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A. ERDÉLYI

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Transformation of a certain series of products of confluent hypergeometric functions. Applications to Laguerre and Charlier polynomials

by

A. Erdélyi

Edinburgh

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1. In this note I propose to discuss some transformations of a certain series of products of Kummer's confluent hypergeometric functions ${}_1F_1$.

The generalized Laguerre polynomials being expressible in terms of Kummer's series ${}_1F_1$, a corresponding transformation of a certain series in products of Laguerre polynomials is readily found. This transformation turns out to convert an *infinite* series of products of Laguerre polynomials into a *finite* one, thus expressing the sum of the infinite series in finite terms. The same holds for the transformation of the corresponding series of products of a generalized Laguerre polynomial with a general ${}_1F_1$.

The Charlier polynomials occur, connected with Poisson's frequency function, in Mathematical Statistics of seldom events. They are expressible in terms of particular cases of ${}_1F_1$, and also expressible in terms of generalized Laguerre polynomials. It seems that this connection has been overlooked till now. The application of the general transformation formula to Charlier polynomials yields, among other results, the expansion of Wicksell which was studied also in a recent paper of Meixner.

2. For the sake of brevity we put

$$\Phi(a, c, x) = \frac{\Gamma(a)}{\Gamma(c)}\, {}_1F_1(a; c; x) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(c+m)} \frac{x^m}{m!}. \tag{1}$$

We shall use the integral representation

$$\Gamma(c-a)\Phi(a, c, x) = \int_0^1 e^{ux} u^{a-1} (1-u)^{c-a-1} du, \tag{2}$$

valid provided that

$$\Re(c) > \Re(a) > 0.$$

This integral representation is easily checked expanding exp (ux) on the right of (2) and integrating term by term.

Kummer's transformation [1]) of ${}_1F_1$ in our notations runs:

$$(3) \qquad \Gamma(c-a)\Phi(a, c, x) = e^x\Gamma(a)\Phi(c-a, c, -x).$$

It can be derived from (2), replacing u by $1 - u$ on the right of this equation.

For large positive values of r the asymptotic equations

$$(4) \qquad \Phi(a, c+r, x) = \frac{\Gamma(a)}{\Gamma(c+r)}\{1+O(r^{-1})\}$$

and

$$(5) \qquad \Phi(a+r, c+r, x) = \frac{\Gamma(a+r)}{\Gamma(c+r)}e^x\{1+O(r^{-1})\}$$

hold for all fixed finite values of x, real or complex.

In that what follows, a and $c - a$ are supposed not to be zero or a negative integer. In section 4 these restrictions are removed.

3. Now we can begin with the transformation of the infinite series

$$(6) \qquad S \equiv \sum_{r=0}^{\infty} \Phi(a+r, c+r, x)\Phi(b+r, d+r, y)\frac{z^r}{r!}.$$

From (5) it is seen that (6) is equiconvergent with ${}_2F_2(a, b; c, d; z)$, and therefore convergent for all finite values of x, y, z and the parameters, with the only restriction that neither a nor b is equal to 0 or to a negative integer. We shall transform (6) with the preliminary restrictions

$$(7) \qquad \Re(c) > \Re(a) > 0, \ \Re(d) > \Re(b) > 0.$$

Later on these restrictions can be removed.

With the restrictions (7) the integral representations (2) for the Φ-functions in S are valid with any non-negative integer value of r, and hence

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1) See e.g. E. T. Whittaker & G. N. Watson, Modern Analysis [1927], § 16.11 (I).

$$S=\sum_{r=0}^{\infty}\frac{1}{\Gamma(c-a)}\int_0^1 e^{ux}u^{a+r-1}(1-u)^{c-a-1}du\,\frac{1}{\Gamma(d-b)}\int_0^1 e^{vy}v^{b+r-1}(1-v)^{d-b-1}dv\,\frac{z^r}{r!}$$

$$=\frac{1}{\Gamma(c-a)\Gamma(d-b)}\int_0^1\int_0^1 e^{ux+vy}u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{d-b-1}\left\{\sum_{r=0}^{\infty}\frac{(uvz)^r}{r!}\right\}du\,dv$$

$$=\frac{1}{\Gamma(c-a)\Gamma(d-b)}\int_0^1\int_0^1 e^{ux+yv+uvz}\,u^{a-1}\,v^{b-1}\,(1-u)^{c-a-1}\,(1-v)^{d-b-1}\,du\,dv.$$

Summation under the signs of integration is permissible by reason of the absolute and uniform convergence of the infinite series in the $\{\ldots\}$, in the domain of integration.

Now,

$$e^{ux+vy+uvz}$$
$$=e^{-z}\cdot e^{u(x+z)}\cdot e^{v(y+z)}\cdot e^{(1-u)(1-v)z}$$
$$=e^{-z}\cdot e^{u(x+z)}\cdot e^{v(y+z)}\cdot\sum_{r=0}^{\infty}\frac{\{(1-u)(1-v)z\}^r}{r!},$$

and therefore, term by term integration being permissible by the same reason as before,

$$S=e^{-z}\sum_{r=0}^{\infty}\frac{1}{\Gamma(c-a)}\int_0^1 e^{u(x+z)}\,u^{a-1}\,(1-u)^{c+r-a-1}\,du$$

(8) $$\times\frac{1}{\Gamma(d-b)}\int_0^1 e^{v(y+z)}\,v^{b-1}\,(1-v)^{d+r-b-1}\,dv\,\frac{z^r}{r!}$$

$$=e^{-z}\sum_{r=0}^{\infty}(c-a)_r(d-b)_r\Phi(a,\,c+r,\,x+z)\Phi(b,\,d+r,\,y+z)\,\frac{z^r}{r!},$$

according to (2). In (8) the usual notation

$$(c)_0=1,\quad (c)_r=c(c+1)\cdots(c+r-1)=\frac{\Gamma(c+r)}{\Gamma(c)}\qquad (r=1,\,2,\,\ldots)$$

is used.

Comparing (8) with (6), the transformation formula

(9) $$\sum_{r=0}^{\infty}\Phi\,(a+r,\,c+r,\,x)\Phi(b+r,\,d+r,\,y)\,\frac{z^r}{r!}$$
$$=e^{-z}\sum_{r=0}^{\infty}(c-a)_r(d-b)_r\Phi(a,\,c+r,\,x+z)\Phi(b,\,d+r,\,y+z)\,\frac{z^r}{r!}$$

at once follows. The series on the right of this relation is, it is seen from (4), absolutely convergent for all finite values of x, y, z and the parameters, save non-positive integer values of a and b.

With a slight modification of the notation we write instead of (9)

$$\sum_{r=0}^{\infty} (c-a)_r(d-b)_r \Phi(a, c+r, x)\, \Phi(b, d+r, y) \frac{z^r}{r!} = e^z \sum_{r=0}^{\infty} \Phi(a+r, c+r, x-z)\Phi(b+r, d+r, y-z) \frac{z^r}{r!}. \tag{10}$$

So far (10) has only been proved with the restrictions (7). Now, both infinite series in (10) being absolutely convergent, (10) is valid, by the theory of analytic continuation, with the only restriction that neither a nor b is equal to a non-positive integer.

4. Some other forms of (10) may be written out. Transforming the left of (10) according to Kummer's transformation (3), we obtain

$$\Gamma(a)\Gamma(b) \sum_{r=0}^{\infty} \Phi(c-a+r, c+r, -x)\Phi(d-b+r, d+r, -y) \frac{z^r}{r!} = \Gamma(c-a)\Gamma(d-b)e^{z-x-y} \sum_{r=0}^{\infty} \Phi(a+r,c+r,x-z)\Phi(b+r,d+r,y-z) \frac{z^r}{r!}. \tag{11}$$

Using Kummer's transformation on the right of (10) we get

$$\sum_{r=0}^{\infty} \Gamma(c-a+r)\Gamma(d-b+r)\Phi(a, c+r, x)\Phi(b, d+r, y) \frac{z^r}{r!} = e^{x+y-z} \sum_{r=0}^{\infty} \Gamma(a+r)\Gamma(b+r)\Phi(c-a,c+r,z-x)\Phi(d-b,d+r,z-y)\frac{z^r}{r!}. \tag{12}$$

In both the equations (11) and (12) the parameters are only subject to the conditions expressed on the end of section 2.

Using Kummer's transformation on both sides of (10), there results a transformation formula which is, save for notation, identical with (10).

Using Kummer's transformation only for the first Φ-functions on both sides of (10), we arrive at

$$\sum_{r=0}^{\infty} (d-b)_r\Phi(c-a+r, c+r, -x)\Phi(b, d+r, y)\frac{z^r}{r!} = \sum_{r=0}^{\infty} (a)_r\Phi(c-a, c+r, z-x)\Phi(b+r, d+r, y-z) \frac{z^r}{r!}.$$

With a slight change of notation we get the symmetrical form of (10), viz.

$$(13)\quad \sum_{r=0}^{\infty} (d-b)_r \Phi(a+r,\, c+r,\, x-z)\Phi(b,\, d+r,\, y)\,\frac{z^r}{r!}$$
$$= \sum_{r=0}^{\infty} (c-a)_r \Phi(a,\, c+r,\, x)\Phi(b+r,\, d+r,\, y-z)\,\frac{z^r}{r!}.$$

Some other forms of (10), evidently obtainable by similar transformations, are left to the reader.

The restriction, that none of the quantities a, b, $c-a$, $c-b$ is allowed to be equal to zero or to a negative integer, can easily be removed expressing the Φ-functions, by means of (1), in terms of ${}_1F_1$ and then dividing the transformation formula by a suitable product of Gamma-functions. Doing so with (10) [in this case we must divide the whole equation by $\Gamma(a)\Gamma(b)$] we obtain

$$(14)\quad \sum_{r=0}^{\infty} \frac{(c-a)_r(d-b)_r}{\Gamma(c+r)\Gamma(d+r)}\, {}_1F_1(a;\, c+r;\, x)\, {}_1F_1(b;\, d+r;\, y)\,\frac{z^r}{r!}$$
$$= e^z \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{\Gamma(c+r)\Gamma(d+r)}\, {}_1F_1(a+r;\, c+r;\, x-z)\, {}_1F_1(b+r;\, d+r;\, y-z)\,\frac{z^r}{r!}.$$

In this form our transformation formula is valid without any restrictions.

A similar operation on (13) yields

$$(15)\quad \sum_{r=0}^{\infty} \frac{(a)_r(d-b)_r}{\Gamma(c+r)\Gamma(d+r)}\, {}_1F_1(a+r;\, c+r;\, x-z)\, {}_1F_1(b;\, d+r;\, y)\,\frac{z^r}{r!}$$
$$= \sum_{r=0}^{\infty} \frac{(c-a)_r(b)_r}{\Gamma(c+r)\Gamma(d+r)}\, {}_1F_1(a;\, c+r;\, x)\, {}_1F_1(b+r;\, d+r;\, y-z)\,\frac{z^r}{r!}.$$

5. In this section another transformation of the same series may be obtained. For the sake of simplicity let us assume again that the preliminary restrictions (7) are fulfilled. Being so, the integral representation

$$S = \frac{1}{\Gamma(c-a)\Gamma(d-b)} \int_0^1\!\!\int_0^1 e^{ux+vy+uvz}\, u^{a-1} v^{b-1} (1-u)^{c-a-1} (1-v)^{d-b-1}\, du\, dv,$$

obtained in section 3, is valid. In this integral representation we put, supposing $z \neq 0$,

$$e^{ux+vy+uvz} = e^{-\frac{xy}{z} + z\left(u+\frac{y}{z}\right)\left(v+\frac{x}{z}\right)}$$
$$= e^{-\frac{xy}{z}} \sum_{r=0}^{\infty} \frac{1}{r!} \left\{\left(u+\frac{y}{z}\right)\left(v+\frac{x}{z}\right) z\right\}^r.$$

and integrate term by term. This process is permissible by reason of the absolute and uniform convergence of this series in the domain of integration. Doing so, we obtain

$$=e^{-\frac{xy}{z}}\sum_{r=0}^{\infty}\frac{1}{\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}\left(u+\frac{y}{z}\right)^r du\,\frac{1}{\Gamma(d-b)}\int_0^1 v^{b-1}(1-v)^{d-b-1}\left(v+\frac{x}{z}\right)^r dv$$

The integrals occuring in this equation represent ordinary hypergeometric polynomials, being [2])

$$\frac{1}{\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}\left(u+\frac{y}{z}\right)^r du=\frac{\Gamma(a)}{\Gamma(c)}\left(\frac{y}{z}\right)^r {}_2F_1\left(-r,a;c;-\frac{z}{y}\right).$$

Hence we obtain

$$(16)\quad \begin{aligned}\frac{S}{\Gamma(a)\Gamma(b)}&=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{\Gamma(c+r)\Gamma(d+r)}\,{}_1F_1(a+r;\;c+r;\;x)\,{}_1F_1(b+r;\;d+r;\;y)\,\frac{z^r}{r!}\\ &=e^{-\frac{xy}{z}}\sum_{r=0}^{\infty}\frac{\left(\frac{xy}{z}\right)^r}{\Gamma(c)\,\Gamma(d)r!}\,{}_2F_1\left(-r,\;a;\;c;\;-\frac{z}{y}\right){}_2F_1\left(-r,\;b;\;d;\;-\frac{z}{x}\right).\end{aligned}$$

This result is true, with the sole restriction $z \neq 0$, for all values of the variables and parameters.

It seems, perhaps, that $x = 0$ and $y = 0$ should also be excluded. This is by no means true, because

$$\lim_{x\to 0}\left\{x^r\,{}_2F_1\left(-r,\;b;\;d;\;-\frac{z}{x}\right)\right\}=\frac{(b)_r}{(d)_r}z^r$$

and the similar limit for $y \to 0$ yield finite values. For c or d equal to 0 or to a negative integer the hypergeometric series have no significance at all, but the limits

$$\lim_{c\to -m}\left\{\frac{1}{\Gamma(c+r)}\,{}_1F_1(a+r;\;c+r;\;x)\right\},\;\lim_{c\to -m}\left\{\frac{1}{\Gamma(c)}\,{}_2F_1\left(-r,a;c;-\frac{z}{y}\right)\right\}\text{ etc.}$$

$$(m=0,\,1,\,2,\,\ldots)$$

exist in each case.

Using Kummer's transformation on the left hand side, and some transformation formulae of Jacobi's polynomials on the right, other forms of (16) are obtainable.

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[2]) Modern Analysis, § 14.6 Example 1.

6. Some particular cases of the transformation formulae obtained hitherto are of some importance. Merely to take a few examples let us put $x = y = z$ in (14). There results

$$(17)\quad \sum_{r=0}^{\infty} \frac{(c-a)_r(d-b)_r}{(c)_r(d)_r r!}\, {}_1F_1(a;\, c+r;\, z)\, {}_1F_1(b;\, d+r;\, z) z^r = e^z {}_2F_2(a, b;\, c, d;\, z).$$

Thus we have expressed the sum of the infinite series of products of confluent hypergeometric functions in terms of a generalised hypergeometric series.

The assumption $x = y = z$ in (15) yields

$$(18)\quad \sum_{r=0}^{\infty} \frac{(a)_r(d-b)_r}{\Gamma(c+r)\Gamma(d+r)}\, {}_1F_1(b;\, d+r;\, z) \frac{z^r}{r!} = \sum_{r=0}^{\infty} \frac{(b)_r(c-a)_r}{\Gamma(c+r)\Gamma(d+r)}\, {}_1F_1(a;\, c+r;\, z) \frac{z^r}{r!}.$$

In (16) we put $x = y = -z$ and remark that

$$ {}_2F_1(-r, a;\, c;\, 1) = \frac{(c-a)_r}{(c)_r},$$

thus obtaining

$$\sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r(d)_r}\, {}_1F_1(a+r;\, c+r;\, -z)\, {}_1F_1(b+r;\, d+r;\, -z) \frac{z^r}{r!} = e^{-z} {}_2F_2(c-a, d-b;\, c, d;\, z),$$

a relation which is equivalent to (17).

7. Now we can proceed to applications of the results, obtained in the previous sections, to transformations of certain infinite series in products of Laguerre polynomials. m and n may be non negative integers throughout.

The generalized Laguerre polynomial

$$(19)\quad L_m^{(\alpha)}(x) = \sum_{r=0}^{m} \binom{m+\alpha}{m-r} \frac{(-x)^r}{r!}$$

is usually defined by either of the two generating functions:

$$(20)\quad (1-u)^{-\alpha-1} e^{-\frac{ux}{1-u}} = \sum_{m=0}^{\infty} L_m^{(\alpha)}(x) u^m \qquad (|u| < 1),$$

$$(21)\quad (vx)^{-\frac{1}{2}\alpha} e^v J_\alpha(2\sqrt{vx}) = \sum_{m=0}^{\infty} \frac{L_m^{(\alpha)}(x) v^m}{\Gamma(\alpha+m+1)}.$$

There is, however, a third generating function

$$(22)\quad (1+w)^{\alpha} e^{-xw} = \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x) w^n \qquad (|w| < 1),$$

which deserves, in consideration of its simplicity, to be noticed.

Although I can not remember to have ever seen (22) explicitly, this generating function can hardly be unknown, for it is closely connected with the well-known expression

$$(23) \qquad L_m^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{m!} \frac{d^m(e^{-x} x^{\alpha+m})}{dx^m}$$

for Laguerre polynomials.

Finally the connection between Laguerre polynomials and confluent hypergeometric functions must be mentioned. For the purposes of the present article it is the best to write this connection in the form

$$(24) \qquad {}_1F_1(-m; \alpha+1; x) = \frac{m!\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} L_m^{(\alpha)}(x).$$

This formula can easily be checked replacing Laguerre's polynomial by its explicite form (19).

8. Particular values of the parameters in the general transformation formulae of sections 3—6 yield by means of (24) a great number of transformations of finite and infinite series of products of Laguerre polynomials. A few examples of such formulae may be written out.

We begin putting $a = -m$, $b = -n$, $c = \alpha + 1$, $d = \beta + 1$ in (14). In consequence of these substitutions, the series on the right of (14) becomes a *terminating* one, having only min $(m, n)+1$ terms. [min (m, n) denotes the smallest of the non-negative integers m, n.] After some algebra we get

$$(25) \qquad \sum_{r=0}^{\infty} L_m^{(\alpha+r)}(x) L_n^{(\beta+r)}(y) \frac{z^r}{r!} = e^z \sum_{r=0}^{\min(m,n)} L_{m-r}^{(\alpha+r)}(x-z) L_{n-r}^{(\beta+r)}(y-z) \frac{z^r}{r!},$$

thus expressing the sum of the infinite series on the left in *finite* terms.

We get another example with the same values of the parameters in (15). In this case *both* the series terminate, yielding

$$(26) \qquad \sum_{r=0}^{m} L_{m-r}^{(\alpha+r)}(x-z) L_n^{(\beta+r)}(y) \frac{(-z)^r}{r!} = \sum_{r=0}^{n} L_m^{(\alpha+r)}(x) L_{n-r}^{(\beta+r)}(y-z) \frac{(-z)^r}{r!}.$$

This is an interesting equality which transforms a series of m

terms into a series of n terms. Putting $n = 0$ in this formula we obtain

$$(27) \qquad \sum_{r=0}^{m} L_{m-r}^{(\alpha+r)}(x-z)\frac{(-z)^r}{r!} = L_m^{(\alpha)}(x).$$

The same values of a, b, c, d substituted into (17) yield

$$(28) \qquad \begin{aligned} &\sum_{r=0}^{\infty} L_m^{(\alpha+r)}(z)\, L_n^{(\beta+r)}(z)\frac{z^r}{r!} \\ &\quad = \frac{\Gamma(\alpha+m+1)\,\Gamma(\beta+n+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,m!\,n!}\, e^z\, {}_2F_2(-m, -n; \alpha+1, \beta+1; z). \end{aligned}$$

The hypergeometric series on the right of this equation is a terminating one.

Mixed series, i. e. series of products of a confluent hypergeometric function and a Laguerre polynomial are also obtainable from our results. Let us put, for instance, $b = -n$, $d = \beta + 1$ in (14), thus getting

$$(29) \qquad \begin{aligned} &\sum_{r=0}^{\infty} (c-a)_r\, \Phi(a, c+r, x)\, L_n^{(\beta+r)}(y)\frac{z^r}{r!} \\ &\quad = e^z \sum_{r=0}^{n} \Phi(a+r, c+r, x-z)\, L_{n-r}^{(\beta+r)}(y-z)\frac{(-z)^r}{r!}. \end{aligned}$$

Some similar formulae, obtainable in the same manner, are left to the reader.

9. In Mathematical Statistics of seldom events Poisson's frequency function [3])

$$(30) \qquad \psi(x; a) \equiv \psi_0(x; a) = \frac{a^x e^{-a}}{x!}$$

very often occurs. Frequency distribution functions having not exactly the shape of $\psi(x; a)$ are, according to Tchébycheff [4]) and Charlier [5]), to be represented [6]) in terms of $\psi(x; a)$ and its finite differences with respect to the non-negative integer x. Thus functions $\psi_n(x; a)$ are defined by the equation

$$(31) \qquad \psi_n(x; a) = (-\Delta)^n \psi(x-n; a) \qquad [\Delta f(x) \equiv f(x+1) - f(x)].$$

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[3]) Poisson, Recherches sur la probalité des jugements [1837]. L. v. Bortkievicz, Das Gesetz der großen Zahlen [1898].

[4]) Tchébycheff [Bull. de l' Acad. Saint-Pétersbourg 1859].

[5]) C. V. Charlier [Arkiv för Mat. 2 (1905/6), Nos 8 and 20].

[6]) See also H. Pollaczek-Geiringer [Zeitschr. für angew. Math. und Mech. 8 (1926), 292—309].

Jordan [7]) has pointed out that the same set of functions can be obtained differentiating $\psi_0(x; a)$ with respect to a, (31) being equivalent to

$$\psi_n(x; a) = D^n \psi(x; a) \qquad \left[D \equiv \frac{d}{da}\right]. \tag{32}$$

A computation of $\psi_n(x; a)$ by means of (31) of (32) shows that $\frac{\psi_n}{\psi}$ is a polynomial of degree n or x, whichever is the less, in $\frac{1}{a}$. For this reason it is usual to put

$$\psi_n(x; a) = \psi(x; a) p_n(x; a). \tag{33}$$

Here

$$p_n(x; a) = \sum_{r=0}^{\min(n,x)} (-)^{n-r} \binom{n}{r} \binom{x}{r} r!\, a^{-r} = (-)^{x-n} p_x(n; a) \tag{34}$$

is called Charlier's polynomial.

Now, at the first glance it is seen from (34) that Charlier's polynomial is expressible in terms of the hypergeometric series ${}_2F_0$, that is to say, in terms of Whittaker's confluent hypergeometric function $W_{k,m}(z)$. On the other hand we know [8]) that all solutions of Whittaker's differential equation expressible in finite terms must be connected with Laguerre polynomials. Hence *Charlier polynomials must be connected with Laguerre polynomials*. Though this connection is a very simple one and therefore unlikely to be perfectly unknown, I have not found any reference to it [9]).

The simplest expedient to find out this connection is Doetsch's generating function [10])

$$\sum_{n=0}^{\infty} p_n(x; a) \frac{z^n}{n!} = \left(1 + \frac{z}{a}\right)^x e^{-z}. \tag{35}$$

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[7]) Ch. Jordan [Bull. Soc. Math. de France **54** (1926), 101—137], 110.

[8]) A. Erdélyi [Monatshefte für Math. und Phys. **46** (1937), 1—9].

[9]) Since this paper was written, Dr. A. C. Aitken kindly pointed out to me, that he was aware of the existence of the connection between these two systems of polynomials too.

[10]) G. Doetsch [Math. Annalen **109** (1933), 257—266], 260, equation (C_0). See also J. Meixner [Journal London Math. Soc. **9** (1934), 1—9]. This generating function is only another expression for

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} D^n \psi(x; a) = \psi(x; a+z),$$

which is a consequence of Taylor's theorem.

Comparing (35) with (22) and using (24) we at once obtain

$$(36)\quad p_n(x; a) = n!\, a^{-n} L_n^{(x-n)}(a) = \frac{x!\, a^{-n}}{(x-n)!}\, {}_1F_1(-n; x-n+1; a).$$

This connection is of a certain importance, because it enables us to write out almost the whole theory of Charlier polynomials specialising some results relative to Laguerre polynomials and to confluent hypergeometric functions.

10. I conclude writing out some particular cases of the expansions obtained in section 8, which yield formulae with Charlier polynomials. M and N denote in this section non-negative integers.

Putting $\alpha = M - m, \beta = N - n$ in (25), we obtain after some algebra

$$(37)\quad \begin{aligned} &\sum_{r=0}^{\infty} p_m(M+r; x)\, p_n(N+r; y) \frac{z^r}{r!} \\ &= \left(1-\frac{z}{x}\right)^m \left(1-\frac{z}{y}\right)^n e^z \sum_{r=0}^{\min(m,n)} r! \binom{m}{r}\binom{n}{r} p_{m-r}(M; x-z) p_{n-r}(N; y-z) \left\{\frac{z}{(x-z)(y-z)}\right\}^r. \end{aligned}$$

Hence we have expressed the sum of the infinite series on the left of this equation in *finite* terms. A further simplification can be attained, putting $M = N = 0$. From (35) it is seen that

$$(38)\quad p_n(0; x) = (-)^n.$$

Thus (37) yields with $M = N = 0$:

$$(39)\quad \begin{aligned} &\sum_{r=0}^{\infty} p_m(r; x)\, p_n(r; y) \frac{z^r}{r!} \\ &= \left(\frac{z}{x}-1\right)^m \left(\frac{z}{y}-1\right)^n e^z \sum_{r=0}^{\min(m,n)} r! \binom{m}{r}\binom{n}{r} \left\{\frac{z}{(x-z)(y-z)}\right\}^r \\ &= \left(\frac{z}{x}-1\right)^m \left(\frac{z}{y}-1\right)^n e^z {}_2F_0\left(-m, -n; \frac{z}{(x-z)(y-z)}\right) \\ &= \left(1-\frac{z}{x}\right)^m \left(\frac{z}{y}-1\right)^n e^z p_m\left(n; \frac{(x-z)(y-z)}{-z}\right) \\ &= \left(\frac{z}{x}-1\right)^m \left(1-\frac{z}{y}\right)^n e^z p_n\left(m; \frac{(x-z)(y-z)}{-z}\right). \end{aligned}$$

This result was, according to Aitken and Gonin [11]), proved by

[11]) A. C. Aitken & H. T. Gonin [Proc. Royal Soc. Edinburgh 55 (1935), 114—125], 115.

Wicksell [12]) and Campbell [13]). It has been recently rediscovered by Meixner [14]).

Now let us put $\alpha = M - m$, $\beta = N - n$ in (28). There occurs

$$(40)\quad \begin{aligned}\sum_{r=0}^{\infty} p_m(M+r; z)\, p_n(N+r; z) \frac{z^{m+n+r}}{r!} \\ = \frac{M!\,N!\,e^z}{(M-m)!\,(N-n)!}\, {}_2F_2(-m, -n; M-m+1, N-n+1; z).\end{aligned}$$

This is obviously a generalisation of the relation expressing the orthogonality of Charlier polynomials. Indeed, putting $M=N=0$ in (40), the first max (m, n) terms of the right of (40) vanish. Now ${}_2F_2$ has only min $(m, n) + 1$ terms, and therefore all its terms vanish if $m \gtrless n$. Only for $m = n$ the $(m+1)$st term of ${}_2F_2$ remains, giving

$$\lim_{\substack{M\to 0\\ N\to 0}} \frac{M!N!}{(M-m)!(N-m)!}\, {}_2F_2(-m, -m; M-m+1, N-m+1; z) = z^m m!.$$

Thus, the limiting form $M \to 0$, $N \to 0$ of (40) runs

$$(41)\qquad \sum_{r=0}^{\infty} p_m(r; z) p_n(r; z) \frac{z^{m+n+r}}{r!} = e^z z^m \delta_{mn}\, m!.$$

δ_{mn} is Kronecker's symbol, being equal to 0 if $m \gtrless n$, and equal to 1 if $m = n$. The same relation can be obtained [15]) as the limiting form $x = y \to z$ of (39).

The same assumption, $\alpha = M-m$ and $\beta = N-n$ in (26) yields

$$(42)\quad \begin{aligned}\left(1-\frac{z}{x}\right)^m \sum_{r=0}^{m} \binom{m}{r} p_{m-r}(M; x-z)\, p_n(N+r; y) \left(1-\frac{x}{z}\right)^{-r} \\ = \left(1-\frac{z}{y}\right)^n \sum_{r=0}^{n} \binom{n}{r} p_m(M+r; x) p_{n-r}(N; y-z) \left(1-\frac{y}{z}\right)^{-r}.\end{aligned}$$

Putting $M = N = 0$ in this equation we obtain

$$(43)\quad \left(\frac{z}{x}-1\right)^m \sum_{r=0}^{m} \binom{m}{r} p_n(r; y) \left(\frac{x}{z}-1\right)^{-r} = \left(\frac{z}{y}-1\right)^n \sum_{r=0}^{n} \binom{n}{r} p_m(r; x) \left(\frac{y}{z}-1\right)^{-r}.$$

Let us now put $\alpha = M - m$ in (27), thus obtaining

$$(44)\qquad \sum_{r=0}^{m} \binom{m}{r} p_{m-r}(M; x-z) \left(1-\frac{x}{z}\right)^{-r} = \left(1-\frac{z}{x}\right)^{-m} p_m(M; x).$$

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[12]) S. D. Wicksell [Svenska Aktuarieföreningens Tidskr. **1916**, 165—213], 192.

[13]) J. T. Campbell [Proc. Edinburgh Math. Soc. (2) **4** (1932), 18—26], 20.

[14]) J. Meixner [Math. Zeitschrift **44** (1938), 531—535], equation (14).

[15]) See e.g. J. Meixner [Math. Zeitschrift **44** (1938), 531—535], equation (16).

Another group of formulae, concerning series in products of Charlier polynomials and Laguerre polynomials, or products of Charlier polynomials and confluent hypergeometric functions is also obtainable from the formulae of section 8.

Let us put $\beta = N - n$ in (25) in order to obtain

$$
\begin{aligned}
(45)\quad & \sum_{r=0}^{\infty} L_m^{(\alpha+r)}(x) p_n(N+r; y)\frac{z^r}{r!} \\
& = \left(1-\frac{z}{y}\right)^n e^z \sum_{r=0}^{n} \binom{n}{r} L_{m-r}^{(\alpha+r)}(x-z)\, p_{n-r}(N; y-z)\left(\frac{y}{z}-1\right)^{-r},
\end{aligned}
$$

a generalisation of (37). Again put $\beta = N - n$ in (29) thus finding

$$
\begin{aligned}
(46)\quad & \sum_{r=0}^{\infty} (c-a)_r\, \Phi(a, c+r, x)\, p_n(N+r; y)\, \frac{z^r}{r!} \\
& = \left(1-\frac{z}{y}\right)^n e^z \sum_{r=0}^{n} \binom{n}{r} \Phi(a+r, c+r, x-z)\, p_{n-r}(N; y-z)\left(1-\frac{y}{z}\right)^{-r},
\end{aligned}
$$

a further generalisation of (45).

These few examples show clearly that formulae concerning Charlier's polynomials can easily be found by specialising parameters in formulae with Laguerre polynomials or with confluent hypergeometric functions. I hope to have soon an opportunity to point out that the whole theory of Charlier polynomials including differential and finite difference equations, recurrence formulae, generating functions and asymptotic expansions can be derived from the theory of confluent hypergeometric functions.

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