqq T \leqq \frac{1}{2}A$, with Q_1, and we should ignore the Q_2, as we did here. But the relationship between S and T referred to above makes it clear that this Q_1 is really our old $Q^{0\prime\prime}(S)$, apart from a constant factor involving p_1 and p_2; whence by a second reflection, Q_2 is the same as $Q^{0\prime}(R)$. Thus, finally, the true nature of the limit situation is made apparent.

For x's on the left, our stretched function $Q^k(t)$ approaches uniformly the limit function $Q^{0\prime}(R)$, while for x's on the right, we employ the stretched function $Q^k(T)$, which approaches uniformly its limit function, Q_1, essentially $Q^{0\prime\prime}(S)$. Thus there are only 2 — and not 4 — distinct limit functions to be found for the stretched variable. Specific properties of these functions can be found by use of the power series expansions, which also afford a proof of the convergence of the limit functions as $t \to \infty$.

When we consider the interior, the appropriate transformations and the numerical results obtained here enable us to give explicit form to the limit functions approached non-uniformly by the original unstretched functions p and q.

We return to our numerical work. Since $P^0(T)$ is formally identical with $P^0(t)$, then here again $P^0(T)$ begins with the value $+1$ at $T = 0$, becomes 0 at $T = .941$, and decreases monotonically to $-.47271$ as $T \to \infty$. Hence, as before, we can choose a value of T, say $T = r$, $> .941$, such that $P^0(r) < 0$, and choose a k, k'', such that for all $k \leqq k''$, T will remain $\leqq \frac{1}{2}A$ and

$$(5.21) \qquad 2P^0(r) \leqq P^k(r) \leqq \tfrac{1}{2}P^0(r).$$

Numerically, we may take $l = r$, whence $P^0(l) = P^0(r)$, because of the formal identity. Also, as before, our k'' can be determined such that $P^k(T)$ remains negative for $r \leqq T \leqq \frac{1}{2}A$, for $k \leqq k''$. Therefore, similarly,

$$(5.22) \qquad p^k(x) < 0 \text{ for } 0 \leqq x \leqq x_r,$$

where $x_r = -2r/A + 1$. Again we note that $x_r \to +1$ as $k \to 0$. Thus we have

$p^k(x_l) = p_1 P^k(l) \leqq \frac{1}{2}p_1 P^0(l)$, a constant independent of k; and $p^k(x_r) = p_2 P^k(r) \leqq \frac{1}{2}p_2 P^0(r)$, a constant also independent of k.

Now in our equation $p_{xx} = \frac{1}{2}q^2$, $q \neq 0$ for non-trivial solutions. Hence p is always concave upward. Therefore, for all x between x_l and x_r, and for $k \leqq k^*$, the smaller of k' and k'', $p^k(x) \leqq$ the larger of $\frac{1}{2}p_1P^0(l)$ and $\frac{1}{2}p_2P^0(r)$. If we denote the absolute value of the larger of these two values by the constant c, we can then conclude

LEMMA 5.1: *There exists a constant c, positive and independent of k, and a value k^*, such that for all $k \leqq k^*$*

$$p^k(x) \leqq -c \tag{5.23}$$

for $x_l \leqq x \leqq x_r$, the end points being described above.

We now give some results for $q(x)$, also needed in the next section. First, since in the interval of Lemma 5.1, $p^k(x) < 0$ for $k \leqq k^*$, the differential equation $kq_{xx} + pq = 0$, with q taken as > 0 (since its sign is arbitrary), shows that q is concave upward here; it has points of inflection at the zeros of $p^k(x)$. Thus there exists a minimum value of q between these zeros, say at $x = m$.

Next, from Theorem 4.2, $Q^k(t) \to Q^{0\prime}(R)$ for a fixed $t(< \frac{1}{2}a)$ as $k \to 0$. Therefore there exists a value of k, say k', and a positive constant d' such that

$$0 < Q^k(t) \leqq d' \text{ for } k \leqq k' \text{ and } l \leqq t \leqq \tfrac{1}{2}a.$$

Similarly, we have a k'' and a d'' such that

$$0 < Q^k(T) \leqq d'' \text{ for } k \leqq k'' \text{ and } r \leqq T \leqq \tfrac{1}{2}A.$$

Returning to our original variables, we have

LEMMA 5.2: *There exists a positive constant d and a $k = k^*$ such that for all $k \leqq k^*$*

$$0 < q^k(x) \leqq d/\sqrt{k} \tag{5.24}$$

for $x_l \leqq x \leqq x_r$.

Here d is the smaller of d', d'', and k^* the smaller of k', k''.

We now turn, in § 6, to a study of the interior.

§ 6. Limit state in the interior.

While the limit procedure in § 4 concerns the boundary layer, we deal in this section with the limit procedure in the interior of the plate as $k \to 0$. For this study, we return to the original equations in the unstretched variables. These equations were, we recall,

$$\text{A} \qquad p_{xx} = \tfrac{1}{2}q^2$$

$$\text{B} \qquad kq_{xx} + pq = 0 \qquad -1 \leqq x \leqq +1$$

while the boundary conditions were

$$\text{C} \qquad p(-1) = p_1,\ p(+1) = p_2,$$

$$\text{D} \qquad q_x(-1) = 0,\ q_x(+1) = 0.$$

An integral of A may be written in the form

$$p^k(x) = \tfrac{1}{2}\int_{-1}^{x} q^2(\tilde{x})[x - \tilde{x}]d\tilde{x} + c'x + c''$$

In calculating the constants c' and c'', we employ the value $x = \mathrm{m}$, the abscissa of the minimum of $q^k(x)$, whose existence was shown at the close of § 5. These constants are

$$c' = p_x^k(m) - \tfrac{1}{2}\int_{-1}^{m} q^2(\tilde{x})d\tilde{x}$$

and

$$c'' = p^k(m) - mp_x^k(m) + \tfrac{1}{2}\int_{-1}^{m} \tilde{x}q^2(\tilde{x})d\tilde{x}.$$

Using these values, we have the representation

$$(6.01)\quad p^k(x) = \tfrac{1}{2}\int_{m}^{x} [x - \tilde{x}]q^2(\tilde{x})d\tilde{x} + (x - m)p_x^k(m) + p^k(m);$$

whence

$$(6.02)\qquad p_x^k(x) = \tfrac{1}{2}\int_{m}^{x} q^2(\tilde{x})d\tilde{x} + p_x^k(m).$$

Our subsequent discussion will be based on the following formulas, which hold for all $k > 0$:

$$(6.03)\quad p^k(x') - p^k(x'') = \tfrac{1}{2}\int_{m}^{x'} (x' - \tilde{x})q^2d\tilde{x} - \tfrac{1}{2}\int_{m}^{x''} [x'' - \tilde{x}]q^2d\tilde{x} + (x' - x'')p_x^k(m);$$

$$(6.04)\qquad kq^k(x) = \int_{x}^{m} [x - \tilde{x}]p(\tilde{x})q(\tilde{x})d\tilde{x} + kq^k(m);$$

$$(6.05)\quad k[q^k(x') - q^k(x'')] = \int_{x'}^{x''} [\tilde{x} - x'](-p)qd\tilde{x} - k[x'' - x']q_x^k(x'');$$

$$(6.06)\quad k[q^k(x'') - q^k(x')] = \int_{x'}^{x''} (x'' - \tilde{x}](-p)qd\tilde{x} + k[x'' - x']q_x^k(x').$$

We are now prepared to prove

THEOREM 6.1: *As* $k \to 0$, $k^{-\frac{1}{2}}q^k(x) \to 0$ *uniformly in every interior interval* $x_- \leqq x \leqq x_+$.

We recall the quantities x_l and x_r introduced in § 5, Pp. 156 and 158. They were the transforms of certain values of the stretched variables and hence dependent upon k. We noted that they approached -1 and $+1$, respectively, as $k \to 0$.

To prove our theorem we consider two possibilities regarding the position of m — it is either to the left of x_r or to the right of x_l. Assuming first the latter, we have

$$q_x^k(x) \leqq 0 \qquad \text{for } x_l \leqq x \leqq m;$$

whence

$$q^k(x') \geqq q^k(x'') \text{ for } x_l \leqq x' < x'' \leqq m,$$

and

$$q^k(x) \leqq d/\sqrt{k} \text{ for } x_l \leqq x \leqq x_r, \text{ from (5.24).}$$

Hence in (6.05) we decrease the right member when we replace $-p$ by c [by (5.23)], $q(\tilde{x})$ by $q(x'')$, and omit the second term. We have then

$$q^k(x') - q^k(x'') > \frac{cq^k(x'')}{k} \int_{x'}^{x''} (\tilde{x} - x')d\tilde{x}.$$

Considering that $q^k(x) > 0$, and that $q^k(x) \leqq d/\sqrt{k}$, the left member of this inequality is $\leqq d/\sqrt{k}$.

$$\therefore d/\sqrt{k} > \frac{cq^k(x'')}{k} \cdot \tfrac{1}{2}(x'' - x')^2,$$

or

$$\text{(6.07)} \qquad \frac{q^k(x'')}{\sqrt{k}} < \frac{2d}{c(x'' - x')^2} \qquad x_l \leqq x' < x'' \leqq m$$

If $m \leqq x_r$, we consider the interval $m \leqq x \leqq x_r$. Here $q_x^k(x) \geqq 0$, whence $q^k(x') \leqq q^k(x'')$ for $m \leqq x' < x'' \leqq x_r$; as before, $q^k(x) \leqq d/\sqrt{k}$. Hence, in (6.06) we decrease the right member when we replace $-p$ by c, $q(\tilde{x})$ by $q(x')$, and omit the second term. Now we have

$$q^k(x'') - q^k(x') > \frac{cq^k(x')}{k} \int_{x'}^{x''} (x'' - \tilde{x})d\tilde{x}.$$

Again, the left member is $\leqq d/\sqrt{k}$.

$$\therefore \ d/\sqrt{k} > \frac{cq^k(x')}{k} \cdot \tfrac{1}{2}(x'' - x')^2, \text{ or}$$

$$\text{(6.08)} \qquad \frac{q^k(x')}{\sqrt{k}} < \frac{2d}{c(x'' - x')^2} \qquad m \leqq x' < x'' \leqq x_r.$$

In (6.07) let us set $x' = x_l$, and let $x'' = x$, ranging over the interval $x_l < x \leqq m$; in (6.08) set $x'' = x_r$ and let $x' = x$, ranging over the interval $m \leqq x < x_r$. We have then the two relations, valid for every x considered:

$$\frac{q^k(x)}{\sqrt{k}} < \frac{2d}{c(x - x_l)^2} \qquad x \text{ in } x_l < x \leqq m \tag{6.09}$$

$$\frac{q^k(x)}{\sqrt{k}} < \frac{2d}{c(x_r - x)^2} \qquad x \text{ in } m \leqq x < x_r. \tag{6.10}$$

Hence we have for $x_l < x < x_r$

$$\frac{q^k(x)}{\sqrt{k}} < \frac{2d}{c}\left[\frac{1}{(x - x_l)^2} + \frac{1}{(x_r - x)^2}\right]. \tag{6.11}$$

Let us now take any interval in the interior: $-1 < x_- \leqq x \leqq x_+ < +1$. Since x_l and x_r depend upon k, then corresponding to any $n > 0$, with $x_- - n > -1$ and $x_+ + n < +1$, we can find a k_n such that, for all $k \leqq k_n$, $x_l < x_- - n$ and $x_r > x_+ + n$. Then from (6.11) we have that for all $k \leqq k_n$,

$$\frac{q^k(x)}{\sqrt{k}} < \frac{4d}{cn^2} \qquad -1 < x_- \leqq x \leqq x_+ < +1 \tag{6.12}$$

This inequality proves Theorem 6.1.

Our other theorem in this section is

THEOREM 6.2: *As $k \to 0$, $p^k(x) \to p^0(x)$ uniformly in every interior interval $x_- \leqq x \leqq x_+$, where $p^0(x)$ is the linear function*

$$p^0(x) = \tfrac{1}{2}(.47271)[(p_1 - p_2)x - (p_1 + p_2)].$$

In (6.03) we had

$$p^k(x') - p^k(x'') + (x'' - x')p^k_x(m) = \tfrac{1}{2}\int_m^{x'} (x' - \tilde{x})q^2 d\tilde{x} - \tfrac{1}{2}\int_m^{x''} (x'' - \tilde{x})q^2 d\tilde{x}.$$

1. Suppose $x_l \leqq x' < x'' \leqq m$.

The right side of the equation above may be rewritten as $\tfrac{1}{2}\int_{x'}^{x''} (\tilde{x} - x')q^2 d\tilde{x} + \tfrac{1}{2}\int_{x''}^{m} (x'' - x')q^2 d\tilde{x}$. Here each integral is $\geqq 0$. Now an immediate consequence of (5.23—4), i.e., of $p^k(x) \leqq -c$ and $0 < q^k(x) \leqq d/\sqrt{k}$ for $x_l \leqq x \leqq x_r$, is that $q^2 \leqq [d/c\sqrt{k}][-pq]$. Since the right side of (6.03) as rewritten above is $\geqq 0$ for the assumed positions of x' and x'', we increase the right side when we replace q^2 by $[d/c\sqrt{k}][-pq]$. Thus

$$p^k(x') - p^k(x'') + (x'' - x')p_x^k(m) \leqq \tfrac{1}{2}[d/c\sqrt{k}] \cdot \left[\int_m^{x'} (x' - \tilde{x})(- pq)d\tilde{x} - \int_m^{x''} (x'' - \tilde{x})(- pq)d\tilde{x}\right]$$

$$\leqq \tfrac{1}{2}[d/c\sqrt{k}] \cdot \left[\int_{x'}^{m} (x' - \tilde{x})pqd\tilde{x} - \int_{x''}^{m} (x'' - \tilde{x})pqd\tilde{x}\right]$$

$$\leqq \tfrac{1}{2}dc^{-1}\sqrt{k}[q^k(x') - q^k(x'')], \quad \text{from } (6.04).$$

2. Suppose next that $x_l \leqq x' \leqq m < x'' \leqq x_r$.

In this case the integrals on the right side of (6.03) may be written in the form $\tfrac{1}{2}\int_{x'}^{m} (\tilde{x} - x')q^2d\tilde{x} - \tfrac{1}{2}\int_m^{x''} (x'' - \tilde{x})q^2d\tilde{x}$. Since here both integrals are positive, we increase the right side when we drop the second term, and replace q^2 by $[d/c\sqrt{k}][- pq]$, as before. Then

$$p^k(x') - p^k(x'') + (x'' - x')p_x^k(m) \leqq \tfrac{1}{2}[d/c\sqrt{k}] \int_{x'}^{m} (x' - \tilde{x})pqd\tilde{x}$$

$$\leqq \tfrac{1}{2}dc^{-1}\sqrt{k}[q^k(x') - q^k(m)], \quad \text{as before.}$$

3. Finally, suppose $m \leqq x' < x'' \leqq x_r$.

Again our integrals on the right are $\tfrac{1}{2}\int_m^{x'} (x' - \tilde{x})q^2d\tilde{x} - \tfrac{1}{2}\int_m^{x''} (x'' - \tilde{x})q^2d\tilde{x}$ — both positive, so that we may once more drop the second term and make the replacement for q^2. Then as before

$$(6.13) \qquad p^k(x') - p^k(x'') + (x'' - x')p_x^k(m) \leqq \tfrac{1}{2}dc^{-1}\sqrt{k}[q^k(x') - q^k(m)].$$

Since m is the minimum point for q, $q^k(m) \leqq q^k(x'')$ for any x''. Hence we may use (6.13) for all three cases, i.e., we have (6.13) for $x_l \leqq x' < x'' \leqq x_r$.

From (6.13) we prove Theorem 6.2. Several steps are necessary.

1. Let us choose $x' = x_l$ and $x'' = x_r$. We have

$$(x_r - x_l)p_x^k(m) \leqq \frac{d\sqrt{k}}{2c}[q^k(x_l) - q^k(m)] - [p^k(x_l) - p^k(x_r)].$$

Since $x_l \to -1$ and $x_r \to 1$ as $k \to 0$, the quantities in the last bracket approach p_1 and p_2, respectively; from (6.07—8), the quantities in the first bracket are bounded; finally, $x_r - x_l$ remains bounded as $k \to 0$. Hence $p_x^k(m)$ also remains bounded as $k \to 0$. Therefore we can choose some convergent subsequence of k's so that $p_x^k(m) \to M$, a fixed number, as $k \to 0$.

2. In (6.13), now set $x' = x_l$ and $x'' = x^*$, a fixed x. Then

$$\left| p^k(x_l) - p^k(x^*) + (x^* - x_l) p_x^k(m) \right| \leqq \frac{d\sqrt{k}}{2c} [q^k(x_l) - q^k(m)].$$

From (6.07—8), $\sqrt{k}q^k(x_l)$ and $\sqrt{k}q^k(m)$ both $\to 0$ with k. For the sequence of k's being considered, $p_x^k(m) \to M$, a fixed quantity; $x^* - x_l$ is bounded; and $p^k(x_l)$ is fixed by definition, lying between $\frac{1}{2}p_1P^0(l)$ and $2p_1P^0(l)$ [cf. P. 156, (5.15)]. Hence $p^k(x^*)$ is bounded for any fixed x^*. Therefore we can now select a sub-sub-sequence of k's such that $p^k(x^*)$ converges to a limit U.

3. Finally, set $x' = x$, $x'' = x^*$, both fixed values of x. Then (6.13) becomes

$$| p^k(x) - p^k(x^*) + (x^* - x)p_x^k(m)| \leqq \frac{d\sqrt{k}}{2c} [q^k(x) - q^k(x^*)].$$

For fixed interior x's, (6.12) shows that the right side $\to 0$ as $k \to 0$. We have already shown that $p^k(x^*)$ and $p_x^k(m)$ approach limits U and M, respectively. Also, $x^* - x$ is bounded. Hence $p^k(x)$ converges to a limit, say $p^0(x)$, as $k \to 0$, for interior x's. More specifically, we have

$$p^0(x) - U + (x^* - x)M = 0,$$

or

$$p^0(x) = U - M(x^* - x),$$

so that $p^0(x)$ is a linear function. To determine its precise form, we return again to (6.13). Recalling that $p^k(x) = p_1P^k(t)$ and that $p_1Q^k(t) = \sqrt{k}q^k(x)$, we have

$$| p_1P^k(t') - p^k(x) + (x - x')p_x^k(m) | \leqq \tfrac{1}{2}\frac{d}{c} [p_1Q^k(t') - \sqrt{k}q^k(m)],$$

where $t' > l$, but is a fixed value, and x is a fixed interior x. Then in the limit, as $k \to 0$, we have

$$| p_1P^0(t') - p^0(x) + (x + 1)M | \leqq \tfrac{1}{2}\frac{d}{c} p_1Q^0(t'),$$

from Theorem 4.3, and the results of the preceding paragraphs. Now let $t' \to \infty$. From (5.11), we know the right side goes to 0. Hence

$$p^0(x) = p_1P^0(\infty) + M(x + 1).$$

For $x = -1$,

$$p^0(-1) = p_1P^0(\infty)$$
$$= -.47271p_1.$$

To calculate the other end point, we use the transformation T [cf. Pp. 157]. We have

$$|p_2P^k(T')-p^k(x)+(x-x')p_x^k(m)| \leqq \tfrac{1}{2}\frac{d}{c}[p_2Q^k(T')-\sqrt{k}q^k(m)],$$

where $T' > r = l$, but is fixed, and x is a fixed interior x. In the limit,

$$|p_2P^0(T')-p^0(x)+(x-1)M| \leqq \tfrac{1}{2}\frac{d}{c}p_2Q^0(T').$$

Again, we let $T' \to \infty$. We have

$$p^0(x) = p_2P^0(\infty) + M(x-1).$$

If $x = 1$,

$$\begin{aligned} p^0(1) &= p_2P^0(\infty) \\ &= -.47271p_2. \end{aligned}$$

Thus $p^0(x)$ is a linear function whose values at $x = -1$ and $x = +1$ are $-.47271p_1$ and $-.47271p_2$, respectively. These values enable us to calculate M explicitly; for, using $x = -1$ in the last limit relation (or $x = +1$ in the first), we obtain

$$p^0(-1) = -.47271p_2 - 2M,$$

i.e.,

$$-.47271p_1 = -.47271p_2 - 2M.$$

Hence

$$M = \lim_{k\to 0} p_x^k(m) = \tfrac{1}{2}(.47271)(p_1-p_2).$$

Therefore, finally,

$$p^0(x) = \tfrac{1}{2}(.47271)\,.\,[(p_1-p_2)x-(p_1+p_2)].$$

This statement completes the proof of Theorem 6.2, and, indeed, our study of the problem for the case where both boundary values are positive. We consider the remaining two possibilities, referred to in the introduction, in the following, and concluding, section.

§ 7. Cases II and III: one or both boundary values negative.

In discussing the solution of our problem with one or both boundary values negative, we find we are able to utilize many of the results obtained in our study of Case I. Only slight modifications are required to give us the desired solution. We shall find that there is no boundary layer phenomenon at an end where the boundary value is negative.

Case II:

$$p_1 > 0,\ p_2 < 0.$$

We start with the same equations

$$p_{xx} = \tfrac{1}{2}q^2 \text{ and } kq_{xx} + pq = 0,\ -1 \leqq x \leqq 1,$$

and the boundary conditions

$$p(-1) = p_1 > 0,\ p(1) = p_2 < 0,\ q_x(-1) = q_x(1) = 0.$$

Since we have already studied a stretching of the positive end, here we shall make a transformation taking the right — and negative — end off to infinity. We require the same transformation as in § 4:

$$t = (x + 1)\sqrt{p_1/k},\ P = p/p_1,\ Q = \sqrt{k}q/p_1.$$

These transform our original equations into

$$P_{tt} = \tfrac{1}{2}Q^2,\ Q_{tt} + PQ = 0,$$

and the boundary conditions into

$$P(0) = 1,\ P(a) = p_2/p_1,\ Q_t(0) = Q_t(a) = 0,$$

where a has the same meaning as before, in (4.04). These equations are identical with the ones used in § 4, except that the ratio p_2/p_1 is now negative. The remainder of § 4 follows precisely as before.

Similarly, the material of § 5 through P. 156 is also valid here; at this point, however, an important simplification enters the discussion. As before, $P^0(t)$ is a monotonically decreasing function with its zero at $t = .941$. Hence we can choose a value of t, say $t = l, > .941$, such that $P^0(l) < 0$; again because of the uniform convergence of $P^k(t)$ to $P^{0\prime}(R)[= P^0(t)$ for $0 \leqq t \leqq \frac{1}{2}a]$ for t fixed and $\leqq \frac{1}{2}a$, as $k \to 0$, we can find a value of $k = k'$ such that for all $k \leqq k'$ all t's under consideration will be $\leqq \frac{1}{2}a$ and $P^k(l) < 0$. Since $P_{tt} = \frac{1}{2}Q^2$, P is still everywhere concave upward, and for $k \neq 0$, $P^k(0) = 1$, $P^k(a) = p_2/p_1 < 0$. Hence for $0 < k \leqq k'$, P^k starts out at $+ 1$, crosses the axis somewhere to the left of $t = l$, and becomes and *remains* negative, assuming the negative value p_2/p_1 at $t = a$. Accordingly, we have here (5.17) with the interval of validity extending on the right to $+ 1$ instead of to 0. This leads us then to

Lemma 7.1: *There exists a constant c, positive and independent of k, and a value k^*, such that for all $k \leqq k^*$, $p^k(x) \leqq - c$ for $x_l \leqq x \leqq + 1$, x_l being the transform of $t = l$.*

Since in the interval of this Lemma, $p^k(x)$ is negative for $0 < k \leqq k^*$, the differential equation $kq_{xx} + pq = 0$, with q taken as > 0 (since its sign is arbitrary), shows that q_x is an increasing function throughout this interval. But $q_x(+ 1) = 0$; hence q_x must be negative up to $x = 1$, and the minimum of q is at this point; i.e., in this case $m = 1$.

Lemma 5.2 carries over here, too, except that our interval is again extended, as in Lemma 5.1. We have

LEMMA 7.2: *There exists a positive constant d and a $k = k^*$ such that, for all $k \leqq k^*$, $0 < q^k(x) \leqq d/\sqrt{k}$ for $x_l \leqq x \leqq +1$.*

Turning to § 6, we have substantially the same results here as through (6.12), but with $+1$ replacing m. Theorem 6.1 still holds, but here $p^0(x)$ is a different linear function. Instead, we have

THEOREM 7.1: *As $k \to 0$, $p^k(x) \to p^0(x)$ uniformly in every interior interval $-1 < x_- \leqq x \leqq +1$, where $p^0(x)$ is the linear function*

$$p^0(x) = \tfrac{1}{2}[(p_2 + .47271p_1)x + p_2 - .47271p_1].$$

As on Pp. 162—163, we can show here that

$$(7.01) \qquad \begin{aligned} &| p^k(x') - p^k(x'') + (x'' - x')p^k_x(1) | \\ &\leqq \tfrac{1}{2}\frac{d\sqrt{k}}{c}[q^k(x') - q^k(x'')], \end{aligned}$$

with $x_l \leqq x' < x'' \leqq 1$. Now as in step 1 on P. 163, if we choose $x' = x_l$ and $x'' = 1$, we can show that $p^k_x(1)$ remains bounded as $k \to 0$, so that we can choose a convergent subsequence of k's so that $p^k_x(1) \to M$, a fixed number, as $k \to 0$. Then, taking $x' = x_l$ and $x'' = x^*$, some fixed x, we obtain the result that $p^k(x^*)$ is bounded for any fixed x^*, and that we can accordingly select a subsequence of k's such that $p^k(x^*)$ converges to a limit, U, as $k \to 0$. Next, as in step 3 on P. 164, we set $x' = x$ and $x'' = x^*$, both fixed values of x, and similarly deduce the existence of our linear limit function

$$p^0(x) = U - M(x^* - x).$$

To determine which linear function we have, we return to (7.01), recalling that $p^k(x) = p_1P^k(t)$, and that $p_1Q^k(t) = \sqrt{k}q^k(x)$. We have

$$(7.02) \qquad \begin{aligned} &| p_1P^k(t') - p^k(x) + (x - x')p^k_x(1) | \\ &\leqq \tfrac{1}{2}\frac{d}{c}[p_1Q^k(t') - \sqrt{k}q^k(x)]. \end{aligned}$$

When $k \to 0$, we have

$$| p_1P^0(t') - p^0(x) + (x + 1)M | \leqq \tfrac{1}{2}\frac{d}{c}p_1Q^0(t').$$

Now let $t' \to \infty$. Then

$$p_1P^0(\infty) - p^0(x) + M(x + 1) = 0.$$

For $x = -1$, $p^0(-1) = p_1P^0(\infty) = -.47271p_1$.

We return now to (7.02) and substitute $+1$ for x. At this stage in Case I we could not do this, since our inequalities were valid only up to x_r, so that we could use the end point values only after the limiting process had been effected. Here, then, we have

$$| p_1 P^k(t') - p^k(1) + (1 - x') p_x^k(1) | \leqq \tfrac{1}{2} \frac{d}{c} [p_1 Q^k(t') - \sqrt{k} q^k(1)].$$

Now we let $k \to 0$. Since $x = 2t/a - 1$, then for a fixed t, say t', $x' \to -1$ as $k \to 0$. Hence we have

$$| p_1 P^0(t') - p_2 + M(1+1) | \leqq \tfrac{1}{2} \frac{d}{c} p_1 Q^0(t').$$

Finally we let $t' \to \infty$. Then

$$p_1 P^0(\infty) - p_2 + 2M = 0,$$

whence

$$M = \tfrac{1}{2}[p_2 - p_1 P^0(\infty)].$$

Therefore $p^0(x) = p_1 P^0(\infty) + \frac{1}{2}(x+1)[p_2 - p_1 P^0(\infty)]$, or, using the value of $P^0(\infty) = -.47271$,

$$p^0(x) = \tfrac{1}{2}[(p_2 + .47271 p_1)x + p_2 - .47271 p_1],$$

the limit function for Case II.

Case III:

$$p_1 < 0, \; p_2 < 0.$$

Since in this case it is our conjecture that there is no boundary layer phenomenon, we should expect to find the stretching procedure unnecessary. This is indeed the situation.

We begin with the formulation of the problem in terms of the functionals $H^k(q)$, $D^k(q)$, $K^k(q)$, and $W^k(q) = D - H + K$, exactly as given in § 1. We establish the same theorems in precisely the same manner. Instead of the theorems of § 2, however, we have here

THEOREM 7.2: *The only solution of the problem B^k is $q \equiv 0$.* From the differential equation $p_{xx} = \frac{1}{2}q^2$ and the boundary conditions $p(-1) = p_1$, $p(1) = p_2$, we have the

COROLLARY: *In case III, $p^k(x)$ is the linear function*

$$p^k(x) = \tfrac{1}{2}[(p_2 - p_1)x + (p_2 + p_1)].$$

To prove Theorem 7.2, we observe that the functional H is always negative when both p_1 and p_2 are negative. This may be more readily seen if one writes the coefficient of q^2 in the integrand in the form $p_1(1 - x) + p_2(1 + x)$. Hence the functional W is non-negative. We have already shown(P. 125) that in this event, $q \equiv 0$ is the only solution of the minimum problem M^k. Theorem 7.2 then follows from Theorem 1.2.

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