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by

A. C. Zaanen

Delft.

§ 1. Introduction.

Suppose that Δ is a bounded or unbounded interval in real m-dimensional Euclidian space R_m. It is therefore permitted that $\Delta = R_m$. By $\Delta \times \Delta$ we mean the set of all points (x, y) in the product space $R_m \times R_m$ for which $x \in \Delta$, $y \in \Delta$.

$L_p(\Delta)$ $(1 \leqq p < \infty)$ is the Lebesgue space of all measurable complex-valued functions $f(x)$ for which $\| f \|_p = \left(\int_\Delta | f |^p \, dx\right)^{1/p}$ $< \infty$ (all integrals are Lebesgue integrals). If no confusion is possible we write shortly L_p. $L_\infty(\Delta)$ is the Lebesgue space of all measurable $f(x)$ for which $\| f \|_\infty = \text{ess. u. b.} \, | f(x) | < \infty$. Every $L_p(\Delta)$ $(1 \leqq p \leqq \infty)$ is a complete Banach space provided $f + g$ and αf (α a complex number) are defined in the natural way. The spaces $L_p(\Delta \times \Delta)$, having functions $T(x, y)$ on $\Delta \times \Delta$ as elements, are introduced similarly.

Let $1 \leqq p \leqq \infty$, $1/p + 1/q = 1$ (hence $q = \infty$ for $p = 1$ and $q = 1$ for $p = \infty$). It is well-known that $f(x) \in L_p$ if and only if fg is summable over Δ for every $g(x) \in L_q$, and that

$$\| f \|_p = \text{l.u.b.} \int_\Delta | fg | \, dx \ \text{ for } \ \| g \|_q \leqq 1.$$

For $1 \leqq p < \infty$ we may even replace l.u.b. by max. Consider now $L_2(\Delta \times \Delta)$. Then similarly $T(x, y) \in L_2(\Delta \times \Delta)$, that is

$$|\!|\!| T |\!|\!|_2 = \left(\int_{\Delta\times\Delta} | T(x, y) |^2 \, dx \, dy\right)^{1/2} < \infty,$$

if and only if $T(x, y) S(x, y)$ is summable over $\Delta \times \Delta$ for every $S(x, y) \in L_2(\Delta \times \Delta)$. In particular, if $T(x, y) \in L_2(\Delta \times \Delta)$ and $f, g \in L_2(\Delta)$, then $T(x, y) f(y) g(x)$ is summable over $\Delta \times \Delta$. The inverse of this last statement is not true however. $T(x, y) = | x - y |^{-\alpha}$ with $1/2 \leqq \alpha < 1$ is a counter example. The class

$L_{22}(\Delta \times \Delta)$ of all $T(x, y)$ such that $T(x, y)f(y)g(x)$ is summable over $\Delta \times \Delta$ for $f,\ g \in L_2(\Delta)$ is therefore larger than $L_2(\Delta \times \Delta)$. Defining the norm $\| T \|_{22}$ by

$$\| T \|_{22} = \text{l.u.b.} \int_{\Delta\times\Delta} | T(x, y)f(y)g(x) | \, dx\, dy \quad \text{for} \quad \| f \|_2 \leqq 1, \ \| g \|_2 \leqq 1,$$

we shall prove that L_{22} is a Banach space with this norm. An easy application of Schwarz's inequality shows that $\| T \|_{22} \leqq ||| T |||_2$.

Extending these remarks to spaces L_p $(1 \leqq p \leqq \infty)$, L_{22} has its analog in the class L_{pq} of all $T(x, y)$ such that $T(x, y)f(y)g(x)$ is summable over $\Delta \times \Delta$ for all $f \in L_p$, $g \in L_q$. We shall prove that L_{pq} is a complete Banach space with norm

$$\| T \|_{pq} = \text{l.u.b.} \int_{\Delta\times\Delta} | T(x, y)f(y)g(x) | \, dx dy \quad \text{for} \quad \| f \|_p \leqq 1, \ \| g \|_q \leqq 1.$$

Using Hölder's inequality it is easily seen that $\| T \|_{pq} \leqq ||| T |||_p$, where

$$||| T |||_p = \| t(x) \|_p, \ t(x) = \| T(x, y) \|_q = \| T_x(y) \|_q.$$

Hence,

$$||| T |||_p = \begin{cases} \left[\int_\Delta \left(\int_\Delta | T(x, y) |^q dy\right)^{p/q} dx\right]^{1/p}, & 1 < p < \infty, \\ \int_\Delta \| T(x, y) \|_\infty dx & , \ p = 1, \\ \left\| \int_\Delta | T(x, y) | \, dy \right\|_\infty & , \ p = \infty. \end{cases}$$

We make two remarks. Firstly it follows already from the simple inequality $\| T \|_{pq} \leqq ||| T |||_p$ that it is natural to introduce $||| T |||_p$ into the theory and not the ordinary norm $\left(\int_{\Delta\times\Delta} | T(x, y) |^p dx dy\right)^{1/p}$ as one might think at first by comparison with the L_2-case. Another and even stronger reason for doing so will be mentioned below. Secondly we observe that we have used the measurability of $t(x) = \| T_x(y) \|_q$. For $1 < p \leqq \infty$ this measurability is a consequence of Fubini's Theorem and for $p = 1$ it is implied by

$$\| T_x(y) \|_\infty = \lim_{r\to\infty} \left[\{m(\Delta_r)\}^{-1} \int_{\Delta_r} | T(x, y) |^r dy\right]^{1/r},$$

where Δ_r is the common part of Δ and the interval $[-r, r; \ldots; -r, r]$, and where $m(\Delta_r)$ is the measure of Δ_r.

It follows from the definition of $L_{pq}(\Delta \times \Delta)$ that $T(x, y) \in L_{pq}$

if and only if $\int_\Delta | T(x, y)f(y) | dy \in L_p(\Delta)$ whenever $f \in L_p(\Delta)$. We shall prove that in this case the linear integraltransformation T with domain L_p, defined by $Tf = \int_\Delta T(x, y)f(y)dy$, has its range in L_p and is bounded. Concisely expressed, if $T(x, y) \in L_{pq}$ then T with kernel $T(x, y)$ is bounded on L_p into L_p (compare Banach [1], p. 87). If moreover $1 < p < \infty$ and $||| T |||_p < \infty$ than T is even completely continuous (that is, T transforms bounded sets in L_p into compact sets). This was proved by Hille and Tamarkin in 1934 [2]; the case $p = 2$ was known earlier. On account of the importance of completely continuous integral transformations for the theory of integral equations, this is the other and stronger reason to introduce $||| T |||_p$. For $p = 1$ the inequality $||| T |||_1 < \infty$ does not necessarily imply the complete continuity of T (cf. the example of von Neumann in [2]). In 1940 however Dunford and Pettis [3] and Phillips [4] proved independently of each other that in this case the iterated transformation $T^2 = TT$ is completely continuous. For $p = \infty$ it is possible, that although $||| T |||_\infty < \infty$, neither T itself nor any of its iterates T^n $(n = 2, 3, \ldots)$ is completely continuous. An example (cf. [2]) on the linear interval $\Delta = [0, 1]$ is furnished by

$$T(x, y) = \begin{cases} x^{-1}, & 0 \leqq y \leqq x, \\ 0, & x < y \leqq 1. \end{cases}$$

Then $||| T |||_\infty = \left\| \int_\Delta | T(x, y) | dy \right\|_\infty = \left\| \int_0^x x^{-1} dy \right\|_\infty = 1$, and for $\mu \geqq 1$, $f(x) = x^{\mu-1}$, we find

$$Tf = \int_0^x x^{-1} y^{\mu-1} dy = \mu^{-1} x^{\mu-1} = \mu^{-1} f, \quad T^n f = \mu^{-n} f \quad (n = 2, 3, \ldots).$$

This shows that all values $\lambda = \mu^{-1}$ $(0 < \lambda \leqq 1)$ belong to the pointspectrum of T^n where n is a positive integer. Then, by a well-known theorem, T^n is not completely continuous.

It is our object to extend all these results to a class of function spaces which contains the Lebesgue spaces $L_p(\Delta)$ as special cases. We want to include e.g. the space of all $f(x)$ for which $\int_\Delta \Phi |f| dx < \infty$, where $\Phi(u)$, $u \geqq 0$, is roughly a non-negative convex function with $\Phi(0) = 0$ which behaves for large u like $u \log u$ or, more generally, like $u^p \log^r u$ $(p \geqq 1, r \geqq 0)$. For this purpose it will be useful to work in Orlicz spaces. These spaces, introduced by

Orlicz in 1932 [5], were also considered by Zygmund [6]. Their original definition was such that all spaces $L_p(\Delta)$, $1 < p < \infty$, were included but not $L_1(\Delta)$ and $L_\infty(\Delta)$. A somewhat more general definition which remedied this defect was given by Zaanen [7]. Recently Morse and Transue [8] have considered a generalization into a different direction.

We shortly give the definition of the Orlicz space $L_\Phi(\Delta)$. Let $v = \varphi(u)$, $u \geqq 0$, satisfying $\varphi(0) = 0$, be non-decreasing. We suppose furthermore that $\varphi(u)$ is left-continuous (hence $\varphi(u) = \varphi(u-)$ for $u > 0$) and not vanishing identically. The function $\varphi(u) = 1$ for $u > 0$ is an example. $u = \psi(v)$ is the left-continuous inverse, suitably defined for those v for which $v = \varphi(u)$ has an interval of constancy. If $\lim_{u\to\infty} \varphi(u) = l < \infty$, then $\psi(v) = \infty$ for $v > l$. In the example above $\psi(v) = 0$ for $0 \leqq v \leqq 1$ and $\psi(v) = \infty$ for $v > 1$. Writing $\Phi(u) = \int_0^u \varphi(u')du'$, $\Psi(v) = \int_0^v \psi(v')dv'$, we have Young's inequality $uv \leqq \Phi(u) + \Psi(v)$, $u \geqq 0$, $v \geqq 0$. The class $L_\Phi^*(\Delta)$ is now the class of all measurable $f(x)$ for which $\int_\Delta \Phi|f|\,dx < \infty$. The class $L_\Psi^*(\Delta)$ is defined similarly. In the example above $\Phi(u) = u$, $\Psi(v) = 0$ for $0 \leqq v \leqq 1$ and $\Psi(v) = \infty$ for $v > 1$; the class $L_\Phi^*(\Delta)$ is therefore identical with $L_1(\Delta)$ and $L_\Psi^*(\Delta)$ is the class of all $f(x)$ satisfying $\|f\|_\infty \leqq 1$. This shows already (Φ and Ψ are interchangeable) that L_Φ^* and L_Ψ^* are not necessarily linear classes. For this reason linear classes L_Φ and L_Ψ are defined containing L_Φ^* and L_Ψ^* as subclasses. If $f(x)$ is measurable on Δ we write

$$\|f\|_\Phi = \text{l.u.b.} \int_\Delta |fg|\,dx \text{ for } \int_\Delta \Psi|g|\,dx \leqq 1,$$

$$\|f\|_\Psi = \text{l.u.b.} \int_\Delta |fh|\,dx \text{ for } \int_\Delta \Phi|h|\,dx \leqq 1.$$

The Orlicz space $L_\Phi(\Delta)$ is now the class of all $f(x)$ satisfying $\|f\|_\Phi < \infty$ and similarly $L_\Psi(\Delta)$ is the class of all $f(x)$ for which $\|f\|_\Psi < \infty$. Choosing $\Phi(u) = u^p/p$ $(1 \leqq p < \infty)$ the corresponding Orlicz space L_Φ contains the same functions as L_p and $\|f\|_\Phi = q^{1/q}\|f\|_p$ ($q^{1/q} = 1$ for $p = 1$). The complementary space contains the same functions as $L_q(\Delta)$ and $\|f\|_\Psi = p^{1/p}\|f\|_q$. It has been proved that L_Φ and L_Ψ are complete Banach spaces with norms $\| f \|_\Phi$ and $\| f \|_\Psi$, and that $f(x) \in L_\Phi$ if and only if fg is summable over Δ for every $g \in L_\Psi^*$.

The class $L_{\Phi\Psi}$ is introduced as the class of all $T(x, y)$, measurable on $\Delta \times \Delta$, such that $T(x, y)f(y)g(x)$ is summable over $\Delta \times \Delta$ for all $f \in L_\Phi$, $g \in L_\Psi$. We shall prove that $L_{\Phi\Psi}$ is a complete Banach space with norm

$$\| T \|_{\Phi\Psi} = \text{l.u.b.} \int_{\Delta\times\Delta} | T(x, y)f(y)g(x) | \, dx\, dy$$

$$\text{for} \int_\Delta \Phi | f | \, dy \leq 1, \quad \int_\Delta \Psi | g | \, dx \leq 1.$$

In analogy with the L_p-case we should await $\| T \|_{\Phi\Psi} \leqq ||| T |||_\Phi$, where

$$||| T |||_\Phi = \| t(x) \|_\Phi, \quad t(x) = \| T(x, y) \|_\Psi = \| T_x(y) \|_\Psi.$$

Here however a difficulty arises, because it is not à priori evident that $t(x)$ is a measurable function of x. If $t(x)$ should not be measurable, $\| t(x) \|_\Phi$ is not defined. To overcome this difficulty we denote by $t_{\text{maj}}(x)$ an arbitrary measurable majorant of $t(x)$, hence $t_{\text{maj}}(x) \geqq t(x)$ almost everywhere in Δ. $||| T |||_\Phi$ is now defined as g.l.b. $\| t_{\text{maj}}(x) \|_\Phi$ over all majorants. Evidently, for $\Phi(u) = u^p/p$ $(1 \leqq p < \infty)$, $||| T |||_\Phi$ is except for a constant factor equal to $||| T |||_p$.

We shall prove that it follows from $T(x, y) \in L_{\Phi\Psi}$ that the integral transformation T with kernel $T(x, y)$ is bounded on L_Φ into L_Φ. If moreover there exists a constant M such that $\Phi(2u) \leqq M\Phi(u)$, $\Psi(2v) \leqq M\Psi(v)$ for all $u \geqq 0$, $v \geqq 0$ and if $||| T |||_\Phi < \infty$, then T is completely continuous (cf. Zaanen [9]). This is a statement which covers the case L_p $(1 < p < \infty)$ mentioned above. In the case L_1 we have already seen that $||| T |||_1 < \infty$ implies the complete continuity of T^2. There remains a gap to bridge. e.g. for those spaces L_Φ with $\Phi(u)$ behaving like $u \log u$ for large u. We shall prove:

THEOREM A. *If there exists a constant M such that $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$ and if $||| T |||_\Phi < \infty$, then T^2 is completely continuous (on L_Φ into L_Φ).*

Let $T(x, y) \in L_{\Phi\Psi}$ and consider the integral equation $Tf - \lambda f = g$ in the space L_Φ. If $\lambda \neq 0$ is in the resolvent set of T, that is if $T - \lambda I$ (I is the identical transformation) has a bounded inverse $R_\lambda = (T - \lambda I)^{-1}$ with domain L_Φ, then

$$Tf - \lambda f = g, \; f = R_\lambda g.$$

The set of transformations R_λ where λ runs through the resolvent set of T is called the resolvent of T. In most of the classical treatises on integral equations a somewhat different terminology may be found. Writing first $Tf - \lambda f = g$ for $\lambda \neq 0$ as $f - \lambda^{-1}Tf = -\lambda^{-1}g$ and then putting $\lambda^{-1} = \mu$ and $-\lambda^{-1}g = -\mu g = g_1$, we find $f - \mu Tf = g_1$. The transformation H_λ is now introduced by $R_\lambda = -\lambda^{-1}I - \lambda^{-2}H_\lambda = -\mu I - \mu^2 H_\lambda$ so that $f = R_\lambda g = -\mu g - \mu^2 H_\lambda g = g_1 + \mu H_\lambda g_1$. Hence

$$f - \mu Tf = g_1, \; f = g_1 + \mu H_\lambda g_1.$$

It may be asked now whether it follows from $T(x, y) \in L_{\Phi\Psi}$ that H_λ is also an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$. Should this be so, we could call $H_\lambda(x, y)$ in accordance with the classical custom the resolvent kernel of the integral equation. The answer however is still rather unsatisfactory. Roughly stated we can prove only that $H_\lambda(x, y) \in L_{\Phi\Psi}$ for large values of $|\lambda|$. If however $|||\, T^n \,|||_\Phi < \infty$ for a positive integer n we can show the following statement to hold:

Theorem B. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, $T(x, y) \in L_{\Phi\Psi}$ and $|||\, T^n \,|||_\Phi < \infty$ for an integer $n \geqq 1$. Then, if $\lambda \neq 0$ is not in the pointspectrum of T, H_λ is an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$ and*

$$H_\lambda(x,y) = T(x,y) + \mu T_2(x,y) + \dots + \mu^{n-2}T_{n-1}(x,y) + \mu^{n-1}K_\lambda(x,y)$$

where $|||\, K_\lambda \,|||_\Phi < \infty$. The functions $T_p(x, y)$, $p = 1, 2, \dots$, are here the kernels of T^p. In particular $|||\, T \,|||_\Phi < \infty$ implies $|||\, H_\lambda \,|||_\Phi < \infty$.

The next question to be answered is what conditions are sufficient to ensure that $H_\lambda(x, y)$ be the quotient of two power series in $\mu = \lambda^{-1}$, the coefficients of these series being the well-known Fredholm determinant expressions. For the L_2-case Carleman proved in 1921 [10] that $|||\, T \,|||_2 < \infty$ is sufficient. Smithies in 1944 [11] gave a considerably simpler proof. For the L_p-case $(1 < p < \infty)$ Nikovič in 1948 [12] announced the result that $|||\, T \,|||_p < \infty$ together with $\left[\int_\Delta \left(\int_\Delta |\, T(x, y) \,|^p \, dx\right)^{q/p} dy\right]^{1/q} < \infty$ is sufficient. All methods of proof are based on the approximation of $T(x, y)$ by continuous or degenerate kernels. We shall state, and prove, a result which includes all these cases. For this purpose it is necessary to introduce for a measurable $T(x, y)$ besides

$||| T |||_\Phi$ the number $||| T |||_\Phi^{inv}$ by

$$||| T |||_\Phi^{inv} = ||| T^* |||_\Psi, \; T^*(x, y) = T(y, x).$$

Then we have:

THEOREM C. *Let* $\Phi(2u) \leqq M\Phi(u)$ *for all* $u \geqq 0$, $||| T |||_\Phi < \infty$ *and* $||| T |||_\Phi^{inv} < \infty$. *Then, if* $\lambda \neq 0$ *is not in the pointspectrum of* T, *we have* $H_\lambda(x, y) = H'_\lambda(x, y)/\delta(\mu)$ *where* $H'_\lambda(x, y) = \Sigma_0^\infty H_n(x, y)\mu^n$ *and* $\delta(\mu) = 1 + \Sigma_1^\infty \delta_n \mu^n$. *The coefficients* δ_n *and* $H_n(x, y)$ *are the (modified) Fredholm expressions. Both series converge for all* μ, *the series for* $H'_\lambda(x, y)$ *almost everywhere in* $\Delta \times \Delta$. *This last series even converges relative to the norm* $||| \cdot |||_\Phi$.

Our proof will be free of approximation methods. That this is possible is probably due to our first proving Theorem B.

The reader will have observed the importance of $||| T |||_\Phi$ for the problems in question. An integral transformation T in $L_2(\Delta)$ satisfying $||| T |||_2 < \infty$ was called of „*finite norm*" by Stone ([13], p. 66) and Smithies [11]. By considering the L_p-case for $p \neq 2$ it is seen however that $||| T |||_p$ is essentially a double norm. We shall therefore call $||| T |||_\Phi$ the *double-norm* of T relative to L_Φ, and an integral transformation T with $||| T |||_\Phi < \infty$ will be said to be of *finite double-norm* relative to L_Φ. This has the additional advantage of avoiding confusion with the ordinary norm $|| T ||$ of T. A transformation T for which $||| T |||_\Phi < \infty$, $||| T |||_\Phi^{inv} < \infty$ will be called *completely of finite double-norm* relative to L_Φ. Note that for $L_\Phi \equiv L_2$ we have $||| T |||_\Phi = ||| T |||_\Phi^{inv}$ so that in this case any transformation of finite double-norm is also completely of finite double-norm.

In § 2 we shall list some properties of Orlicz spaces and prove Theorem A. In § 3 kernels belonging to $L_{\Phi\Psi}$ are considered, and in § 4 we introduce the Banach space of all kernels of finite double-norm. Theorem B will be proved in § 5. In § 6 the main properties of an abstract completely continuous linear transformation are listed, and this makes it possible to prove our principal theorem, Theorem C, in § 7. Finally, in § 8, we show that under somewhat stronger hypotheses the expansions for the resolvent kernel converge uniformly.

§ 2. Properties of Orlicz spaces and the proof of Theorem A.

We suppose that $\Phi(u)$ and $\Psi(v)$ are complementary in the sense described in the introduction and that $L_\Phi(\Delta)$, $L_\Psi(\Delta)$ are the corresponding complementary Orlicz spaces. We list here some properties of these spaces. For the proofs we refer to [7] and [6].

1⁰. L_Φ and L_Ψ are complete Banach spaces with norms $\| f \|_\Phi$ and $\| f \|_\Psi$ respectively.

2⁰. If $f \in L_\Phi^*$, then $f \in L_\Phi$ and $\| f \|_\Phi \leqq \int_\Delta \Phi \, | f | \, dx + 1$. Furthermore $\| f \|_\Phi = 0$ if and only if $f(x) = 0$ almost everywhere in Δ. Similarly for L_Ψ.

3⁰. If $f \in L_\Phi$ and $\| f \|_\Phi \neq 0$, then $\int_\Delta \Phi \, [| f | / \| f \|_\Phi] dx \leqq 1$. Similarly for L_Ψ.

4⁰. There exists a positive p such that $\Psi(p) \leqq 1$.

5⁰. $\int_\Delta | fg | \, dx \leqq \| f \|_\Phi \| g \|_\Psi$ for measurable f and g.

6⁰. If f is measurable on Δ and fg is summable over Δ for every $g \in L_\Psi^*$, then $f \in L_\Phi$. In the same way $g \in L_\Psi$ if fg is summable over Δ for every $f \in L_\Phi^*$.

In case $\Phi(u)$ satisfies the additional condition that there exists a constant M such that $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, the following extra properties hold (proofs in [7] and [9]):

7⁰. L_Φ and L_Φ^* (but not necessarily L_Ψ and L_Ψ^*) contain the same functions.

8⁰. If there exists a non-negative integer l such that $M^l \int_\Delta \Phi \, | f | \, dx \leqq 1$, then $\| f \|_\Phi \leqq 2/2^l$. In particular, if $\lim \int_\Delta \Phi \, | f_n | \, dx = 0$, then $\lim \| f_n \|_\Phi = 0$.

9⁰. L_Φ is separable. More precisely, the set of all rational simple functions is dense in L_Φ. A rational simple function is a finite linear combination with rational complex coefficients of functions $g_i(x)$, each $g_i(x)$ being the characteristic function of a rational bounded interval.

10⁰. Every bounded linear functional $g^*(f)$ in L_Φ is of the form

$$g^*(f) = \int_\Delta f(x) g(x) dx$$

with $g(x) \in L_\Psi$ and $\| g(x) \|_\Psi / 2 \leqq \| g^* \| \leqq \| g(x) \|_\Psi$. It follows that $g^* \longleftrightarrow g(x)$ is a one-to-one correspondence between $(L_\Phi)^*$ and L_Ψ which preserves addition and multiplication by complex numbers.

The measure of the measurable set $X \subset R_m$ will be denoted by $m(X)$.

Definition. *If $f(x)$ is defined on Δ and $X \subset \Delta$, then the function $f_X = f(x)_X$ is defined by*

$$f_X = \begin{cases} f(x), & x \in X, \\ 0 & \text{elsewhere in } \Delta. \end{cases}$$

Lemma 1. *If $p > 0$ is such that $\Psi(p) \leqq 1$ (such a p exists by 4^0) and X is a measurable bounded subset of Δ, then*

$$\int_X |f|\, dx \leqq p^{-1}(m(X) + 1) \| f_X \|_\Phi .$$

Proof. Using 5^0 and 2^0 we find

$$\int_X |f|\, dx = \int_X |p^{-1} f|\, p\, dx \leqq \| p^{-1} f_X \|_\Phi \, \|(p)_X\|_\Psi \leqq$$

$$p^{-1} \| f_X \|_\Phi \Big(\int_X \Psi(p) dx + 1 \Big) \leqq p^{-1}(m(X) + 1) \| f_X \|_\Phi .$$

Lemma 2. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, and let all $f(x)$ of the set $\{f\}$ belong to L_Φ, and hence to L^*_Φ by 7^0. We furthermore suppose that to each $\eta > 0$ there is assigned a number $\tau(\eta) > 0$ such that for all $f \in \{f\}$ we have $\int_X \Phi |f|\, dx < \eta$ if only $m(X) < \tau$. Moreover, in the case that Δ is unbounded, we assume that for each $\eta > 0$ there exists a bounded subinterval $E_\eta \subset \Delta$ such that $\int_{\Delta - E_\eta} \Phi |f|\, dx < \eta$ for all $f \in \{f\}$. The set functions $\int_X \Phi |f|\, dx$ are therefore uniformly absolutely continuous.*

Then, if $\varepsilon > 0$ is given, there exists a number $\delta(\varepsilon)$ and (if Δ is unbounded) there also exists a bounded subinterval $\Delta_\varepsilon \subset \Delta$ such that for all $f \in \{f\}$ we have $\| f_X \|_\Phi < \varepsilon$ if only $m(X) < \delta$, and also $\| f_{\Delta - \Delta_\varepsilon} \|_\Phi < \varepsilon$. Furthermore there is a constant A such that $\| f \|_\Phi \leqq A$ for all $f \in \{f\}$.

Proof. If $\varepsilon > 0$ is given we choose the positive integer l such that $2/2^l < \varepsilon$. Then take $\eta = M^{-l}$ and determine $\tau(\eta)$. This $\tau(\eta)$ may be chosen as $\delta(\varepsilon)$. Indeed, if $m(X) < \delta(\varepsilon) = \tau(\eta)$ then $\int_X \Phi|f|\, dx < \eta = M^{-l}$, hence $\| f_X \|_\Phi \leqq 2/2^l < \varepsilon$ by 8^0. Similarly the interval E_η may be taken as Δ_ε.

Now take $\eta = 1$ in the hypothesis. Then there exists a bounded set $E_1 \subset \Delta$ such that $\int_{\Delta - E_1} \Phi |f|\, dx < 1$ for all $f \in \{f\}$. Now determine N such that E_1 may be covered by N sets of measure smaller than $\tau(1)$. Then $\int_{E_1} \Phi |f| \, dx < N$ for all $f \in \{f\}$. This yields $\int_{\Delta} \Phi |f| \, dx < N + 1$; hence $\| f \|_{\Phi} < N + 2 = A$ for all $f \in \{f\}$ by 2^0.

The proof of Theorem A rests essentially upon the theorem which follows now.

THEOREM 1. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, and let the set $\{f\}$ of functions $f(x) \in L_{\Phi}$ have the property that the set functions $\int_X \Phi |f| \, dx$ are uniformly absolutely continuous as defined above. Then this set $\{f\}$ is sequentially weakly compact, that is, it contains a sequence f_n converging weakly to a function $f_0 \in L_{\Phi}$; expressed as a formula*

$$\lim \int_{\Delta} f_n g \, dx = \int_{\Delta} f_0 g \, dx$$

for every $g \in L_{\Psi}$.

PROOF. Let A, $\delta(\varepsilon)$, Δ_{ε} have the same meaning as in Lemma 2, and consider the set of all bounded linear functionals $f^*(g) = \int_{\Delta} fg \, dx$ in L_{Ψ} where f runs through $\{f\}$. Obviously $| f^*(g) | \leqq \| f \|_{\Phi} \| g \|_{\Psi} \leqq A \| g \|_{\Psi}$ for all f^* in this set.

We now let $g(x)$ run through the characteristic functions of all bounded rational intervals $\Delta_r \subset \Delta$; the set of these $g(x)$ is countable and each $g(x) \in L_{\Psi}$. Then, on account of the diagonal process, there exists a sequence $f_n \in \{f\}$ such that $\int_{\Delta} f_n g \, dx = \int_{\Delta_r} f_n \, dx$ converges for every Δ_r. This implies that $\int_{\Sigma} f_n \, dx$ converges for every Σ which is a finite sum of non-overlapping Δ_r.

It follows that $f_n^*(g) = \int_{\Delta} f_n g \, dx$ converges for every $g(x)$ which is a rational simple function. If now $\Psi(2v) \leqq M_1 \Psi(v)$ for all $v \geqq 0$, the set of all rational simple functions is dense in L_{Ψ} by 9°, hence $f_n^*(g)$ convergent for every $g \in L_{\Psi}$. The limit $f_0^*(g)$ is again a bounded linear functional in L_{Ψ} so that $f_0^*(g) = \int_{\Delta} f_0 g \, dx$ with $f_0 \in L_{\Phi}$ by 10°. This completes the proof for this special case. Observe that the uniform absolute continuity has not been used but only $\| f \|_{\Phi} \leqq A$.

Let the set $O \subset \Delta$ be open and bounded, and let $\varepsilon > 0$ be given. Then there exists a set $\Sigma \subset O$ of the kind above with $m(O - \Sigma) <$

$\delta(\varepsilon)$. Hence, using Lemma 1,

$$\left|\int_O (f_n - f_m)dx\right| \leqq \left|\int_\Sigma\right| + \int_{O-\Sigma} |f_n - f_m|\, dx \leqq$$

$$\left|\int_\Sigma\right| + p^{-1}(m(O)+1)\,\|(f_n - f_m)_{O-\Sigma}\|_\Phi \leqq \left|\int_\Sigma\right| + 2p^{-1}(m(O)+1)\varepsilon,$$

which shows that $\int_O f_n dx$ converges. In a similar way we find that $\int_X f_n dx$ converges for every bounded measurable $X \subset \Delta$.

Assume now that the measurable set $A_1 \subset \Delta$ is bounded. For $X \subset A_1$ we define $F(X) = \lim \int_X f_n dx$. Then $F(\Sigma_1^n X_i) = \Sigma_1^n F(X_i)$ for sets X_i no two of which have common points. Furthermore $m(X) < \delta(\varepsilon)$ implies $\left|\int_X f_n dx\right| \leqq p^{-1}(\delta+1)\,\|(f_n)_X\|_\Phi < p^{-1}(\delta+1)\varepsilon$, hence $\lim F(X) = 0$ for $\lim m(X) = 0$. It follows that the set function $F(X)$ on A_1 is additive and absolutely continuous, so that, by the Radon-Nikodym Theorem (cf. [14], Ch. 1, § 14),

$$\lim \int_X f_n dx = F(X) = \int_X f_0\, dx,$$

where f_0 is summable over A_1. Since $\Delta = \Sigma_1^\infty A_n$ where all A_n are measurable, bounded and non-overlapping, we may extend $f_0(x)$ on the whole set Δ in such a way that $\lim \int_X f_n dx = \int_X f_0 dx$ for every bounded measurable set $X \subset \Delta$. Then also $\lim \int_\Delta (f_n - f_0) g dx = 0$ for every simple function $g(x)$ (a simple function assumes only a finite set of values and vanishes outside a bounded interval).

Next, let $g(x)$ be measurable, essentially bounded (hence $\|g\|_\infty < \infty$) and vanishing outside a bounded interval $\Delta_1 \subset \Delta$. We may assume $g(x) \geqq 0$ without loss of generality. There exists a sequence of simple functions $g_n(x)$ with $g(x) \geqq g_n(x) \geqq 0$, $g(x) = \lim g_n(x)$ and all $g_n(x)$ vanishing outside Δ_1. Furthermore, if $\varepsilon > 0$ is given, there exists a positive $\delta'(\varepsilon)$ such that $\int_X |f_0|\, dx < \varepsilon$ if only $m(X) < \delta'(\varepsilon)$, $X \subset \Delta_1$. Take $\delta'' = \min\,[\delta(\varepsilon), \delta'(\varepsilon)]$. Then Egoroff's Theorem guarantees the existence of an index $N(\varepsilon)$ and a set $Q \subset \Delta_1$ with $m(Q) < \delta''$ and $|g_n(x) - g(x)| < \varepsilon$ for $n \geqq N$, $x \in \Delta_1 - Q$. Hence

$$\left|\int_{\Delta} (f_i - f_0)(g - g_N)\,dx\right| = \left|\int_{\Delta_1}\right| \leqq 2\,\|g\|_\infty \int_{Q} |f_i - f_0|\,dx + \varepsilon \int_{\Delta_1 - Q} |f_i - f_0|\,dx$$

$$\leqq 2\,\|g\|_\infty \left\{\varepsilon + \int_{Q} |f_i|\,dx\right\} + \varepsilon\left\{\int_{\Delta_1} |f_0|\,dx + \int_{\Delta_1} |f_i|\,dx\right\}$$

$$\leqq 2\,\|g\|_\infty \{\varepsilon + p^{-1}(\delta'' + 1)\varepsilon\} + \varepsilon\left\{\int_{\Delta_1} |f_0|\,dx + p^{-1}(m(\Delta_1) + 1)A\right\},$$

and this shows that $\lim \int_{\Delta} f_i g\,dx = \int_{\Delta} f_0 g\,dx$.

By Δ_n we shall denote the interval $[-n, n; -n, n; \ldots; -n, n]$ $(n = 1, 2, \ldots)$ in R_m. Supposing that $g(x) \in L_\Psi$ we define $g_n(x)$ $(n = 1, 2, \ldots)$ by

$$g_n(x) = \begin{cases} |g(x)| / \operatorname{sgn} f_0(x), & |g(x)| \leqq n,\ x \in \Delta \,.\, \Delta_n, \\ 0 & \text{elsewhere in } \Delta. \end{cases}$$

Then $g_n(x) f_0(x) = |g_n(x) f_0(x)|$, $\lim |g_n(x)| = |g(x)|$ and $\|g_n\|_\Psi \leqq \|g\|_\Psi$ on account of $|g_n(x)| \leqq |g(x)|$. Furthermore, since $g_n(x)$ is bounded and vanishing outside Δ_n, $\int_{\Delta} g_n f_0\,dx = \lim_i \int_{\Delta} g_n f_i\,dx$ with $\left|\int_{\Delta} g_n f_i\,dx\right| \leqq \|f_i\|_\Phi \|g_n\|_\Psi \leqq A\,\|g\|_\Psi$, hence $\int_{\Delta} g_n f_0\,dx \leqq A\,\|g\|_\Psi$ for all n. It follows by Fatou's Theorem that

$$\int_{\Delta} |g f_0|\,dx \leqq \liminf \int_{\Delta} |g_n f_0|\,dx = \liminf \int_{\Delta} g_n f_0\,dx \leqq A\,\|g\|_\Psi,$$

which shows on account of 6° that $f_0 \in L_\Phi$.

It remains to prove that $\lim \int_{\Delta} (f_i - f_0) g\,dx = 0$ for every $g \in L_\Psi$. Let first $g(x) = 0$ outside a bounded interval Δ_1, and define $g_n(x)$ $(n = 1, 2, \ldots)$ by

$$g_n(x) = \begin{cases} g(x), & |g(x)| \leqq n,\ x \in \Delta \,.\, \Delta_1, \\ 0 & \text{elsewhere in } \Delta. \end{cases}$$

Then $|g_n(x)| \leqq |g(x)|$, $g(x) = \lim g_n(x)$ and $\|g_n\|_\Psi \leqq \|g\|_\Psi$. If $\varepsilon > 0$ is given, there exists a positive $\delta_1(\varepsilon)$ such that $\|(f_0)_X\|_\Phi < \varepsilon$ if only $m(X) < \delta_1(\varepsilon)$, $X \subset \Delta_1$. Take $\delta_2 = \min[\delta(\varepsilon), \delta_1(\varepsilon)]$. Then Egoroff's Theorem guarantees the existence of a set $Q \subset \Delta_1$ and an index $N(\varepsilon)$ such that $m(Q) < \delta_2$ and $|g_n(x) - g(x)| < \varepsilon$ for $n \geqq N$, $x \in \Delta_1 - Q$. Hence

$$\left|\int_\Delta (f_i - f_0)(g - g_N)dx\right| \leqq 2 \| g \|_\Psi \| (f_i - f_0)_Q \|_\Phi + \varepsilon \int_{\Delta_1 - Q} | f_i - f_0 | dx$$
$$\leqq 2 \| g \|_\Psi \{\varepsilon + \varepsilon\} + \varepsilon \left\{ \int_{\Delta_1} | f_0 | dx + p^{-1}(m(\Delta_1) + 1)A \right\},$$

which shows that $\lim \int_\Delta (f_i - f_0) g\, dx = 0$ for this $g(x)$.

Let now $g \in L_\Psi$ without any restriction. If $\varepsilon > 0$ is given there exists a bounded interval Δ_ε^* such that $\| (f_0)_{\Delta - \Delta_\varepsilon^*} \|_\Phi < \varepsilon$. Let Q be a bounded interval which contains Δ_ε and Δ_ε^*. Then $\| (f_i - f_0)_{\Delta - Q} \|_\Phi < 2\varepsilon$ for all f_i, hence $\left| \int_{\Delta - Q} (f_i - f_0) g\, dx \right| < 2\varepsilon \| g \|_\Psi$. Since $\lim \int_Q (f_i - f_0) g\, dx = 0$ we finally find

$$\lim \int_\Delta f_i g\, dx = \int_\Delta f_0 g\, dx.$$

This completes the proof. Observe that the existence of the interval Δ_ε has only been used in the last lines.

Theorem A. *If $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$ and if the linear integral transformation T with kernel $T(x, y)$ satisfies $||| T |||_\Phi < \infty$, then T^2 is completely continuous on L_Φ into L_Φ; that is, if $\{f\}$ is a bounded set of functions $f(x) \in L_\Phi$, then $\{T^2 f\}$ is sequentially compact. This means that $\{T^2 f\}$ contains a sequence converging according to the L_Φ-norm.*

Proof. Let $\| f \|_\Phi \leqq B$ for all $f \in \{f\}$. For any $f \in L_\Phi$ and for almost every $x \in \Delta$ we have

$$| h(x) | = \left| \int_\Delta T(x, y) f(y) dy \right| \leqq \| T_x(y) \|_\Psi \| f \|_\Phi,$$

hence $\| h \|_\Phi \leqq ||| T |||_\Phi \| f \|_\Phi$. This shows already that T is a bounded transformation. Note in particular that $\| Tf \|_\Phi \leqq B ||| T |||_\Phi$ for all $f \in \{f\}$. Let now $t_{\text{maj}}(x)$ be a measurable majorant of $\| T_x(y) \|_\Psi$ such that $\| t_{\text{maj}} \|_\Phi < \infty$. A majorant of this kind exists since $||| T |||_\Phi < \infty$. Then, on account of 7^0, also $\int_\Delta \Phi | B\, t_{\text{maj}}(x) | dx < \infty$. It follows that to each $\eta > 0$ there are assigned a number $\tau(\eta) > 0$ and a bounded subinterval $E_\eta \subset \Delta$ such that $\int_X \Phi | Bt_{\text{maj}} | dx < \eta$ if only $m(X) < \tau$ and such that $\int_{\Delta - E_\eta} \Phi | Bt_{\text{maj}} | dx < \eta$. But, since for any $h = Tf$, $f \in \{f\}$, we have

$$| h(x) | \leqq \| f \|_\Phi t_{\text{maj}}(x) \leqq B\, t_{\text{maj}}(x),$$

this implies $\int_{\Delta - E_\eta} \Phi \mid h \mid dx < \eta$ and $\int_X \Phi \mid h \mid dx < \eta$ for $m(X) < \tau$.

Theorem 1 is therefore applicable on the set $\{h\}$; there exists a sequence $h_n(x) = Tf_n(x)$ and there exists a function $h_0(x) \in L_\Phi$ such that

$$\lim \int_\Delta h_n(y) g(y) dy = \int_\Delta h_0(y) g(y) dy$$

for every $g \in L_\Psi$. Since $T(x, y) \in L_\Psi$ as a function of y for almost every $x \in \Delta$ we may take $g(y) = T(x, y)$ for these values of x, hence

$$\lim k_n(x) = \lim \int_\Delta T(x, y) h_n(y) dy = \int_\Delta T(x, y) h_0(y) dy = k_0(x)$$

or $\lim \mid k_n(x) - k_0(x) \mid = 0$ almost everywhere in Δ. Then, $\Phi(u)$ being continuous, also

$$(1) \qquad \lim \Phi \mid k_n(x) - k_0(x) \mid = 0$$

almost everywhere in Δ. Furthermore $\mid k_n(x) - k_0(x) \mid \leqq \| h_n - h_0 \|_\Phi \| T_x(y) \|_\Psi \leqq (B \||| T \|||_\Phi + \| h_0 \|_\Phi) \, t_{\text{maj}}(x) = q(x)$, where $q(x) \in L_\Phi$. This implies

$$(2) \qquad \Phi \mid k_n - k \mid \leqq \Phi \mid q \mid, \int_\Delta \Phi \mid q \mid dx < \infty.$$

In view of (1) and (2) Lebesgue's well-known theorem yields now $\lim \int_\Delta \Phi \mid k_n - k_0 \mid dx = 0$, hence by 8^0

$$\lim \| k_n - k_0 \|_\Phi = 0.$$

But $k_n = Th_n = T^2 f_n$, so that the sequence $T^2 f_n$, $f_n \in \{f\}$, converges in L_Φ to a function $k_0 \in L_\Phi$. This was to be proved.

If $\Psi(2v) \leqq M_1 \Psi(v)$ for all $v \geqq 0$, the set $\{f\}$ itself already contains a weakly converging sequence f_n, so that in this case the sequence Tf_n converges. This shows that now T itself is completely continuous (cf. [9]).

§ 3. Integral transformations with kernels belonging to $L_{\Phi\Psi}$ and their resolvents.

We repeat the definitions of $L_{\Phi\Psi}$ and $\| T \|_{\Phi\Psi}$.

DEFINITION. *$L_{\Phi\Psi}$ is the class of all $T(x, y)$, measurable on $\Delta \times \Delta$, having the property that $T(x, y) f(y) g(x)$ is summable over $\Delta \times \Delta$ for all $f \in L_\Phi$, $g \in L_\Psi$.*

DEFINITION. *If $T(x, y)$ is measurable on $\Delta \times \Delta$, then $\| T \|_{\Phi\Psi} =$ l.u.b. $\int_\Delta \int_\Delta \mid T(x, y) f(y) g(x) \mid dx\, dy$ for $\int_\Delta \Phi \mid f \mid dy \leqq 1$, $\int_\Delta \Psi \mid g \mid dx \leqq 1$.*

THEOREM 2. *$T(x, y) \in L_{\Phi\Psi}$ if and only if $\int_{\Delta} | T(x, y)f(y) | dy \in L_{\Phi}$ for every $f \in L_{\Phi}$.*

PROOF. Follows immediately from 6^0 and 5^0.

THEOREM 3. *If $T(x, y) \in L_{\Phi\Psi}$, then the integral transformation T with kernel $T(x, y)$ is bounded on L_{Φ} into L_{Φ}. Similarly T_a with kernel $| T(x, y) |$ is bounded on L_{Φ} into L_{Φ}. We have*

$$\| T \| \leqq \| T_a \| \leqq \| T \|_{\Phi\Psi} \leqq 2 \| T_a \|.$$

PROOF. Since $T(x, y)f(y)g(x)$ is summable over $\Delta \times \Delta$ for all $f \in L_{\Phi}$, $g \in L_{\Psi}$, it follows from 6^0 that $\int_{\Delta} T(x, y)f(y)\,dy \in L_{\Phi}$. The integral transformation T with domain L_{Φ} has therefore its range in L_{Φ} as well. Similarly $\int_{\Delta} T(x, y)g(x)dx \in L_{\Psi}$ for $g \in L_{\Psi}$.

We shall prove now that T is closed, that is, supposing that $\lim \| f_n - f \|_{\Phi} = 0$, $\lim \| Tf_n - k \|_{\Phi} = 0$, we shall prove that $k = Tf$. Indeed, for every $g \in L_{\Psi}$ we have

$$\begin{aligned}\int_{\Delta} kg\,dx &= \lim \int_{\Delta} \left(\int_{\Delta} T(x, y)f_n(y)\,dy \right) g(x)\,dx \\ &= \lim \int_{\Delta} f_n(y) \left(\int_{\Delta} T(x, y)g(x)\,dx \right) dy \\ &= \int_{\Delta} f(y) \left(\int_{\Delta} T(x, y)g(x)\,dx \right) dy = \int_{\Delta} \left(\int_{\Delta} T(x, y)f(y)dy \right) g(x)dx,\end{aligned}$$

hence $k = Tf$. But a closed linear transformation, having a complete Banach space as its domain is bounded (cf. [1], p. 41 or [15], p. 30). Hence T is bounded. Similarly, since $T(x, y) \in L_{\Phi\Psi}$ if and only if $| T(x, y) | \in L_{\Phi\Psi}$, the transformation T_a with kernel $| T(x, y) |$ is bounded.

For $f \in L_{\Phi}$ we write $h(x) = Tf(x)$, $k(x) = T_a | f(x) |$. In view of $\left| \int_{\Delta} T(x, y)f(y)dy \right| \leqq \int_{\Delta} | T(x, y) | \,.\, | f(y) | \,dy$ we find $| h(x) | \leqq k(x)$, hence $\| Tf \|_{\Phi} \leqq \| T_a | f | \|_{\Phi}$. Observing that f and $| f |$ have the same L_{Φ}-norm, we obtain $\| T \| \leqq \| T_a \|$.

If $\varepsilon > 0$ is given there exist two functions $f \in L_{\Phi}$ and $g \in L_{\Psi}$, satisfying $\| f \|_{\Phi} = 1$ and $\int_{\Delta} \Psi | g | \,dx \leqq 1$ such that

$$\| T_a \| - \varepsilon < \left| \int_{\Delta \times \Delta} | T(x, y) | \, f(y)g(x)dxdy \right|.$$

But $\| f \|_\Phi = 1$ implies $\int_\Delta \Phi |f| \, dy \leqq 1$ by 3^0, hence

$$\| T_a \| - \varepsilon < \int_{\Delta \times \Delta} | T(x, y) f(y) g(x) | \, dx dy \leqq \| T \|_{\Phi\Psi}.$$

On the other hand, if $\int_\Delta \Phi |f| \, dy \leqq 1$, $\int_\Delta \Psi |g| \, dx \leqq 1$, we have $\| f \|_\Phi \leqq \int_\Delta \Phi |f| \, dy + 1 \leqq 2$, so that

$$\int_{\Delta \times \Delta} | T(x, y) f(y) g(x) | \, dx dy \leqq \left\| \int_\Delta | T(x, y) f(y) | \, dy \right\|_\Phi$$
$$\leqq \| T_a \| \cdot \| f \|_\Phi \leqq 2 \| T_a \|,$$

hence $\| T \|_{\Phi\Psi} \leqq 2 \| T_a \|$. Collecting our results we have $\| T \| \leqq \| T_a \| \leqq \| T \|_{\Phi\Psi} \leqq 2 \| T_a \|$.

Corollary. *$T(x, y) \in L_{\Phi\Psi}$ if and only if $\| T \|_{\Phi\Psi} < \infty$.*

Remarks. 1^0. In the L_p-case ($1 \leqq p \leqq \infty$, $1/p + 1/q = 1$), defining $\| T \|_{pq}$ by

$$\| T \|_{pq} = \text{l.u.b.} \int_{\Delta \times \Delta} | T(x, y) f(y) g(x) | \, dx dy \text{ for } \| f \|_p \leqq 1, \ \| g \|_q \leqq 1,$$

we find similarly $\| T \| \leqq \| T_a \| = \| T \|_{pq}$.

2^0. If $T(x, y) \in L_{\Phi\Psi}$, the kernels $T^*(x, y) = T(y, x)$ and $| T^*(x, y) |$ correspond with linear integral transformations T^* and T_a^* which are bounded on L_Ψ into L_Ψ, and for which

$$\| T^* \| \leqq \| T_a^* \| \leqq \| T \|_{\Phi\Psi} \leqq 2 \| T_a^* \|.$$

Theorem 4. *$\| T \|_{\Phi\Psi} = 0$, $\| T_a \| = 0$, $\| T \| = 0$ and $T(x, y) = 0$ almost everywhere in $\Delta \times \Delta$ are four equivalent statements.*

Proof. $T(x, y) = 0$ trivially implies $\| T \|_{\Phi\Psi} = 0$. But $\| T \|_{\Phi\Psi} = 0$ implies $\| T_a \| = 0$ and this in its turn implies $\| T \| = 0$. It remains to prove that $\| T \| = 0$ implies $T(x, y) = 0$. Obviously $\| T \| = 0$ gives $\int_{\Delta \times \Delta} T(x, y) f(y) g(x) dx dy = 0$ for all $f \in L_\Phi$, $g \in L_\Psi$, hence $\int_{\Delta_1 \times \Delta_2} T(x, y) \, dx \, dy = 0$ where Δ_1 and Δ_2 are arbitrary bounded subintervals of Δ. Then however $\int_S T(x, y) \, dx \, dy = 0$ for every bounded measurable set $S \subset \Delta \times \Delta$, which implies $T(x, y) = 0$.

Theorem 5. *If addition and multiplication by complex numbers are defined in the natural way, the class $L_{\Phi\Psi}$ is a complete Banach space with norm $\| T \|_{\Phi\Psi}$.*

Proof. We have only to prove that $L_{\Phi\Psi}$ is complete. Let therefore the sequence $T_n(x, y) \in L_{\Phi\Psi}$ $(n = 1, 2, \ldots)$ with $\lim \| T_n - T_m \|_{\Phi\Psi} = 0$ be given. We suppose first that Δ is bounded. Then there exist positive numbers p and q such that $m(\Delta)\Phi(p) \leqq 1$ and $m(\Delta)\Psi(q) \leqq 1$, so that, taking $f(y) = p$ and $g(x) = q$, we have $\int_\Delta \Phi |f| \, dy \leqq 1$ and $\int_\Delta \Psi |g| \, dx \leqq 1$. It follows that

$$\lim \int_{\Delta\times\Delta} | T_n(x, y) - T_m(x, y) | \, dx \, dy = 0$$

for $m, n \to \infty$, from which we infer by a well-known argument that a subsequence $T_k(x, y)$ $(k = n_1, n_2, \ldots)$ converges pointwise to a measurable function $T(x, y)$. Letting now in

$$\int_{\Delta\times\Delta} | (T_n - T_m) f(y) g(x) | \, dx \, dy \leqq \varepsilon,$$

holding for $m, n \geqq N(\varepsilon)$, the index m run through the subsequence $k = n_1, n_2, \ldots$, Fatou's Theorem yields

$$\int_{\Delta\times\Delta} | (T_n - T) f(y) g(x) | \, dx \, dy \leqq \varepsilon$$

for $n \geqq N(\varepsilon)$. Hence $\lim \| T_n - T \|_{\Phi\Psi} = 0$. If Δ is an unbounded interval, it is necessary to introduce an extra diagonalprocess in order to obtain the subsequence $T_k(x, y)$.

Theorem 6. *If $T_1(x, y)$ and $T_2(x,y)$, corresponding with the transformations T_1 and T_2, both belong to $L_{\Phi\Psi}$, then $T_3 = T_1 T_2$ has the kernel $T_3(x, y) = \int_\Delta T_1(x, z) T_2(z, y) \, dz$ belonging to $L_{\Phi\Psi}$.*

Proof. It follows from our hypothesis that

$$\int_\Delta | T_1(x, z) | \left(\int_\Delta | T_2(z, y) | \cdot | f(y) | \, dy \right) dz \in L_\Phi.$$

for every $f \in L_\Phi$, hence

$$\int_{\Delta\times\Delta\times\Delta} | T_1(x, z) T_2(z, y) f(y) g(x) | \, dx \, dy \, dz < \infty$$

for arbitrary $f \in L_\Phi$, $g \in L_\Psi$. For Δ_1 an arbitrary bounded subset of Δ this gives $\int_{\Delta_1\times\Delta_1\times\Delta} | T_1(x, z) T_2(z, y) | \, dx \, dy \, dz < \infty$, hence

$$\int_{\Delta_1\times\Delta_1\times\Delta} T_1(x, z) \, T_2(z, y) \, dx \, dy \, dz = \int_{\Delta_1\times\Delta_1} \left(\int_\Delta T_1(x, z) \, T_2(z, y) \, dz \right) dx \, dy,$$

where $T_3(x, y) = \int_\Delta T_1(x, z) T_2(z, y)\, dz$ is measurable in $\Delta_1 \times \Delta_1$, and therefore in $\Delta \times \Delta$. It is now easily seen that $T_3 = T_1 T_2$ has $T_3(x, y)$ as its kernel and that

$$\int_\Delta | T_3(x, y) f(y) |\, dy \leqq \int_\Delta \left(\int_\Delta | T_1(x, z) T_2(z, y) |\, dz \right) | f(y) |\, dy \in L_\Phi,$$

hence $T_3(x, y) \in L_{\Phi\Psi}$.

Corollary. *If T has the kernel $T(x, y) \in L_{\Phi\Psi}$, all transformations $T^n (n = 2, 3, \ldots)$ are integral transformations with the iterated kernels $T_n(x, y) = \int_\Delta T_{n-1}(x, z) T(z, y) dz \in L_{\Phi\Psi}$.*

Theorem 7. *If T_1 and T_2 have kernels $T_1(x, y) \in L_{\Phi\Psi}$ and $T_2(x, y) \in L_{\Phi\Psi}$, and T_{1a}, T_{2a} are the transformations with kernels $| T_1(x, y) |$ and $| T_2(x, y) |$, then $\| (T_1 T_2)_a \| \leqq \| T_{1a} T_{2a} \| \leqq \| T_{1a} \| \cdot \| T_{2a} \|$. In particular $\| (T^n)_a \| \leqq \| T_a \|^n$ $(n = 2, 3, \ldots)$ for $T(x, y) \in L_{\Phi\Psi}$, and hence $\| T^n \|_{\Phi\Psi} \leqq 2 \| T \|^n_{\Phi\Psi}$.*

Proof. We have only to prove that $\| (T_1 T_2)_a \| \leqq \| T_{1a} T_{2a} \|$. The transformation $(T_1 T_2)_a$ has the kernel $\left| \int_\Delta T_1(x, z) T_2(z, y) dz \right| \leqq \int_\Delta | T_1(x, z) T_2(z, y) |\, dz$ which is the kernel of $T_{1a} T_{2a}$. The rest of the proof is similar to that of $\| T \| \leqq \| T_a \|$ in Theorem 3.

Theorem 8. *Suppose that $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$ and that $T(x, y) \in L_{\Phi\Psi}$ is the kernel of T. Then, if $g^* \in (L_\Phi)^*$ is represented by $g(x) \in L_\Psi$ according to 10^0, and T^* is the adjoint of T (hence $(T^* g^*)(f) = g^*(Tf)$ for all $f \in L_\Phi$, $g^* \in (L_\Phi)^*$), the functional $k^* = T^* g^*$ is represented by*

$$k(x) = \int_\Delta T(y, x) g(y) dy.$$

Hence T^* corresponds with the kernel $T^*(x,y) = T(y, x)$ of a bounded transformation on L_Ψ into L_Ψ. If T_1 and T_2 have the kernels $T_1(x, y) \in L_{\Phi\Psi}$ and $T_2(x, y) \in L_{\Phi\Psi}$ and if $T_3 = T_1 T_2$, then T_3^* corresponds with $T_3^*(x, y) = T_3(y, x) = \int_\Delta T_1(y, z) T_2(z, x)\, dz$.

Proof. We have

$$k^*(f) = (T^* g^*)(f) = g^*(Tf) =$$

$$\int_{\Delta \times \Delta} T(y, x) f(x) g(y)\, dx\, dy = \int_\Delta \left(\int_\Delta T(y, x) g(y)\, dy \right) f(x) dx,$$

hence, since $k(x)$ is uniquely determined,

$$k(x) = \int_{\Delta} T(y, x) g(y) dy.$$

The statement on $T_3^* = (T_1 T_2)^*$ follows now immediately from what we proved in Theorem 6 on $T_3 = T_1 T_2$.

Let E be an abstract complete Banach space and T a bounded linear transformation on E into E. By I we denote the identical transformation and, provided the complex number λ belongs to the resolvent set of T, the bounded linear transformation $(T - \lambda I)^{-1}$ is denoted by R_λ. The following two statements are well-known:

If $|\lambda| > \|T\|$, then λ belongs to the resolvent set of T and

$$R_\lambda = -\lambda^{-1} I - \lambda^{-2} T - \lambda^{-3} T^2 - \ldots,$$

where this series converges uniformly (that is, if S_n is its n-th partial sum, then $\lim_{n\to\infty} \| R_\lambda - S_n \| = 0$).

If λ_0 (either $> \|T\|$ or $\leqq \|T\|$ in absolute value) belongs to the resolvent set of T, then all λ in a sufficiently small neighbourhood of λ_0 also belong to the resolvent set of T and, writing $R_0 = (T - \lambda_0 I)^{-1}$, $R_\lambda = (T - \lambda I)^{-1}$, we have

$$R_\lambda = R_0 + (\lambda - \lambda_0) R_0^2 + (\lambda - \lambda_0)^2 R_0^3 + \ldots,$$

where this series converges uniformly, certainly for $|\lambda - \lambda_0| < \|R_0\|^{-1}$.

Both formulas are easily proved on multiplying by $T - \lambda I = (T - \lambda_0 I) - (\lambda - \lambda_0) I$.

As in the introduction we define H_λ for any $\lambda \neq 0$ in the resolvent set of T by $R_\lambda = -\lambda^{-1} I - \lambda^{-2} H_\lambda$, hence $H_\lambda = -\lambda I - \lambda^2 R_\lambda$, and $H_\lambda = T + \lambda^{-1} T^2 + \lambda^{-2} T^3 + \ldots$ for $|\lambda| > \|T\|$. Writing $\mu = \lambda^{-1}$ we have already seen in the introduction that $I + \mu H_\lambda = (I - \mu T)^{-1}$.

THEOREM 9. *If $\lambda \neq 0$ belongs to the resolvent set of T, then*

$$(T - \lambda I) H_\lambda = H_\lambda (T - \lambda I) = -\lambda T$$

or

$$H_\lambda = -\lambda T R_\lambda = -\lambda R_\lambda T.$$

If T^n is completely continuous for an integer $n \geqq 1$, the same holds for $(H_\lambda)^n$.

If $\lambda_0 \neq 0$ belongs to the resolvent set of T and we write $H_0 = H_{\lambda_0}$, $\mu_0 = \lambda_0^{-1}$, $\mu = \lambda^{-1}$, then

$$H_\lambda = H_0 + (\mu - \mu_0) H_0^2 + (\mu - \mu_0)^2 H_0^3 + \ldots$$

uniformly for all λ in a sufficiently small neighbourhood of λ_0, certainly for $|\mu - \mu_0| < \| H_0 \|^{-1}$.

PROOF. The first statement follows immediately from the definition of H_λ. The second statement follows from $(H_\lambda)^n = (-\lambda)^n R_\lambda^n T^n$.

For $|\mu - \mu_0| < \| H_0 \|^{-1}$ the series $X_\lambda = H_0 + (\mu - \mu_0) H_0^2 + \dots$ obviously converges uniformly. Since $(\mu T - I) H_\lambda = -T$, we find

$$(\mu T - I) X_\lambda = (\mu_0 T - I) \Sigma_0^\infty (\mu - \mu_0)^k H_0^{k+1} + (\mu - \mu_0) T \Sigma_0^\infty (\mu - \mu_0)^k H_0^{k+1} =$$
$$- T \Sigma_0^\infty (\mu - \mu_0)^k H_0^k + T \Sigma_1^\infty (\mu - \mu_0)^k H_0^k = -T$$

or $X_\lambda = H_\lambda$.

We return to the space $L_{\Phi\Psi}$.

THEOREM 10. *If T has the kernel $T(x, y) \in L_{\Phi\Psi}$ and $|\lambda| > \| T_a \|$ (so that λ belongs to the resolvent set of T on account of $\| T_a \| \geqq \| T \|$), then H_λ is an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$. The series*

$$T(x, y) + \lambda^{-1} T_2(x, y) + \lambda^{-2} T_3(x, y) + \dots$$

(Neumann series) converges pointwise to $H_\lambda(x, y)$ almost everywhere in $\Delta \times \Delta$.

PROOF. Since $|\lambda| > \| T_a \| \geqq \| T \|$, we have $H_\lambda = T + \lambda^{-1} T^2 + \lambda^{-2} T^3 + \dots$ uniformly, and the partial sums of this series are integral transformations with kernels belonging to $L_{\Phi\Psi}$. The series $\Sigma |\lambda|^{-n} \| T_a \|^{n+1}$ also converges. From

$$\| \Sigma_p^q \lambda^{-n} T^{n+1} \|_{\Phi\Psi} \leqq \Sigma_p^q |\lambda|^{-n} \| T^{n+1} \|_{\Phi\Psi} \leqq$$
$$2 \Sigma_p^q |\lambda|^{-n} \| (T^{n+1})_a \| \leqq 2 \Sigma_p^q |\lambda|^{-n} \| T_a \|^{n+1}$$

(cf. Theorem 3 and Theorem 7) follows then that the series for H_λ converges in $L_{\Phi\Psi}$. The completeness of the space $L_{\Phi\Psi}$ guarantees now that H_λ has a kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$.

The convergence of $\Sigma_0^\infty \| \lambda^{-n} T^{n+1} \|_{\Phi\Psi}$ implies that $\Sigma \int_{\Delta_1 \times \Delta_1} | T_{n+1}(x, y) \lambda^{-n} | \, dx\, dy < \infty$ for every bounded interval $\Delta_1 \subset \Delta$, hence $\Sigma | T_{n+1}(x, y) \lambda^{-n} | < \infty$ almost everywhere in $\Delta_1 \times \Delta_1$. It follows that $\Sigma_0^\infty \lambda^{-n} T_{n+1}(x, y)$ converges almost everywhere in $\Delta \times \Delta$. That its sum function is $H_\lambda(x, y)$ may be seen by applying Fatou's Theorem as in Theorem 5.

REMARK. If $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, so that the adjoint transformation T^* corresponds by Theorem 8 with the kernel

$T(y, x)$, the transformation H_λ^* in $(L_\Phi)^*$, defined by $(T^* - \lambda I)^{-1} = - \lambda^{-1} I - \lambda^{-2} H_\lambda^*$, corresponds with the kernel $H_\lambda(y, x)$. We have used here that λ is in the resolvent set of T^* if and only if λ is in the resolvent set of T.

THEOREM 11. *If T has the kernel $T(x, y) \in L_{\Phi\Psi}$ and if λ_0, belonging to the resolvent set of T, has the property that $H_0 = H_{\lambda_0}$ is an integral transformation with kernel $H_0(x, y) \in L_{\Phi\Psi}$, then H_λ is an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$ for all λ satisfying $| \lambda^{-1} - \lambda_0^{-1} | < \| (H_0)_a \|^{-1}$.*

PROOF. For $| \lambda^{-1} - \lambda_0^{-1} | < \| (H_0)_a \|^{-1}$ we have $\Sigma | \lambda^{-1} - \lambda_0^{-1} |^k \| (H_0)_a \|^{k+1} < \infty$ and $H_\lambda = \Sigma_0^\infty (\lambda^{-1} - \lambda_0^{-1})^k H_0^{k+1}$ uniformly by Theorem 9. The desired result follows now in a similar way as in the preceding theorem.

REMARK. The series $\Sigma_0^\infty (\lambda^{-1} - \lambda_0^{-1})^k H_{0, k+1}(x, y)$ converges almost everywhere in $\Delta \times \Delta$ to $H_\lambda(x, y)$. Proof as in the preceding theorem.

It seems difficult to say more about the character of the transformation H_λ without making additional hypotheses. Our next theorem will show that under certain conditions H_λ is an integral transformation with kernel $H_\lambda(x, y)$ for every $\lambda \neq 0$ in the resolvent set of T. It will not follow from our proof however that this $H_\lambda(x, y)$, as a function on $\Delta \times \Delta$, is measurable and hence even less that $H_\lambda(x, y) \in L_{\Phi\Psi}$.

THEOREM 12. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$ and $T(x, y) \in L_{\Phi\Psi}$. Let furthermore $T(x, y) = T_x(y) \in L_\Psi$ for almost every $x \in \Delta$. Then, if $\lambda \neq 0$ belongs to the resolvent set of T, there exists a function $H_\lambda(x, y)$, belonging to L_Ψ as a function of y for almost every $x \in \Delta$, such that $g = H_\lambda f$ is given by*

$$g(x) = \int_\Delta H_\lambda(x, y) f(y)\, dy.$$

PROOF. Note first that $T(x, y) \in L_{\Phi\Psi}$ does not always imply $T_x(y) \in L_\Psi$. If L_Φ is the Lebesgue space L_2 we have an example in $T(x, y) = | x - y |^{-\alpha}$, $1/2 \leqq \alpha < 1$.

By hypothesis $T(x, y) = T_x(y) \in L_\Psi$ for all $x \in \Delta - E_0$ where E_0 is of measure zero. Using now the one-to-one correspondence between all functionals $g^* \in (L_\Phi)^*$ and all functions $g(x) \in L_\Psi$ we apply the transformation $- \lambda R_\lambda^*$ to $T_x(y) \in L_\Psi$, and we obtain a function $H_\lambda(x, y) = H_{\lambda, x}(y) \in L_\Psi$ for all $x \in \Delta - E_0$. Hence $(- \lambda R_\lambda^*) T_x(y) = H_{\lambda, x}(y)$. Then, if $f \in L_\Phi$ is arbitrary, we have for $x \in \Delta - E_0$

$$\int_{\Delta} H_\lambda(x, y) f(y)\, dy = \{(-\lambda R_\lambda^*) T_x(y)\} f$$
$$= \{T_x(y)\} (-\lambda R_\lambda f) = \int_{\Delta} T(x, y) h(y) dy,$$

where we have written $h = -\lambda R_\lambda f$. But evidently $h \in L_\Phi$, so that $g(x) = \int_{\Delta} T(x, y) h(y)\, dy \in L_\Phi$ as well. Since $g(x)$ and $\int_{\Delta} H_\lambda(x, y) f(y) dy$ are identical for $x \in \Delta - E_0$ we may say therefore that

$$\int_{\Delta} H_\lambda(x, y) f(y) dy = g(x) = Th = T(-\lambda R_\lambda f) = -\lambda T R_\lambda f = H_\lambda f.$$

This completes the proof.

REMARK. If in particular $|\lambda| > \| T_a \|$ it is to be expected that the function $H_\lambda(x, y)$ in the present theorem is identical almost everywhere in $\Delta \times \Delta$ with the function $H_\lambda(x, y)$ found in Theorem 10. To show that this is true denote for a moment the $H_\lambda(x, y)$ of the present theorem by $K_\lambda(x, y)$. Then

$$\int_{\Delta} \left(\int_{\Delta} \{H_\lambda - K_\lambda\} f(y)\, dy \right) g(x)\, dx = 0$$

for all $f \in L_\Phi$, $g \in L_\Psi$. Note that it is not permitted to replace this repeated integral by a double integral because we do not know that $H_\lambda(x, y)$ is measurable. It follows that for arbitrary bounded subintervals Δ_1 and Δ_2 of Δ

$$\int_{\Delta_1} \left(\int_{\Delta_2} \{H_\lambda - K_\lambda\}\, dy \right) dx = 0.$$

Keeping first Δ_2 fixed this implies $\int_{\Delta_2} \{H_\lambda - K_\lambda\}\, dy = 0$ for almost every $x \in \Delta$. The set of exceptional x however may depend on Δ_2. Nevertheless, observing that the set of all rational Δ_2 is countable, we may say that for almost every $x \in \Delta$

$$\int_{\Delta_2} \{H_\lambda - K_\lambda\}\, dy = 0$$

for all rational Δ_2 simultaneously, and this is sufficient for drawing the conclusion that $H_\lambda - K_\lambda = 0$ almost everywhere in $\Delta \times \Delta$.

§ 4. The space D_Φ of all kernels of finite double-norm.

Suppose that $T(x, y)$ is measurable in $\Delta \times \Delta$. For the definition of the double-norm $|||\, T\, |||_\Phi$ of $T(x, y)$ relative to the Orlicz space L_Φ we refer to the introduction.

Theorem 13. *If $T(x, y)$ is measurable, then $\|\, T\, \|_{\Phi\Psi} \leqq |||\, T\, |||_\Phi$. Hence, if $T(x, y)$ is of finite double-norm ($|||\, T\, |||_\Phi < \infty$), then $T(x, y) \in L_{\Phi\Psi}$.*

Proof. For $\int_\Delta \Phi\, |\, f\, |\, dy \leqq 1$ we have

$$\int_\Delta |\, T(x, y) f(y)\, |\, dy \leqq \|\, T_x(y)\, \|_\Psi \leqq t_{\text{maj}}(x),$$

where $t_{\text{maj}}(x)$ is an arbitrary measurable majorant of $t(x) = \|\, T_x(y)\, \|_\Psi$. Hence, provided also $\int_\Delta \Psi\, |\, g\, |\, dx \leqq 1$,

$$\int_{\Delta\times\Delta} |\, T(x, y) f(y) g(x)\, |\, dx\, dy \leqq \int_\Delta |\, t_{\text{maj}}(x) g(x)\, |\, dx \leqq \|\, t_{\text{maj}}\, \|_\Phi.$$

Since this holds for all $t_{\text{maj}}(x)$, we find $\|\, T\, \|_{\Phi\Psi} \leqq |||\, T\, |||_\Phi$.

Definition. *The class D_Φ is the class of all measurable $T(x, y)$ satisfying $|||\, T\, |||_\Phi < \infty$.*

Theorem 14. *For elements T, T_1, T_2 of D_Φ we have*

$$|||\, \alpha T\, |||_\Phi = |\, \alpha\, | \,.\, |||\, T\, |||_\Phi \quad \textit{for all complex } \alpha,$$
$$|||\, T_1 + T_2\, |||_\Phi \leqq |||\, T_1\, |||_\Phi + |||\, T_2\, |||_\Phi,$$

$|||\, T\, |||_\Phi = 0$ if and only if $T(x, y) = 0$ almost everywhere in $\Delta \times \Delta$.

Proof. Trivial.

Theorem 15. *The class D_Φ is a complete Banach space with norm $|||\, T\, |||_\Phi$.*

Proof. We have only to prove that D_Φ is complete. For this purpose let the sequence $T_n(x, y) \in D_\Phi$ $(n = 1, 2, \ldots)$ with $\lim |||\, T_n - T_m\, |||_\Phi = 0$ be given. Since this implies $T_n(x, y) \in L_{\Phi\Psi}$ and $\lim \|\, T_n - T_m\, \|_{\Phi\Psi} = 0$, there exists a subsequence $T_k(x, y)$ $(k = n_1, n_2, \ldots)$ converging almost everywhere in $\Delta \times \Delta$ to a measurable $T(x, y)$ (cf. the proof of Theorem 5). Hence for n fixed, by Fatou's Theorem,

$$\int_\Delta |\, T(x, y) - T_n(x, y)\, | \,.\, |\, f(y)\, |\, dy$$
$$\leqq \liminf \int_\Delta |\, T_k(x, y) - T_n(x, y)\, | \,.\, |\, f(y)\, |\, dy$$

for almost every $x \in \Delta$, where k runs through $n_1, n_2, \ldots$ and

$f \in L_\Phi$. If in particular $\int_\Delta \Phi \mid f \mid dy \leqq 1$, the integral on the right does not exceed $\| T_k(x, y) - T_n(x, y) \|_\Psi$, hence

$$d(x) = \| T(x, y) - T_n(x, y) \|_\Psi$$
$$\leqq \liminf \| T_k(x, y) - T_n(x, y) \|_\Psi = \liminf d_k(x).$$

Let now $m_k(x)$ be a measurable majorant of $d_k(x)$ $(k = n_1, n_2, \ldots)$. Then $m(x) = \liminf m_k(x)$ is a measurable majorant of $d(x)$. The majorants $m_k(x)$ may be chosen so that $\| m_k(x) \|_\Phi \leqq 2 ||| T_k - T_n |||_\Phi$. For $\int_\Delta \Psi \mid g \mid dx \leqq 1$ we have now

$$\int_\Delta m(x) \,|g(x)|\, dx = \int_\Delta \liminf m_k(x) \,|g(x)|\, dx \leqq \liminf \int_\Delta m_k(x) \,|g(x)|\, dx$$
$$\leqq \liminf \| m_k \|_\Phi \leqq \liminf 2 \; ||| T_k - T_n |||_\Phi,$$

hence $\| m \|_\Phi \leqq \liminf 2 \; ||| T_k - T_n |||_\Phi$, so that certainly

$$||| T - T_n |||_\Phi \leqq \liminf 2 \; ||| T_k - T_n |||_\Phi.$$

This shows that $||| T - T_n |||_\Phi$ tends to zero as $n \to \infty$.

THEOREM 16. *If $T(x, y) \in D_\Phi$, $S(x, y) \in L_{\Phi\Psi}$ and $V = TS$, then $V(x, y) \in D_\Phi$ and*

$$||| V |||_\Phi \leqq ||| T |||_\Phi \, \| S \|_{\Phi\Psi}.$$

In particular, if T_1 and $T_2 \in D_\Phi$, then $T_1 T_2 \in D_\Phi$ and $||| T_1 T_2 |||_\Phi \leqq ||| T_1 |||_\Phi \; ||| T_2 |||_\Phi$. If $T \in D_\Phi$, then $||| T^n |||_\Phi \leqq ||| T |||_\Phi^n$ $(n=1, 2, \ldots)$.

PROOF. We observe first that the transformation with kernel $S(y, x)$ is bounded on L_Ψ into L_Ψ by Theorem 6, Remark, and that its bound does not exceed $\| S \|_{\Phi\Psi}$. Since both $T(x, y)$ and $S(x, y)$ belong to $L_{\Phi\Psi}$, the transformation $V = TS$ is an integral transformation with kernel $V(x, y) = \int_\Delta T(x, z)\, S(z, y)\, dz \in L_{\Phi\Psi}$ by Theorem 6. Then $V_x(y) = V(x, y) = \int_\Delta S(z, y)\, T_x(z)\, dz$ for almost every $x \in \Delta$. Hence, by what we have observed, $\| V_x(y) \|_\Psi \leqq \| S \|_{\Phi\Psi} \, \| T_x(y) \|_\Psi$ for these values of x. This implies $||| V |||_\Phi \leqq ||| T |||_\Phi \, \| S \|_{\Phi\Psi}$.

Before proceeding we recall the definition of $||| T |||_\Phi^{\text{inv}}$. We have $||| T |||_\Phi^{\text{inv}} = ||| T^* |||_\Psi$ where $T^*(x, y) = T(y, x)$. We shall say that $T(x, y)$ is *inversely of finite double-norm* if $||| T |||_\Phi^{\text{inv}} < \infty$. In this case $T^*(x, y) \in L_{\Psi\Phi}$, hence $T(x, y) \in L_{\Phi\Psi}$.

THEOREM 17. *Suppose that $T_1 \in D_\Phi$ and that T_2 is inversely of finite double-norm. Then, defining the trace $\tau(T_1T_2)$ of T_1T_2 by*

$$\tau(T_1T_2) = \int_{\Delta\times\Delta} T_1(x, y)\, T_2(y, x)\, dx\, dy,$$

we have

$$|\, \tau(T_1T_2)\, | \leqq |||\, T_1\, |||_\Phi\, |||\, T_2\, |||_\Phi^{\mathrm{inv}}.$$

PROOF. We have

$$\int_\Delta |\, T_1(x, y)\, T_2(y, x)\, |\, dy \leqq \|\, T_1(x, y)\, \|_\Psi\, \|\, T_2(y, x)\, \|_\Phi \leqq m_1(x)\, m_2(x),$$

where $m_1(x)$ and $m_2(x)$ are measurable majorants. Hence

$$|\, \tau(T_1T_2)\, | \leqq \int_\Delta m_1(x)\, m_2(x)\, dx \leqq \|\, m_1\, \|_\Phi\, \|\, m_2\, \|_\Psi$$

for all m_1, m_2, which implies $|\, \tau(T_1T_2)\, | \leqq |||\, T_1\, |||_\Phi\, |||\, T_2\, |||_\Phi^{\mathrm{inv}}$.

COROLLARY. *If $S \in D_\Phi$ and T is inversely of finite double-norm, then $V = ST \in D_\Phi$ (cf. Theorem 16) has a finite trace $\tau(V)$ and the kernel $V(x, z) = \int_\Delta S(x, y)T(y, z)dy$ satisfies*

$$|\, \tau(V)\, | = \left|\, \int_\Delta V(x, x)dx\, \right| \leqq |||\, S\, |||_\Phi\, |||\, T\, |||_\Phi^{\mathrm{inv}}.$$

If moreover $S = \Sigma_1^\infty S_i$ where all S_i belong to D_Φ and if the series converges in double-norm (that is, $\lim |||\, S - \Sigma_1^n S_i\, |||_\Phi = 0$ for $n \to \infty$), then $\tau(ST) = \Sigma_1^\infty \tau(S_iT)$.

§ 5. The proof of Theorem B.

In the present paragraph we shall have to deal with a bounded linear transformation T on a Banach space E into the same space E. This transformation T will have the property that one of its iterates T^n ($n = 1, 2, \ldots$) is completely continuous. The spectral properties of a transformation T of this kind will be listed more extensively in the next paragraph but for our present purpose it will be sufficient to know that every complex number $\lambda \neq 0$ is either in the point spectrum or in the resolvent set of T, and that the point spectrum is either empty, finite or countable. In this latter case the points of the point spectrum tend to $\lambda = 0$. We recall that λ is in the resolvent set of the adjoint transformation T^* if and only if λ is in the resolvent set of T.

THEOREM B. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, $T(x, y) \in L_{\Phi\Psi}$ and $|||\, T^n\, |||_\Phi < \infty$ for an integer $n \geqq 1$. Then, if $\lambda \neq 0$ is not* *in the point spectrum of T, the transformation H_λ (defined as before) is an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$, and*

$$H_\lambda(x, y) = T(x, y) + \lambda^{-1} T_2(x, y) + \dots + \lambda^{-(n-2)} T_{n-1}(x, y) + \lambda^{-(n-1)} K_\lambda(x, y)$$

where $||| K_\lambda |||_\Phi < \infty$. The functions $T_p(x, y)$ $(p = 2, 3, \dots)$ are here the kernels of T^p. In particular $||| T |||_\Phi < \infty$ implies $||| H_\lambda |||_\Phi < \infty$.

Proof. Since $||| T^n |||_\Phi < \infty$ and $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$ it follows from Theorem A that T^{2n} is completely continuous on L_Φ into L_Φ. The transformation T has therefore the spectral properties described above. In particular any $\lambda \neq 0$ not in the point spectrum of T is in the resolvent set of T so that H_λ may be defined again by $(T - \lambda I)^{-1} = - \lambda^{-1} I - \lambda^{-2} H_\lambda$. Writing $H_\lambda = T + \lambda^{-1} T^2 + \dots + \lambda^{-(n-2)} T^{n-1} + \lambda^{-(n-1)} K_\lambda$, a direct computation shows immediately that

$$(T - \lambda I)K_\lambda = K_\lambda(T - \lambda I) = - \lambda T^n.$$

The rest of the proof will be divided into three parts.

1°. Suppose that we know already that H_λ is an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$. Then, since $T(x, y)$ and all iterates $T_j(x, y) \in L_{\Phi\Psi}$, the transformation K_λ has also a kernel $K_\lambda(x, y) \in L_{\Phi\Psi}$. The relation $K_\lambda(T - \lambda I) = - \lambda T^n$ then implies

$$\int_\Delta T(y, x) K_\lambda(z, y)\, dy - \lambda K_\lambda(z, x) = - \lambda T_n(z, x)$$

almost everywhere in $\Delta \times \Delta$. But this shows that, for almost every $z \in \Delta$, the function $f(x) = K_\lambda(z, x)$ is a solution of

$$\int_\Delta T(y, x) f(y) dy - \lambda f(x) = - \lambda T_n(z, x) = g(x).$$

Observing that the adjoint transformation T^* in $(L_\Phi)^*$ corresponds by Theorem 8 with the kernel $T(y, x)$, and that $- \lambda T_n(z, x) = g(x) \in L_\Psi$ so that $g(x)$ corresponds with an element $g^* \in (L_\Phi)^*$, our last equation is seen to be equivalent with the equation $(T^* - \lambda I) f^* = g^*$ in $(L_\Phi)^*$. Hence, λ being in the resolvent set of T^*, we have $\| f^* \| \leqq \| R_\lambda^* \| . \| g^* \| \leqq \| R_\lambda \| . \| g^* \|$. By § 2, 10° this implies

$$\| f(x) \|_\Psi \leqq 2 \| f^* \| \leqq 2 \| R_\lambda \| . \| g^* \| \leqq 2 \| R_\lambda \| . \| g(x) \|_\Psi$$

or

$$\| K_\lambda(z, x) \|_\Psi \leqq 2 \| R_\lambda \| . \| \lambda T_n(z, x) \|_\Psi = 2 \| \lambda R_\lambda \| . \| T_n(z, x) \|_\Psi .$$

It follows that

$$||| K_\lambda |||_\Phi \leqq 2 \, \| \lambda R_\lambda \| \, . \, ||| T^n |||_\Phi < \infty.$$

2°. Since $\| K_\lambda \|_{\Phi\Psi} \leqq ||| K_\lambda |||_\Phi$ we find under the same extra hypothesis as in 1° that

$$\| H_\lambda \|_{\Phi\Psi} \leqq \| T \|_{\Phi\Psi} + | \lambda |^{-1} \| T^2 \|_{\Phi\Psi} + \dots$$
$$+ | \lambda |^{-(n-2)} \| T^{n-1} \|_{\Phi\Psi} + 2 | \lambda |^{-(n-1)} \| \lambda R_\lambda \| \, . \, ||| T^n |||_\Phi = F(\lambda).$$

We furthermore recall that $\| (H_\lambda)_a \| \leqq \| H_\lambda \|_{\Phi\Psi}$ by Theorem 3, where $(H_\lambda)_a$ is the transformation with kernel $| H_\lambda(x, y) |$.

3°. Suppose that $\lambda_1 \neq 0$ is an arbitrary point in the resolvent set of T. On account of the first part of the present proof we have only to show that H_{λ_1} is an integral transformation with kernel $H_1(x, y) \in L_{\Phi\Psi}$. By Theorem 10 we know already that for $| \lambda^* | > \| T_a \|$ the transformation H_{λ^*} has a kernel $H_{\lambda^*}(x, y) \in L_{\Phi\Psi}$. Since there are at most a finite number of points λ in the point spectrum of T for which $| \lambda | \geqq | \lambda_1 |$, we may join λ_1 by a straight linesegment with a point λ^* in $| \lambda^* | > \| T_a \|$ in such a way that every point on this linesegment is in the revolvent set of T. The expression $F(\lambda)$ in the second part of the present proof is continuous on this linesegment (cf. the formula for R_λ in § 3); it has therefore a finite non-negative maximum B on the segment. If λ_0 is an arbitrary point on the segment we consider the open set $| \lambda^{-1} - \lambda_0^{-1} | < B^{-1}$ in the λ-plane. Obviously we may cover the closed segment from λ^* to λ_1 by a finite number of these sets in such a way that λ^* is the centre of the first set, and that each centre is in the interior of the preceding set. Suppose now that λ_0 is the centre of one of these sets and that we know already that $H_0 = H_{\lambda_0}$ is an integral transformation with kernel $H_0(x, y) \in L_{\Phi\Psi}$ Then $\| (H_0)_a \| \leqq \| H_0 \|_{\Phi\Psi} \leqq B$ by the second part of the present proof and the definition of B. It follows now from Theorem 11 that H_λ is an integral transformation with kernel $H_\lambda(x, y) \in L_{\Phi\Psi}$ for all λ satisfying $| \lambda^{-1} - \lambda_0^{-1} | < \| (H_0)_a \|^{-1}$; hence, since $B^{-1} \leqq \| (H_0)_a \|^{-1}$, certainly for all λ satisfying $| \lambda^{-1} - \lambda_0^{-1} | < B^{-1}$, that is, for all λ in the particular set around λ_0 which we consider. Since we know that $H_{\lambda^*}(x, y) \in L_{\Phi\Psi}$, a successive application of this argument shows that H_{λ_1} is an integral transformation with kernel $H_1(x, y) \in L_{\Phi\Psi}$.

Remark. Observe the curious twist in the above proof. As we already remarked in the introduction it is probably due to our first establishing Theorem B that the proof of Theorem C may be kept free of approximation methods.

§ 6. Properties of a bounded linear transformation one of whose iterates is completely continuous.

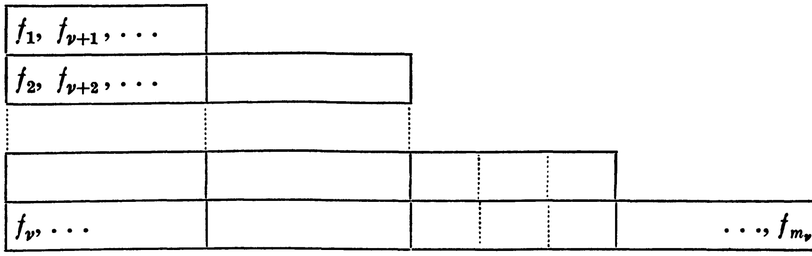
We consider an abstract complete Banach space E. By E^* we denote its adjoint space. Let T be a bounded linear transformation on E into E with the property that T^p is completely continuous for a certain integer $p \geqq 1$. It will be useful for what follows to list here the most important spectral properties of a transformation of this kind. For the proofs we refer to Riesz [16], Schauder [17], Banach [1], Ch. X. § 2 and Zaanen [18].

We observe in the first place that the adjoint transformation T^* (on E^* into E^*) has the property that $(T^*)^p = (T^p)^*$ is also completely continuous. The complex number λ is a characteristic value of T whenever there exists an element $f \neq 0$ satisfying $Tf = \lambda f$ or, equivalently, $(T - \lambda I)f = 0$. The set of all f (with $f = 0$ included) for which $(T - \lambda I)f = 0$ is a linear subspace of E, the nullspace of $T - \lambda I$, also called the characteristic space of λ. The dimension of this space is called the geometric multiplicity of λ for T. The set of all characteristic values of T is the point spectrum of T. The set of all λ such that $T - \lambda I$ has a bounded inverse $R_\lambda = (T - \lambda I)^{-1}$ with domain E is the resolvent set of T.

a) Every $\lambda \neq 0$ belongs either to the resolvent set of T (and of T^*) or to the point spectrum of T (and of T^*). The point spectrum is empty, finite or countable, and in this latter case the points of the point spectrum tend to $\lambda = 0$. Every characteristic value $\lambda \neq 0$ has the same finite geometric multiplicity m_1 for T as for T^*.

b) For any $\lambda \neq 0$ the dimensions m_n of the null spaces of $(T - \lambda I)^n$ $(n = 0, 1, 2, \ldots)$ form a non-decreasing sequence with $m_0 = 0$ and $m_1 = 0$ or $m_1 > 0$ according as λ is in the resolvent set or in the point spectrum of T. This sequence does not tend to infinity because there exists an index $\nu = \nu(\lambda)$ such that $m_n < m_{n+1}$ for $n < \nu$ whereas $m_n = m_\nu < \infty$ for $n \geqq \nu$. The dimension m_ν is called the algebraic multiplicity of λ for T. The differences $m_{n+1} - m_n$ $(n = 0, 1, 2, \ldots)$ are non-increasing. Denoting by m_n^* the corresponding dimensions for T^*, we have $m_n^* = m_n$ for all n. Every $\lambda \neq 0$ has therefore the same finite algebraic multiplicity m_ν for T as for T^*, and also the same finite index.

c) For any characteristic value $\lambda_0 \neq 0$ there exists a base $\{f_1, \ldots, f_{m_\nu}\}$ of the null space M_ν of $(T - \lambda_0 I)^\nu$ which may be arranged into the following pattern



The total number of elements in the last n rows ($n = 1, \ldots, \nu$) is m_n, and the elements in these rows form a base of the null space of $(T - \lambda_0 I)^n$. If f_i and f_{i+1} are in the same column, then $f_{i+1} = (T - \lambda_0 I) f_i$. In the same way there exists a base $\{g_1^*, \ldots, g_{m_\nu}^*\}$ of the null space M_ν^* of $(T^* - \lambda_0 I)^\nu$ which may be arranged into an exactly congruent pattern and which has the same properties relative to the null spaces of $(T^* - \lambda_0 I)^n$ ($n = 1, \ldots, \nu$). Numbering these g_i^* thus that the first column contains $g_1^*, \ldots, g_\nu^*$ from bottom to top, the second column $g_{\nu+1}^*, \ldots$ (also from bottom to top) and so on, the g_i^* may be chosen thus that $g_{i-1}^* = (T^* - \lambda_0 I) g_i^*$ for g_i^* and g_{i-1}^* in the same column, and

$$g_i^*(f_j) = \delta_{ij} \quad (i, j = 1, \ldots, m_\nu).$$

Here $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

d) Denoting for any $\lambda_0 \neq 0$ with index ν the null space of $(T - \lambda_0 I)^\nu$ by M_ν and the linear set of all $g = (T - \lambda_0 I)^\nu f$ by L_ν, this set L_ν is a linear subspace and the whole space E is the direct sum of M_ν and L_ν, that is, any $f \in E$ has a unique decomposition $f = g + h$, $g \in L_\nu$, $h \in M_\nu$. Writing $g = P_L f$, $h = P_M f$, we have $P_L + P_M = I$, $P_L^2 = P_L$, $P_M^2 = P_M$. The transformations P_L and P_M are therefore projections. Since they are closed they are bounded. P_L is called the projection on L_ν along M_ν and P_M in the projection on M_ν along L_ν. We have $g_i^*(P_L f) = 0$ for all g_i^* ($i = 1, \ldots, m_\nu$). Writing $T_M = TP_M$, $T_L = TP_L$, $T_M - \lambda_0 I$ is nilpotent on M_ν into M_ν (because $(T_M - \lambda_0 I)^n f = 0$ for all $f \in M_\nu$ provided $n \geqq \nu$), whereas $T_L - \lambda_0 I$ (on L_ν into L_ν) has a bounded inverse R_0 (note that R_0 is only defined on L_ν). For $\lambda \neq \lambda_0$ but in a sufficiently small neighbourhood of λ_0, we have

$$R_\lambda = \frac{-B_{\nu-1}}{(\lambda - \lambda_0)^\nu} + \frac{-B_{\nu-2}}{(\lambda - \lambda_0)^{\nu-1}} + \ldots + \frac{-B_0}{\lambda - \lambda_0} + R_\lambda P_L,$$

where

$$B_k = (T - \lambda_0 I)^k P_M \quad (k = 0, 1, \ldots, \nu - 1),$$
$$R_\lambda P_L = R_0 P_L + (\lambda - \lambda_0) R_0^2 P_L + (\lambda - \lambda_0)^2 R_0^3 P_L + \ldots$$

Using the above notations we prove two theorems.

THEOREM 18. *Assume that each column of the diagram in c) is completed by an infinite sequence of zeros below. Then the transformations* $B_k = (T - \lambda_0 I)^k P_M$ $(k = 0, 1, \ldots, \nu - 1)$ *satisfy*

$$B_k f = \Sigma_{i=1}^{m_\nu} g_i^*(f) h_i,$$

where h_i *is the element in the diagram which is in the same column as* f_i *but* k *rows below. In particular* $B_0 f = P_M f = \Sigma_{i=1}^{m_\nu} g_i^*(f) f_i$ *and* $B_{\nu-1} f = (T - \lambda_0 I)^{\nu-1} P_M f = \Sigma_{i=1}^{m_\nu - m_{\nu-1}} e_i^*(f) e_i$ *where* e_i *and* e_i^* *are the elements in the last rows of the* $\{f\}$*-diagram and the* $\{g^*\}$*-diagram respectively, ordered from left to right.*

PROOF. It is easily seen that $B_0 f = P_M f = \Sigma_1^{m_\nu} g_i^*(f) f_i$. Indeed, since this is true for every $f \in L_\nu$ on account of $g_i^*(P_L f) = 0$ and also for every $f = f_i$ $(i = 1, \ldots, m_\nu)$, it holds for every $f \in E$. Consider now $B_k = (T - \lambda_0 I)^k P_M$. Observing that $(T - \lambda_0 I)^k$ transforms every f_i of the diagram into the element h_i in the same column but k rows below, we obtain the desired result. Note that $m_\nu - m_{\nu-1}$ is the number of terms in the upper row of the diagram.

THEOREM 19. *Denote by* B_λ *the sum of all terms with negative powers of* $\lambda - \lambda_0$ *in the expression for* R_λ *in d) above. Define again* H_λ *by* $R_\lambda = -\lambda^{-1} I - \lambda^{-2} H_\lambda$ *and write* $S_\lambda = T - H_\lambda$. *Then, for* $\lambda \neq \lambda_0$ *but* $|\lambda - \lambda_0|$ *sufficiently small,*

$$S_\lambda = T - H_\lambda = T - H_\lambda P_L + \lambda P_M + \lambda^2 B_\lambda,$$

where $T - H_\lambda P_L$ *may be expanded in terms of non-negative powers of* $\lambda^{-1} - \lambda_0^{-1}$.

PROOF. Observing that $P_L P_M = O$ (the null transformation) we conclude from the expression for R_λ that

$$H_\lambda P_M = (-\lambda I - \lambda^2 R_\lambda) P_M = -\lambda P_M - \lambda^2 B_\lambda,$$

hence

$$S_\lambda = T - H_\lambda = T - H_\lambda P_L - H_\lambda P_M = T - H_\lambda P_L + \lambda P_M + \lambda^2 B_\lambda.$$

In order to find the expansion for $T - H_\lambda P_L$ we consider $T_L = T P_L$. We have

$$T_L - \lambda I = \begin{cases} -\lambda I \text{ on } M_\nu, \\ T - \lambda I \text{ on } L_\nu. \end{cases}$$

For $|\lambda - \lambda_0|$ sufficiently small ($\lambda = \lambda_0$ included) $T - \lambda I$ has a bounded inverse on L_ν. For $\lambda \neq \lambda_0$ this inverse is R_λ and for

$\lambda = \lambda_0$ it is the transformation R_0 introduced in d) above. Hence

$$(R_L)_\lambda = (T_L - \lambda I)^{-1} = \begin{cases} -\lambda^{-1} P_M + R_\lambda P_L, & \lambda \neq \lambda_0, \\ -\lambda^{-1} P_M + R_0 P_L, & \lambda = \lambda_0. \end{cases}$$

For $\lambda \neq \lambda_0$ this implies

$$(H_L)_\lambda = -\lambda I - \lambda^2 (R_L)_\lambda = -\lambda I + \lambda P_M - \lambda^2 R_\lambda P_L = \\ -\lambda P_L - \lambda^2 R_\lambda P_L = H_\lambda P_L.$$

Furthermore, since λ_0 is in the resolvent set of T_L, the transformation $(H_L)_\lambda$ may be expanded in terms of non-negative powers of $\lambda^{-1} - \lambda_0^{-1}$ by Theorem 9. Denoting $(H_L)_\lambda$ for $\lambda = \lambda_0$ by $(H_L)_0$, we find then

$$T - H_\lambda P_L = T - (H_L)_\lambda = \\ T - (H_L)_0 - (\lambda^{-1} - \lambda_0^{-1})(H_L)_0^2 - (\lambda^{-1} - \lambda_0^{-1})^2 (H_L)_0^3 - \dots$$

This is the desired result. We shall however write the series obtained in a slightly different form. From $H_\lambda = -\lambda R_\lambda T$ (cf. Theorem 9) we conclude

$$H_\lambda = -\lambda(-\lambda^{-1} I - \lambda^{-2} H_\lambda) T = (I + \lambda^{-1} H_\lambda) T,$$

hence with T replaced by T_L, $n \geqq 2$, $\lambda = \lambda_0$,

$$(H_L)_0^n = (H_L)_0^{n-1} \{I + \lambda_0^{-1}(H_L)_0\} T_L = \{(H_L)_0^{n-1} + \lambda_0^{-1}(H_L)_0^n\} T_L.$$

Furthermore, keeping in mind the general relation $S_\lambda = T - H_\lambda = -\lambda^{-1} H_\lambda T$, we find

$$T - (H_L)_0 = T P_M + T_L - (H_L)_0 = T P_M - \lambda_0^{-1}(H_L)_0 T_L.$$

The final result is therefore that

$$T - H_\lambda P_L = T P_M - [(\lambda^{-1} - \lambda_0^{-1})(H_L)_0 + (\lambda^{-1} - \lambda_0^{-1})^2 (H_L)_0^2 + \dots] T_L \\ - \lambda_0^{-1} [(H_L)_0 + (\lambda^{-1} - \lambda_0^{-1})(H_L)_0^2 + (\lambda^{-1} - \lambda_0^{-1})^2 (H_L)_0^3 + \dots] T_L.$$

§ 7. The proof of Theorem C.

We suppose that the Banach space E of § 6 is the space $L_\Phi(\Delta)$, where $\Phi(2u) \leqq M\Phi(u)$ for a constant M and all $u \geqq 0$. If T is an integral transformation with kernel $T(x, y)$ satisfying $|||\, T \,|||_\Phi < \infty$ the transformation T^2 is completely continuous by Theorem A, so that T has all properties mentioned in § 6. The elements $f_1, \dots, f_{m_\nu}$ in the diagram are now functions $f_1(x), \dots, f_{m_\nu}(x)$, all belonging to L_Φ, and the linear functionals $g_1^*, \dots, g_{m_\nu}^*$ such that $g_i^*(f_j) = \delta_{ij}$ are represented by functions $g_1(x), \dots, g_{m_\nu}(x)$ belonging to L_Ψ and such that $\int_\Delta g_i(x) f_j(x)\, dx = \delta_{ij}$. Theorem

18 shows that the transformations $B_k = (T - \lambda_0 I)^k P_M$ $(k = 0, 1, \ldots, \nu - 1)$ are integral transformations with kernels $B_k(x, y) = \Sigma_{i=1}^{m_\nu} g_i(y) h_i(x)$ where $h_i(x)$ is in the same (extended) column as $f_i(x)$ but k rows below. Note that $\int_\Delta B_0(x, x)\, dx = m_\nu$ and $\int_\Delta B_k(x, x)\, dx = 0$ for $k = 1, \ldots, \nu - 1$.

THEOREM 20. *Suppose that $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, and that T is completely of finite double-norm, hence $|||\, T \,|||_\Phi < \infty$, $|||\, T \,|||_\Phi^{\text{inv}} < \infty$. Then, if $\lambda_0 \neq 0$ is a characteristic value of T with algebraic multiplicity m_ν, $\lambda \neq \lambda_0$* and $|\lambda - \lambda_0|$ *sufficiently small, the transformation $S_\lambda = T - H_\lambda$ has a finite trace $\tau(S_\lambda)$ which satisfies*

$$\tau(S_\lambda) = m_\nu(\mu - \mu_0)^{-1} + \Sigma_{k=0}^{\infty} \alpha_k(\mu - \mu_0)^k,\ \mu = \lambda^{-1},\ \mu_0 = \lambda_0^{-1}.$$

If $\lambda_0 \neq 0$ is in the resolvent set of T the first term in the expansion vanishes.

PROOF. Suppose first that $\lambda_0 \neq 0$ is a characteristic value of T with index ν and algebraic multiplicity m_ν. Observing that $P_M = B_0$ is an integral transformation with kernel $B_0(x, y) = \Sigma_1^{m_\nu} f_i(x) g_i(y)$, we see that $T_L = TP_L = T - TP_M$ has the kernel $T(x, y) - \Sigma_1^{m_\nu} g_i(y)\, Tf_i(x)$, so that T_L is completely of finite double-norm. It follows now from Theorem B that $(H_L)_\lambda$ is of finite double-norm for all λ in the resolvent set of T_L, in particular for $\lambda = \lambda_0$ and all λ in a sufficiently small neighbourhood of λ_0. Hence $|||\, (H_L)_0 \,|||_\Phi < \infty$, and, by Theorem 16, $|||\, (H_L)_0^n \,|||_\Phi \leqq |||\, (H_L)_0 \,|||_\Phi^n$. It follows that for $|\mu - \mu_0| < |||\, (H_L)_0 \,|||_\Phi^{-1}$ the series $\Sigma_1^\infty (\mu - \mu_0)^k (H_L)_0^k$ and $\Sigma_0^\infty (\mu - \mu_0)^k (H_L)_0^{k+1}$ converge in double-norm. These series are exactly those between the square brackets in the expansion for $T - H_\lambda P_L$ in Theorem 19. Noting that TP_M has a finite trace, we find therefore by Theorem 17, Corollary, that

$$\tau(T - H_\lambda P_L) = \Sigma_0^\infty \alpha_k(\mu - \mu_0)^k.$$

By Theorem 19 we have $S_\lambda = \lambda^2 B_\lambda + \lambda P_M + (T - H_\lambda P_L)$, and we know already that $\lambda^2 B_\lambda + \lambda P_M$ is an integral transformation with kernel

$$\lambda^2 \left\{ \frac{-B_{\nu-1}(x, y)}{(\lambda - \lambda_0)^\nu} + \ldots + \frac{-B_0(x, y)}{\lambda - \lambda_0} \right\} + \lambda B_0(x, y),$$

hence

$$\tau(\lambda^2 B_\lambda + \lambda P_M) = -\lambda^2 m_\nu(\lambda - \lambda_0)^{-1} + \lambda m_\nu = \\ -\lambda\lambda_0(\lambda - \lambda_0)^{-1} m_\nu = m_\nu(\mu - \mu_0)^{-1}.$$

This leads to the final result that

$$\tau(S_\lambda) = m_\nu(\mu - \mu_0)^{-1} + \Sigma_0^\infty \alpha_k(\mu - \mu_0)^k.$$

If $\lambda_0 \neq 0$ is no characteristic value of T a similar but easier argument (it is not necessary to introduce T_L now) shows that in this case the term with $(\mu - \mu_0)^{-1}$ vanishes.

Theorem 21. *Under the same conditions for $\Phi(u)$ and $T(x, y)$ we have, for $|\lambda|$ sufficiently large,*

$$\tau(S_\lambda) = -(\mu\sigma_2 + \mu^2\sigma_3 + \ldots),$$

where $\sigma_n = \tau(T^n)$ $(n = 2, 3, \ldots)$ and $\mu = \lambda^{-1}$.

Proof. For small $|\mu|$ we have $H_\lambda = T + \mu T^2 + \mu^2 T^3 + \ldots$ (cf. the proof of Theorem 10). This series converges in double-norm for $|\mu| < |||T|||_\Phi^{-1}$. Hence, since $S_\lambda = -\mu H_\lambda T$,

$$\tau(S_\lambda) = -(\mu\sigma_2 + \mu^2\sigma_3 + \ldots).$$

Theorem 22. *Under the same conditions for $\Phi(u)$ and $T(x, y)$ there exists a power series $\delta(\mu) = 1 + \Sigma_1^\infty \delta_n\mu^n$, converging for all complex μ and having the property that its logarithmic derivative $\delta'(\mu)/\delta(\mu)$ satisfies*

$$\delta'(\mu)/\delta(\mu) = \tau(S_\lambda)$$

for all $\mu = \lambda^{-1}$ for which λ is in the resolvent set of T. The sum $\delta(\mu)$ of this power series, the Fredholm determinant of T, has a zero of multiplicity m_ν in $\mu_0 = \lambda_0^{-1}$ if and only if λ_0 is a characteristic value of T with algebraic multiplicity m_ν. The coefficients $\delta_n (n = 1, 2, \ldots)$ of this series satisfy $\delta_n = (-1)^n Q_n/n!$ where

$$Q_n = \det \begin{vmatrix} 0 & n-1 & 0 & - & - & 0 & 0 \\ \sigma_2 & 0 & n-2 & - & - & 0 & 0 \\ - & - & - & - & - & - & - \\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & - & - & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & - & - & \sigma_2 & 0 \end{vmatrix}$$

(*compare* Smithies [11], Theorem 4.4).

Proof. The existence of $\delta(\mu)$ and the statement concerning its zeros follow by a well-known function theoretic argument from the properties of $\tau(S_\lambda)$ proved in Theorems 20 and 21.

Write $\delta_0 = 1$. For small $|\mu|$ we have by Theorem 21

$$\delta'(\mu)/\delta(\mu) = \tau(S_\lambda) = -\Sigma_1^\infty \sigma_{n+1}\mu^n,$$

hence

$$\Sigma_0^\infty (n+1)\delta_{n+1}\mu^n = (-\Sigma_1^\infty \sigma_{n+1}\mu^n)(\Sigma_0^\infty \delta_n\mu^n)$$

or

$$\delta_1 = 0,\ (n+1)\delta_{n+1} = -\Sigma_{m=0}^{n-1} \delta_m\sigma_{n+1-m}\ (n = 1, 2, \ldots).$$

Write $d_0 = 1$, $d_n = (-1)^n Q_n/n!$ $(n = 1, 2, \ldots)$. Then it is easily seen that

$$d_1 = 0,\ (n+1)d_n = -\Sigma_{m=0}^{n-1} d_m \sigma_{n+1-m}\ (n = 1, 2, \ldots).$$

Hence $\delta_n = d_n$ $(n = 0, 1, 2, \ldots)$.

Theorem C. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, and let T be completely of finite double-norm. Then, if $\lambda \neq 0$ is not in the point spectrum of T and $\lambda = \mu^{-1}$, we have $H_\lambda(x, y) = H'_\lambda(x, y)/\delta(\mu)$ where*

$$H'_\lambda(x, y) = \Sigma_0^\infty H_n(x, y)\mu^n,$$

$$\delta(\mu) = 1 + \Sigma_1^\infty \delta_n \mu^n.$$

The coefficients δ_n and $H_n(x, y)$ are the (modified) Fredholm expressions. Both series converge for all μ, the series for $H'_\lambda(x, y)$ almost everywhere in $\Delta \times \Delta$. This series even converges in double-norm.

Proof. Let $\lambda = \mu^{-1} \neq 0$ run through the resolvent set of T, and let $\delta(\mu) = 1 + \Sigma_1^\infty \delta_n \mu^n$ be the power series introduced in Theorem 22. Consider now the transformation H'_λ defined by $H'_\lambda = \delta(\mu)H_\lambda$. If $|\mu|$ is small,

$$H'_\lambda = (\Sigma_0^\infty \delta_n \mu^n)(T + \mu T^2 + \mu^2 T^3 + \ldots) = \Sigma_0^\infty H_n \mu^n.$$

Since $\delta_0 = 1$, $\delta_1 = 0$, we find $H_0 = T$, $H_1 = T^2$. Generally, for $n \geqq 1$,

$$H_n = \delta_n T + \delta_{n-1} T^2 + \ldots + \delta_0 T^{n+1} = \delta_n T + TH_{n-1}.$$

It follows that all H_n are integral transformations with kernels $H_n(x, y)$ of finite double-norm. Furthermore, from $H_n - \delta_n T = \delta_{n-1}T^2 + \ldots + \delta_0 T^{n+1}$ for $n \geqq 1$,

$$\tau(H_n - \delta_n T) = \delta_{n-1}\sigma_2 + \ldots + \delta_0 \sigma_{n+1} = -(n+1)\delta_{n+1}.$$

This formula also holds for $n = 0$, hence for all n.

One may prove now exactly as in Smithies [11], Theorem 5.8 that the recurrence formulas

$$\begin{aligned} H_n &= \delta_n T + TH_{n-1}, \\ -n\delta_n &= \tau(H_{n-1} - \delta_{n-1}T), \end{aligned} \quad (n = 1, 2, \ldots),$$

imply

$$\delta_n = \frac{(-1)^n}{n!} \int_\Delta \ldots \int_\Delta N_n dz_1 \ldots dz_n,$$

$$H_n(x, y) = \frac{(-1)^n}{n!} \int_\Delta \ldots \int_\Delta N_n^*(x, y) dz_1 \ldots dz_n,$$

where N_n is the $n \times n$ determinant with elements $T(z_i, z_j)$

$(i, j = 1, \ldots, n)$ but with the elements $T(z_i, z_i)$ on the main diagonal replaced by zeros, and where $N_n^*(x, y)$ is the determinant

$$\begin{vmatrix} T(x, y) & T(x, z_1) \ldots\ldots T(x, z_n) \\ T(z_1, y) & \cdots\cdots\cdots\cdots \\ \vdots & \\ T(z_n, y) & N_n \end{vmatrix}.$$

It remains only to prove the statements concerning the convergence of $\Sigma_0^\infty H_n \mu^n$. Evidently all partial sums are of finite double-norm. Let $\lambda_0 = \mu_0^{-1} \neq 0$ be a characteristic value of T with index ν and algebraic multiplicity m_ν. Then it follows easily from what we have proved in Theorem 19 that H_λ has for small $|\lambda - \lambda_0|$ an expansion in terms of powers of $\mu - \mu_0$ with exponents $\geqq -\nu$, the expansion converging in double-norm. Hence, since $\delta(\mu)$ has a zero of multiplicity $m_\nu \geqq \nu$ in $\mu = \mu_0$, the transformation $H'_\lambda = \delta(\mu) H_\lambda$ has an expansion in terms of non-negative powers of $\mu - \mu_0$. The same is trivially true whenever $\lambda_0 \neq 0$ is in the resolvent set of T. These facts imply that the radius of convergence of $\Sigma H_n \mu^n$ (which represents H'_λ for small $|\mu|$) is infinite, since the theorem that the sum of a power series with a finite radius of convergence has a singularity on the circle of convergence remains true in the case that the coefficients are elements of a Banach space (here the space D_Φ of all kernels of finite double-norm). The proof of this theorem as it is reproduced e.g. in Hurwitz-Courant [19], Part 1, Ch. 3, § 5, may be taken over practically without modifications. Hence $H'_\lambda = \Sigma_0^\infty H_n \mu^n$ in double-norm for all μ. The proof that $H'_\lambda(x, y) = \Sigma_0^\infty H_n(x, y) \mu^n$ pointwise almost everywhere in $\Delta \times \Delta$ is similar to that in Theorem 10.

Remark. It hardly needs observing that for any $\lambda = \mu^{-1}$ in the resolvent set of T the equation $f - \mu T f = g$, $g \in L_\Phi$, has the solution.

$$f(x) = g(x) + \mu H_\lambda g(x) = g(x) + \frac{\mu}{\delta(\mu)} \Sigma_0^\infty \mu^n \int_\Delta H_n(x, y) g(y) dy,$$

where this series converges according to the L_Φ-norm, and also pointwise almost everywhere in Δ.

The proof of Theorem C rests essentially upon the existence and convergence everywhere of the power series $\delta(\mu) = \Sigma_0^\infty \delta_n \mu^n$. In order to establish these properties of $\delta(\mu)$ we had to use an argument derived from the theory of complex functions. Smithies, in the L_2-case (cf. [11]), could avoid this argument because he could find bounds for the coefficients δ_n by using Hadamard's determinant inequality. It would be interesting to know whether Smithies' method may be extended to the L_Φ-case or at least to the L_p-case $(1 \leqq p < \infty)$.

§ 8. Uniform convergence of the expansions.

Under somewhat stronger hypotheses we may prove that the expansions for $H_\lambda(x, y)$ (cf. Theorem 10 and 11, Remark) and $H'_\lambda(x, y)$ (cf. Theorem C) converge uniformly. For this purpose we define:

Class B. *The kernel* $T(x, y) \epsilon L_{\Phi\Psi}$ *belongs to the class* B *whenever there exists a constant* c *such that, for all* $x \epsilon \Delta$ *and all* $y \epsilon \Delta$,

$$t(x) = \| T(x, y)\|_\Psi \leqq c, s(y) = \| T(x, y) \|_\Phi \leqq c.$$

Class Cm. *In case the interval* Δ *is bounded and closed, the measurable kernel* $T(x, y)$ *belongs to the class* Cm *whenever, for all* $x_1, x_2 \epsilon \Delta$ *and all* $y_1, y_2 \epsilon \Delta$,

$$\lim_{x_2 \to x_1} \| T(x_2, y) - T(x_1, y) \|_\Psi = 0,$$
$$\lim_{y_2 \to y_1} \| T(x, y_2) - T(x, y_1) \|_\Phi = 0.$$

Whenever $T(x, y) \epsilon Cm$ *we shall also say that* $T(x, y)$ *is continuous in mean (relative to* L_Φ*).*

Obviously, if Δ is bounded and $T(x, y) \epsilon B$, then $||| T |||_\Phi < \infty$ and $||| T |||_\Phi^{\text{inv}} < \infty$. Furthermore, if $T(x, y) \epsilon Cm$, the mean continuity is uniform on Δ since Δ is supposed to be bounded and closed in this case. It follows that $t(x) = \| T(x, y) \|_\Psi$ and $s(y) = \| T(x, y) \|_\Phi$ are continuous on Δ, and therefore bounded. Then $T(x, y) \epsilon B$ so that, by what we already observed, $||| T |||_\Phi < \infty$ and $||| T |||_\Phi^{\text{inv}} < \infty$.

Theorem 23. *If* T *has the kernel* $T(x, y) \epsilon B$ *and* $| \lambda | > \| T_a \|$ *(so that* λ *belongs to the resolvent set of* T *on account of* $\| T_a \| \geqq \| T \|$*), then the Neumann series*

$$T(x, y) + \lambda^{-1}T_2(x, y) + \lambda^{-2}T_3(x, y) + \dots$$

converges uniformly in $\Delta \times \Delta$ *to* $H_\lambda(x, y)$. *We have* $H_\lambda(x, y) \epsilon B$, *and* $S_\lambda(x, y) = T(x, y) - H_\lambda(x, y)$ *is bounded on* $\Delta \times \Delta$.

If $T(x, y) \epsilon Cm$, *then* $H_\lambda(x, y) \epsilon Cm$ *and all iterated kernels* $T_p(x, y)$ $(p = 2, 3, \dots)$ *are continuous on* $\Delta \times \Delta$, *so that* $S_\lambda(x, y)$ *is continuous as well.*

Proof. Since $T_2(x, y)$ may be defined now by

$$T_2(x, y) = \int_\Delta T(z, y)T(x, z)dz$$

for all $(x, y) \in \Delta \times \Delta$ (not, as before, only for almost all $(x, y) \in \Delta \times \Delta$), we have, as in the proof of Theorem 16,

$$t_2(x) = \| T_2(x, y) \|_\Psi \leqq \| T \|_{\Phi\Psi} \| T(x, y) \|_\Psi \leqq c \| T \|_{\Phi\Psi}.$$

Generally

$$t_n(x) = \| T_n(x, y) \|_\Psi \leqq c \| T^{n-1} \|_{\Phi\Psi} \quad (n \geqq 2).$$

In the same way

$$s_n(y) = \| T_n(x, y) \|_\Phi \leqq c \| T^{n-1} \|_{\Phi\Psi} \quad (n \geqq 2).$$

It follows that

$$| T_2(x, y) | \leqq t(x)s(y) \leqq c^2,$$

$$| T_n(x, y) | = \left| \int_\Delta T_{n-1}(x, z)T(z, y)dz \right| \leqq t_{n-1}(x)s(y) \leqq c^2 \| T^{n-2} \|_{\Phi\Psi} \quad (n > 2).$$

The uniform convergence of the Neumann series for $| \lambda | > \| T_a \|$ is proved now by a similar argument as in Theorem 10. In the same way it is seen that $\Sigma_0^\infty | \lambda |^{-n} \| T_{n+1}(x, y) \|_\Psi$ and $\Sigma_0^\infty | \lambda |^{-n} \| T_{n+1} (x, y) \|_\Phi$ converge uniformly in Δ. Hence $H_\lambda(x, y) \in B$ and $S_\lambda(x, y)$ bounded.

If $T(x, y) \in Cm$, then $T_2(x, y)$ is a continuous function of x uniformly in y and a continuous function of y uniformly in x. This shows that $T_2(x, y)$ is continuous in $\Delta \times \Delta$. The same holds for $T_p(x, y)$ $(p = 3, 4, \ldots)$. It follows that $S_\lambda(x, y)$ is continuous for $| \lambda | > \| T_a \|$, so that $H_\lambda(x, y) \in Cm$.

Remark. If λ_0 is in the resolvent set of $T(x, y) \in B$, and we know that $H_0(x, y) = H_{\lambda_0}(x, y) \in B$ and that $T(x, y) - H_0(x, y)$ is bounded, then it may be proved similarly that for $| \lambda^{-1} - \lambda_0^{-1} | < \| (H_0)_a \|^{-1}$ the series

$$H_\lambda(x, y) = H_0(x, y) + (\lambda^{-1} - \lambda_0^{-1}) H_{0,2}(x, y) + (\lambda^{-1} - \lambda_0^{-1})^2 H_{0,3}(x, y) + \ldots$$

converges uniformly in $\Delta \times \Delta$, and that all $H_{0,p}(x, y)$ $(p = 2, 3, \ldots)$ are bounded. Hence $H_\lambda(x, y) \in B$ and $T(x, y) - H_\lambda(x, y)$ bounded for these values of λ.

In the same way, if $T(x, y) \in Cm$, $H_0(x, y) \in Cm$ and $T(x, y) - H_0(x, y)$ continuous, then $H_\lambda(x, y) \in Cm$ and $T(x, y) - H_\lambda(x, y)$ continuous for $| \lambda^{-1} - \lambda_0^{-1} | < \| (H_0)_a \|^{-1}$.

Theorem 24. *If $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, $T(x, y) \in B$ and*

$|||\, T\, |||_\Phi < \infty$, then $H_\lambda(x, y) \in B$ and $T(x, y) - H_\lambda(x, y)$ bounded for all λ in the resolvent set of T. If $T(x, y) \in Cm$, then $H_\lambda(x, y) \in Cm$ and $T(x, y) - H_\lambda(x, y)$ continuous for all λ in the resolvent set of T.

PROOF. Follows from the preceding theorem by observing that, starting from a point λ^* in $|\, \lambda^* \,| > \|\, T_a \,\|$, any λ_1 in the resolvent set of T may be reached in a finite number of steps of the kind described in the remark above (cf. the proof of Theorem B).

THEOREM 25. *Let $\Phi(2u) \leqq M\Phi(u)$ for all $u \geqq 0$, and let $T(x, y)$ be completely of finite double-norm. Let moreover $T(x, y) \in B$. Then, if $\lambda \neq 0$ is in the resolvent set of T and $\lambda = \mu^{-1}$, we have, with the notations of Theorem C,*

$$H_\lambda(x, y) = \{\delta(\mu)\}^{-1} \Sigma_0^\infty H_n(x, y)\mu^n$$

uniformly in $\Delta \times \Delta$. Furthermore $H_n(x, y) - \delta_n T(x, y)$ is bounded for $n = 0, 1, 2, \ldots$ If $T(x, y) \in Cm$, then $H_n(x, y) - \delta_n T(x, y)$ is continuous for $n = 0, 1, 2, \ldots$

PROOF. In view of $H_0 = T$, $H_n = \delta_n T + \delta_{n-1} T^2 + \ldots + \delta_0 T^{n+1} = \delta_n T + H_{n-1} T (n \geqq 1)$ we find

$$\begin{aligned} h_n(x) = \|\, H_n(x, y) \,\|_\Psi &\leqq |\, \delta_n \,| \,.\, \|\, T(x, y) \,\|_\Psi + \|\, H_{n-1} \,\|_{\Phi\Psi} \|\, T(x, y) \,\|_\Psi \\ &\leqq (|\, \delta_n \,| + |||\, H_{n-1} \,|||_\Phi) c \quad (n \geqq 1), \end{aligned}$$

hence

$$\begin{aligned} |\, H_0(x, y) - \delta_0 T(x, y) \,| &= 0, \\ |\, H_1(x, y) - \delta_1 T(x, y) \,| &= |\, T_2(x, y) \,| \leqq c^2, \\ |\, H_{n+1}(x, y) - \delta_{n+1} T(x, y) \,| &\leqq \|\, H_n(x, y) \,\|_\Psi \,\|\, T(x, y) \,\|_\Phi \\ &\leqq (|\, \delta_n \,| + |||\, H_{n-1} \,|||_\Phi)\, c^2 \quad (n \geqq 1). \end{aligned}$$

Since $\Sigma\, |\, \delta_n \mu^n \,| < \infty$ and $\Sigma\, |||\, H_n \,|||_\Phi \,|\, \mu^n \,| < \infty$, it follows that $\Sigma_0^\infty\, |\, H_n(x, y) - \delta_n T(x, y) \,| \,.\, |\, \mu \,|^n$ converges uniformly in $\Delta \times \Delta$. The same holds then for

$$H_\lambda(x, y) - T(x, y) = \{\delta(\mu)\}^{-1}\, \Sigma_0^\infty\, [H_n(x, y) - \delta_n T(x, y)]\, \mu^n.$$

The remaining statements are now evident.

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Note (added 24-1-52). The results in this paper are connected with those of Dr A. F. Ruston in his recent paper „On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space", Proc. London math. Soc., II Ser. 53 (1951), 109–124. Dr Ruston shows that for transformations in the trace class the classical methods (Hadamard's inequality) yield the analogs of the classical results.

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