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by

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§ 1. *Introduction.*

By the differential equation

$$u''(x) + \lambda u(x) = 0 \qquad (0 \leqq x \leqq \pi)$$

and the boundary conditions

$$u'(0) = 0, \quad u'(\pi) = 0$$

a *boundary value problem* is defined. As well-known, the problem has solutions, different from zero, only for the values $\lambda_n = n^2$ $(n=0, 1, 2, \ldots)$, called *eigenvalues*. These solutions, called *eigenfunctions*, are the functions $u_n(x) = \cos nx \quad (n=0, 1, 2, \ldots)$.

For the boundary conditions

$$u(0) - u(\pi) = 0, \quad u'(0) - u'(\pi) = 0$$

we find the eigenvalues $\lambda_n = (2n)^2$. Every one of them, except

$\lambda_0 = 0$, to which belongs $u_0(x) = 1$, is *double*; that is to say, to $(2n)^2$ belong the two eigenfunctions $\cos 2nx$ and $\sin 2nx$ (the *trigonometrical system*; by putting $2x = y$, so that $0 \leqq y \leqq 2\pi$, we get the usual notation).

By multiplying every $u_n(x)$ with a suitable constant, we obtain that the integral of $u_n^2(x)$ over $(0, \pi)$ is equal to 1. Then the system $\{u_n(x)\}$ is *orthonormal*, which means that $(u_m, u_n) = 0$ for $m \neq n$ and $= 1$ for $m = n$, (f, g) being defined as the integral of $f(x)\,g(x)$ over $(0, \pi)$.

Further $\{u_n(x)\}$ is *complete*, which means that from $(f, u_n) = 0$ $(n=0, 1, 2, \ldots)$ for a Lebesgue-integrable $f(x)$ follows that $f(x)=0$ ($f(x)$ may differ from zero on a set with Lebesgue-measure zero, but we consider such functions to be not different from $f(x) \equiv 0$).

Putting

$$c_0(x) = \left(\frac{1}{\pi}\right)^{\frac{1}{2}},\ c_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos nx \qquad (n=1, 2, \ldots)$$

we call the orthonormal complete system $\{c_n(x)\}$ the *cosinesystem*, the series $\Sigma\, a_n c_n(x)$ a *cosineseries*. If $a_n = (f, c_n)$ for an integrable $f(x)$ and for all n, then $C(f) \equiv \Sigma\, a_n c_n(x)$ is called the *Fourier-cosineseries* of $f(x)$.

We shall indicate the normalized trigonometrical system by $\{t_n(x)\}$. The series $S(f) \equiv \Sigma\,(f, t_n)\, t_n(x)$ is called the *Fourierseries* of $f(x)$ and there exists an extensive theory about the behaviour of these series.

In the years 1908—1912 E. W. Hobson and A. Haar considered systems of eigenfunctions of more general problems (Sturm-Liouville-problems), defined by

$$u''(x) + [Q(x) + \lambda]u(x) = 0 \qquad (0 \leqq x \leqq \pi),$$

$$\begin{cases} \alpha_1\, u(0) + \alpha_2\, u'(0) = 0, \\ \beta_3\, u(\pi) + \beta_4 u'(\pi) = 0, \end{cases}$$

where $Q(x)$ is continuous in $(0, \pi)$.

The system of normalized eigenfunctions $\{u_n(x)\}$ is orthonormal and complete and the series $\Sigma(f, u_n)u_n(x)$ is indicated by $SL(f)$.

The following theorems hold:

Equiconvergencetheorem of Haar. [1])

If $Q(x)$ is continuous and of bounded variation in $(0, \pi)$, if α_2 and β_4 are $\neq 0$ and the partial sums of $SL(f)$ and $C(f)$ are $s_n(x)$

[1]) A. Haar [Math. Annalen **69** (1910), 331—371], 355.

and $s_n^*(x)$, *then*

$$\lim [s_n(x) - s_n^*(x)] = 0$$

uniformly in $(0, \pi)$.

We shall say that these series are *uniformly equiconvergent* in $(0, \pi)$.

Theorem of Du Bois-Reymond for SL-series. [2])

If, under the same conditions for $Q(x)$, α_2 *and* β_4, *a* SL*-series converges in the whole interval* $(0, \pi)$ *to a finite, integrable* $f(x)$, *then this series is* $SL(f)$.

If a SL-series, converging everywhere in $(0, \pi)$, except perhaps on a set E, to a finite integrable $f(x)$, is $SL(f)$, then E is called a *set of uniqueness* for SL-series. In 1930 A. Zygmund proved the following theorem: [3])

Necessary and sufficient that the set E *in* $(0, \pi)$ *should be a set of uniqueness for* SL*-series is that it should be a set of uniqueness for series* $\sum (a_n \cos nx + b_n \sin nx)$.

The just mentioned theorem of Du Bois-Reymond follows from this one.

All proofs rest on Hobson's asymptotic formula [4])

$$u_n(x) = c_n(x) + \frac{\beta(x)}{n} \sin nx + \frac{\alpha_n(x)}{n^2}. \tag{1}$$

In the cases that the coefficients α_2 and β_4 are not both $\neq 0$, similar theorems can be found.

In this paper we consider boundary value problems, given by

$$u''(x) + [Q(x) + \lambda]u(x) = 0 \qquad (0 \leq x \leq \pi),$$

$$\begin{cases} \alpha_1 u(0) + \alpha_2 u'(0) + \alpha_3 u(\pi) + \alpha_4 u'(\pi) = 0, \\ \beta_1 u(0) + \beta_2 u'(0) + \beta_3 u(\pi) + \beta_4 u'(\pi) = 0. \end{cases}$$

We shall restrict ourselves to the case that the system of eigenfunctions is orthogonal and complete, which turns out to be equivalent with $\alpha_1\beta_2 - \alpha_2\beta_1 = \alpha_3\beta_4 - \alpha_4\beta_3$ (§§ 2, 11).

Formulae analogous to (1) are obtained, showing different forms in different cases, according to the behaviour of the expressions $\alpha_i\beta_j - \alpha_j\beta_i$ $(i, j = 1, 2, 3, 4)$ (§§ 5, 6, 7, 8).

By these formulae it is possible then to prove theorems, cor-

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[2]) A. Haar [Math. Annalen 71 (1912), 38—53].

[3]) A. Zygmund [Studia Math. 2 (1930), 97—170].

[4]) E. W. Hobson [Proc. of the London Math. Soc. (2) 6 (1908), 349—395].

responding with those of Haar, Du Bois-Reymond and Zygmund (§§ 9, 10, 12, 13).

§ 2. *The general boundary value problem.*

We shall consider problems, given by

$$u''(x) + [Q(x) + \lambda]u(x) = 0 \qquad (0 \leqq x \leqq \pi), \tag{2}$$

$$\begin{cases} \alpha_1 u(0) + \alpha_2 u'(0) + \alpha_3 u(\pi) + \alpha_4 u'(\pi) = 0, & (3) \\ \beta_1 u(0) + \beta_2 u'(0) + \beta_3 u(\pi) + \beta_4 u'(\pi) = 0, & (4) \end{cases}$$

where $Q(x)$ is continuous and of bounded variation (the bounded variation is not yet used in this paragraph).

Putting

$$\pi_{ij} = \alpha_i \beta_j - \alpha_j \beta_i \qquad (i, j = 1, 2, 3, 4)$$

we have the following theorem:

THEOREM 1.

Necessary and sufficient that the system of eigenfunctions should be an orthogonal complete system is that $\pi_{12} = \pi_{34}$.

The proof rests on several lemmas.

LEMMA 1.

Considering a set of solutions of (2), it is necessary and sufficient for the orthogonality of every pair of them, not belonging to the same λ, that every solution out of the set satisfies two different boundary conditions (3) and (4), between the coefficients of which exists the relation $\pi_{12} = \pi_{34}$.

Proof.

Necessity. If $u_i(x)$ and $u_j(x)$ are two of the considered solutions of (2), belonging to λ_i and λ_j, then

$$[u_j(x)\, u_i'(x) - u_i(x)\, u_j'(x)]_0^\pi + (\lambda_i - \lambda_j)(u_i, u_j) = 0. \tag{5}$$

So we see that if the condition of orthogonality is satisfied, the second term of (5) is always zero. Now we distinguish between two cases:

1°. There are two functions $u_1(x)$ and $u_2(x)$ in the considered set, for which the rank of the matrix

$$\left\| \begin{matrix} u_1(0) & u_1'(0) & u_1(\pi) & u'(\pi) \\ u_2(0) & u_2'(0) & u_2(\pi) & u_2'(\pi) \end{matrix} \right\| \tag{6}$$

is two. Let $u(x)$ be an arbitrary function of the set. From (5) follows

$$[u(x)u_1'(x) - u_1(x)u'(x)]_0^\pi = 0,$$
$$[u(x)u_2'(x) - u_2(x)u'(x)]_0^\pi = 0,$$

and these are two boundary conditions for $u(x)$, different because (6) has the rank two, while between the coefficients exists the relation

$$[u_1(x)u_2'(x) - u_2(x)u_1'(x)]_0^\pi = 0, \tag{7}$$

that is $\pi_{12} = \pi_{34}$.

2^0. For every pair $u_1(x)$ and $u_2(x)$ of the set the rank of (6) is 1, so

$$u_1'(0)u_2(0) - u_1(0)u_2'(0) = u_1'(\pi)u_2(\pi) - u_1(\pi)u_2'(\pi) = 0. \tag{8}$$

By letting $u_2(x)$ run through all functions of the set, we see that we have two *Sturmian* boundary conditions (one for $x = 0$ and one for $x = \pi$) and such conditions always satisfy $\pi_{12} = \pi_{34} = 0$.

Sufficiency. Now we assume that every function of the set satisfies two different boundary conditions (3) and (4) with $\pi_{12} = \pi_{34}$. If $\pi_{12} = \pi_{34} \neq 0$ and $u_1(x)$ and $u_2(x)$ belong to the set, we can express $u_i(\pi)$ and $u_i'(\pi)$ in $u_i(0)$ and $u_i'(0)$ $(i=1, 2)$. Using the identity $\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23} = 0$ we find then (7). If, however, $\pi_{12} = \pi_{34} = 0$, the conditions (3) and (4) are equivalent with two Sturmian conditions, so that we have (8), from which follows (7). The orthogonality of $u_1(x)$ and $u_2(x)$ for $\lambda_1 \neq \lambda_2$ is concluded from (5) and (7).

LEMMA 2. [5])

If the conditions (3) and (4) are equivalent with two Sturmian conditions (so if $\pi_{12}=\pi_{34}=0$), the problem defined by (2), (3) and (4) has an enumerable number of eigenvalues λ_n $(\lambda_n \to \infty)$. If $\{\lambda_n\}$ $(n=0, 1, 2, \ldots)$ is the sequence of eigenvalues $(\lambda_n<\lambda_{n+1})$, there belongs to every λ_n one eigenfunction with n zeros in $0 < x < \pi$. (Sturmian oscillation theorem.)

LEMMA 3.

It is impossible that an orthogonal complete system of solutions of (2) satisfies three independent boundary conditions. (The completeness is essential here, for the orthogonal, incomplete system $\{\cos 2nx\}$ satisfies

$$u''(x) + \lambda\, u(x) = 0,$$
$$u'(0) = 0, \quad u'(\pi) = 0, \quad u(0) - u(\pi) = 0.)$$

Proof.

Assuming that the orthogonal complete system $\{u_n(x)\}$ of solutions of (2) satisfies three independent boundary conditions, the matrix of their coefficients has the rank 3 and we have four possibilities:

[5]) See e.g. M. BÔCHER, Leçons sur les méthodes de Sturm, Ch. III.

1^0. $u'(0)$, $u(\pi)$ and $u'(\pi)$ can be expressed in $u(0)$.

2^0. $u(0)$, $u(\pi)$ and $u'(\pi)$ can be expressed in $u'(0)$,
while 3^0 and 4^0 arise from 1^0 and 2^0 by changing 0 and π.

If we have 1^0, then

$$\begin{aligned} u'(0) &= a\,u(0), \\ u(\pi) &= b\,u(0), \\ u'(\pi) &= c\,u(0), \end{aligned} \tag{9}$$

so

$$c\,u(\pi) = b\,u'(\pi). \tag{10}$$

We see that $\{u_n(x)\}$ is a subset of the system of eigenfunctions of the Sturm-Liouville-problem, defined by (2), (9) and (10). From lemma 2 it follows that, if $b \neq 0$, not for all these eigenfunctions $u(\pi) = b\,u(0)$. So $\{u_n(x)\}$ is a real subset of this system which is incompatible with the completeness. If $b = 0$, not all eigenfunctions of the SL-problem can satisfy $u'(\pi) = c\,u(0)$, so again $\{u_n(x)\}$ would be a real subset.

In the other cases we arrive at a contradiction in a similar way.

Proof of Theorem 1.

Necessity. From lemma 1 it follows that the system of eigenfunctions satisfies two boundary conditions with $\pi_{12} = \pi_{34}$. From lemma 3 we conclude that the original conditions (3) and (4) depend on these new conditions so that for the coefficients of (3) and (4) also $\pi_{12} = \pi_{34}$.

Sufficiency. To begin with, we have to prove the existence of eigenfunctions. If $\pi_{12} = \pi_{34} = 0$, we refer to lemma 2. If $\pi_{12} = \pi_{34} \neq 0$, the proof was given by G. D. Birkhoff in 1909 [6]). He considered besides the given problem a SL-problem, defined by (2) and

$$\begin{cases} \alpha_1\,u(0) + \alpha_2\,u'(0) = 0, \\ \alpha_3\,u(\pi) + \alpha_4\,u'(\pi) = 0. \end{cases}$$

Calling the eigenvalues (in increasing order) of this problem l_n ($n=0, 1, 2, \ldots$), he came to the conclusion that $l_{n-1} \leqq \lambda_n \leqq l_n$. He also gave a calculation, in the assumption that $\pi_{24} \neq 0$, to find out if there is still a $\lambda_0 \leqq l_0$. It is not difficult to see that a similar reasoning holds for the case that $\pi_{24} = 0$ and the result is that in both cases there exists a $\lambda_0 \leqq l_0$ [6a]). The orthogonality of

[6]) G. D. Birkhoff [Transactions of the Am. Math. Soc. **10** (1909), 259—270].

[6a]) For $\pi_{24}=0$ this is only true if $\alpha_2=\alpha_4=0$, and it is always possible to obtain this by replacing if necessary the first boundary condition by a linear combination of both conditions. We shall only need this particular case.

$u_i(x)$ and $u_j(x)$ for $\lambda_i \neq \lambda_j$ follows from lemma 1 and if there are double eigenvalues ($\lambda_n = l_n = \lambda_{n+1}$), we can choose $u_n(x)$ and $u_{n+1}(x)$ out of the solutions of (2) for $\lambda = l_n$ so as to be orthogonal. As for the completeness, it seems necessary to use a theorem about integral equations for the proof in the general case (in some simple cases there are direct proofs, for the trigonometrical system that of H. Lebesgue [7]), for the system of eigenfunctions of a SL-problem that of H. Prüfer [8])). We shall not need the completeness before § 12 and for this reason we shall postpone the proof until § 11.

§ 3. *An asymptotic formula for the eigenfunctions.*

In this paragraph we shall find an asymptotic formula for the eigenfunctions $u_n(x)$ of a problem defined by (2), (3) and (4) with $\pi_{12} = \pi_{34}$. By an asymptotic formula for $u_n(x)$ we understand a formula from which we can read the behaviour of $u_n(x)$ for large n. We shall not yet find its definite form here, for in different subcases which we consider in the §§ 5, 6, 7, 8, this definite form will be different.

It will be convenient to use Landau's O-symbol, $\alpha_n = O(\beta_n)$ meaning that $|\alpha_n| < k\,|\beta_n|$ for all n and $\alpha_n(x) = O(\beta_n)$ that $|\alpha_n(x)| < k\,|\beta_n|$ for all n, uniformly in x.

As $\lambda_n > 0$ for large n, we put $\lambda = \varrho^2$ ($\varrho > 0$). Further it is no loss of generality to suppose that the integral of $Q(x)$ over $(0, \pi)$ disappears. We can reach this by adding a suitable constant to $Q(x)$, if $Q(x)$ is then transformed into $Q(x) + k$, the eigenvalues λ_n are transformed into $\lambda_n - k$, but the eigenfunctions remain the same.

Now we remark that if $u(x)$ satisfies

$$u''(x) + [Q(x) + \varrho^2]\, u(x) = 0,$$

then

$$u_1(x) = u(x) + \varrho^{-1} \int_0^x Q(t)\, u(t) \sin \varrho(x-t)\,dt$$

satisfies $u_1''(x) + \varrho^2 u_1(x) = 0$, so that $u_1(x) = A \cos(\varrho x - \tau)$ with $-\frac{\pi}{2} < \tau \leqq \frac{\pi}{2}$. Putting $A = 1$ because a constant factor is of no importance, we find

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[7]) See A. Zygmund, Trigonometrical Series, 1.5. As we shall refer to this book oftener, we shall indicate it by Tr. S.

[8]) H. Prüfer [Math. Annalen 95 (1926), 499—518].

$$(11)\qquad u(x) = \cos(\varrho x - \tau) - \varrho^{-1}\int_0^x Q(t)u(t)\sin\varrho(x-t)dt,$$

$$(12)\qquad u'(x) = -\varrho\sin(\varrho x - \tau) - \int_0^x Q(t)u(t)\cos\varrho(x-t)dt.$$

If $U = \max|u(x)|$ in $(0, \pi)$, then

$$U \leqq 1 + U\varrho^{-1}\int_0^\pi |Q(t)|\,dt,$$

so that for

$$\varrho \geqq \int_0^\pi |Q(t)|\,dt + 1 = K$$

we find $U \leqq K$.

Substituting $u_n(t) = \cos(\varrho_n t - \tau_n) + O(\varrho_n^{-1})$ in (11) and remarking that

$$\int_0^x Q(t) {\sin \atop \cos} 2\varrho_n t\,dt = O(\varrho_n^{-1}),$$

because $Q(x)$ is of bounded variation, we obtain

$$(13)\quad u_n(x) = \cos(\varrho_n x - \tau_n) - \frac{\sin(\varrho_n x - \tau_n)}{2\varrho_n}\int_0^x Q(t)dt + O(\varrho_n^{-2}),$$

while substitution in (12) gives

$$(14)\quad u_n'(x) = -\varrho_n\sin(\varrho_n x - \tau_n) - \frac{\cos(\varrho_n x - \tau_n)}{2}\int_0^x Q(t)dt + O(\varrho_n^{-1}).$$

In all cases that will be treated in the §§ 5, 6, 7, 8, ϱ_n and τ_n will have the form

$$(15)\qquad \varrho_n = \bar{\varrho}_n + a_n\bar{\varrho}_n^{-1} + O(\varrho_n^{-2}),\ \ \tau_n = \bar{\tau}_n + b_n\bar{\varrho}_n^{-1} + O(\varrho_n^{-2}),$$

where $\bar{\varrho}_n$, $\bar{\tau}_n$, $a_n(=O(1))$ and $b_n(=O(1))$ depend on n in a simple way. Then using Taylor's formula we see that $u_n(x)$ takes the form

$$(16)\quad \begin{aligned} u_n(x) &= \cos(\bar{\varrho}_n x - \bar{\tau}_n) - \sin(\bar{\varrho}_n x - \bar{\tau}_n)[(\varrho_n - \bar{\varrho}_n)x - (\tau_n - \bar{\tau}_n)] - \\ &\quad - \frac{\sin(\bar{\varrho}_n x - \bar{\tau}_n)}{2\bar{\varrho}_n}\int_0^x Q(t)dt + O(\varrho_n^{-2}) = \\ &= \cos(\bar{\varrho}_n x - \bar{\tau}_n) + \beta_n^*(x)\bar{\varrho}_n^{-1}\sin(\bar{\varrho}_n x - \bar{\tau}_n) + O(\varrho_n^{-2}), \end{aligned}$$

where

$$(17)\qquad \beta_n^*(x) = -a_n x + b_n - \frac{1}{2}\int_0^x Q(t)dt.$$

In the same way we find

$$(18)\quad u_n'(x) = -\bar{\varrho}_n\sin(\bar{\varrho}_n x - \bar{\tau}_n) + \beta_n^*(x)\cos(\bar{\varrho}_n x - \bar{\tau}_n) + O(\varrho_n^{-1}).$$

§ 4. *Some preliminaries.*

As the last part of the preceding paragraph shows, the only thing that remains to be done in order to find the definite form of the asymptotic formula for $u_n(x)$ is to prove that the formulae (15) for ϱ_n and τ_n exist and to calculate $\bar{\varrho}_n$, $\bar{\tau}_n$, a_n and b_n in them. For our later purposes it is of importance to show that a_n and b_n are independent of n or at most depend on n being even or odd.

Substitution in (3) and (4) of $u(0) = \cos \tau$, $u'(0) = \varrho \sin \tau$ (resulting from (11) and (12)), $u(\pi) = \cos(\pi\varrho - \tau) + O(\varrho^{-2})$, $u'(\pi) = -\varrho \sin(\pi\varrho - \tau) + O(\varrho^{-1})$ (resulting from (13) and (14)) gives (omitting the indices n):

(19) $$\alpha_1 \cos \tau + \alpha_2 \varrho \sin \tau + \alpha_3 \cos(\pi\varrho - \tau) - \alpha_4 \varrho \sin(\pi\varrho - \tau) + \alpha_3 O(\varrho^{-2}) + \alpha_4 O(\varrho^{-1}) = 0,$$

(20) $$\beta_1 \cos \tau + \beta_2 \varrho \sin \tau + \beta_3 \cos(\pi\varrho - \tau) - \beta_4 \varrho \sin(\pi\varrho - \tau) + \beta_3 O(\varrho^{-2}) + \beta_4 O(\varrho^{-1}) = 0.$$

LEMMA 4.

If the boundary conditions have the form

$$\begin{cases} \alpha_1 u(0) + \alpha_2 u'(0) = 0 & (\alpha_2 \neq 0), \\ u(\pi) = 0 & \end{cases}$$

and the eigenvalues are $\{l_n\}$ $(n=0, 1, 2, \ldots)$, then

$$l_n^{\frac{1}{2}} = n + \frac{1}{2} + O(n^{-1}).$$

Proof.

(19) and (20) take the form

$$\alpha_2 \varrho \sin \tau = -\alpha_1 \cos \tau,$$
$$\cos(\pi\varrho - \tau) = O(\varrho^{-2}),$$

so

(21) $$\tau = b\varrho^{-1} + O(\varrho^{-2}) \text{ with } b = -\alpha_1\alpha_2^{-1},$$
$$\pi\varrho - \tau = \left(n+\frac{1}{2}\right)\pi + O(\varrho^{-2}) \quad (n \text{ a natural number}),$$

from which follows

(22) $$\varrho = n + \frac{1}{2} + b\pi^{-1}\varrho^{-1} + O(\varrho^{-2}).$$

(21) and (22) can be written in the form (15) (with $a_n = b\pi^{-1}$, $b_n = b$), so we obtain from (16)

$$u(x) = \cos\left(n+\frac{1}{2}\right)x + \beta^*(x)\varrho^{-1} \sin\left(n+\frac{1}{2}\right)x + O(\varrho^{-2}).$$

From this formula we see that the number of zeros of $u(x)$ in $0 < x < \pi$ is n or $n + 1$ for large n; because $u(\pi) = 0$ this number is n. But then lemma 2 teaches us that the index of this $u(x)$ is n and (22) becomes

$$l_n^{\frac{1}{2}} = n + \tfrac{1}{2} + O(n^{-1}).$$

LEMMA 5.

If the boundary conditions have the form

$$u(0) = 0, \quad u(\pi) = 0$$

and the eigenvalues are $\{l_n\}$ $(n=0, 1, 2, \ldots)$, then

$$l_n^{\frac{1}{2}} = n + 1 + O(n^{-1}).$$

Proof.

The proof runs on the same lines as that of lemma 4, so that we shall not repeat it.

After these simple cases we shall now proceed with the general case. We have to distinguish between $\pi_{24} \neq 0$ (§ 5) and $\pi_{24} = 0$ (§§ 6, 7, 8).

§ 5. *The definite form of the asymptotic formula in the case* $\pi_{24} \neq 0$.

From (19) and (20) follows

$$\begin{cases} \alpha_2 \sin \tau - \alpha_4 \sin (\pi\varrho - \tau) = O(\varrho^{-1}), \\ \beta_2 \sin \tau - \beta_4 \sin (\pi\varrho - \tau) = O(\varrho^{-1}), \end{cases}$$

so, because $\pi_{24} \neq 0$:

$$\sin \tau = O(\varrho^{-1}), \quad \sin (\pi\varrho - \tau) = O(\varrho^{-1}).$$

Remembering that $-\frac{\pi}{2} < \tau \leqq \frac{\pi}{2}$, we find

$$\cos \tau = 1 + O(\varrho^{-2}), \quad \pi\varrho - \tau = \pi n + O(\varrho^{-1}), \tag{23}$$

where n is a natural number. There are two subcases: n odd and n even.

Calculation for odd n.

From (19) and (20) follows, using (23)

$$\begin{cases} \alpha_2\, \varrho \sin \tau - \alpha_4\, \varrho \sin(\pi\varrho - \tau) = -(\alpha_1 - \alpha_3) + O(\varrho^{-1}), \\ \beta_2\, \varrho \sin \tau - \beta_4\, \varrho \sin(\pi\varrho - \tau) = -(\beta_1 - \beta_3) + O(\varrho^{-1}), \end{cases}$$

so

$$\varrho \sin \tau = -\pi_{24}^{-1}(\pi_{14} - \pi_{34}) + O(\varrho^{-1}) = b_1 + O(\varrho^{-1}),$$
$$\varrho \sin(\pi\varrho - \tau) = -\pi_{24}^{-1}(\pi_{12} - \pi_{32}) + O(\varrho^{-1}) = c_1 + O(\varrho^{-1}),$$

or

(24) $\tau = b_1\varrho^{-1} + O(\varrho^{-2}), \quad \pi\varrho - \tau = \pi n - c_1\varrho^{-1} + O(\varrho^{-2}),$

so that

(25) $\varrho = n + a_1 n^{-1} + O(n^{-2})$ with $a_1 = \pi^{-1}(b_1 - c_1)$.

Now we have to find the index of this ϱ. If the boundary conditions are Sturmian (so if $\pi_{12}=\pi_{34}=0$) we can use again lemma 2. As (24) and (25) can be written in the form (15), we obtain for $u(x)$:

$$u(x) = \cos nx + \beta_1^*(x)\, n^{-1} \sin nx + O(n^{-2}).$$

This function has n zeros in $0 < x < \pi$ for large n, so it has the index n. If the boundary conditions are not Sturmian we may take $\pi_{12} = \pi_{34} = 1$. Then the eigenfunctions have to satisfy the linear combination of (3) and (4)

$$\pi_{14}\, u(0) + \pi_{24}\, u'(0) + u(\pi) = 0.$$

Now we use Birkhoff's result, already mentioned in the proof of theorem 1. This asserts that calling the eigenvalues of the problem with the boundary conditions

$$\pi_{14}\, u(0) + \pi_{24}\, u'(0) = 0, \qquad u(\pi) = 0$$

l_n $(n=0, 1, 2, \ldots)$, the interval $l_{n-1} \leqq \lambda \leqq l_n$ contains λ_n. But lemma 4 gives $l_n^{\frac{1}{2}} = n + \frac{1}{2} + O(n^{-1})$, so that in (25) $\varrho = \lambda^{\frac{1}{2}}$ has the index n.

Both in the Sturmian and in the general case (24) and (25) take therefore the form

(26) $\tau_n = b_1 n^{-1} + O(n^{-2}), \quad \varrho_n = n + a_1 n^{-1} + O(n^{-2}).$

Calculation for even n.

In the same way as for odd n we find

(27) $\tau_n = b_2 n^{-1} + O(n^{-2}), \quad \varrho_n = n + a_2 n^{-1} + O(n^{-2})$

with

$b_2 = -\pi_{24}^{-1}(\pi_{14}+\pi_{34}),\ a_2 = \pi^{-1}(b_2+c_2)$ where $c_2 = -\pi_{24}^{-1}(\pi_{12}+\pi_{32})$.

Remark.

If the boundary conditions are Sturmian (so if $\pi_{12}=\pi_{34}=0$), we see that $b_1 = b_2$ and $a_1 = a_2$, so that we have no longer to distinguish between odd and even n.

THEOREM 2.

If $Q(x)$ is of bounded variation in $(0, \pi)$, $\pi_{12} = \pi_{34}$ and $\pi_{24} \neq 0$, we have for the normalized eigenfunctions of the problem, defined by (2), (3) *and* (4) *the asymptotic formulae*

$$(28) \quad \begin{cases} u_{2n-1}(x) = c_{2n-1}(x) + \dfrac{\beta_1(x)}{2n-1} \sin (2n-1)x + O(n^{-2}), \\ u_{2n}(x) = c_{2n}(x) + \dfrac{\beta_2(x)}{2n} \sin 2nx + O(n^{-2}), \end{cases}$$

where $\beta_i(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(-a_i x + b_i - \frac{1}{2}\int_0^x Q(t)dt\right) \qquad (i=1, 2).$

If $\pi_{12} = \pi_{34} = 0$, then $\beta_1(x) = \beta_2(x)$ and we get back Hobson's formula (1).

Proof.

As (26) and (27) have the form (15) with $\bar{\varrho}_n = n$ and $\bar{\tau}_n = 0$, substitution in (16) and (17) gives

$$u_n(x) = \cos nx + \beta_n^*(x)n^{-1} \sin nx + O(n^{-2})$$

with

$$\beta_{2k-1}^*(x) = \beta_1^*(x) = -a_1 x + b_1 - \frac{1}{2}\int_0^x Q(t)dt,$$

$$\beta_{2k}^*(x) = \beta_2^*(x) = -a_2 x + b_2 - \frac{1}{2}\int_0^x Q(t)dt$$

If we want to normalize $u_n(x)$ we have to multiply with $(u_n, u_n)^{-\frac{1}{2}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} + O(n^{-2})$ and the result is that we obtain (28).

If $\pi_{12} = \pi_{34} = 0$, then $a_1 = a_2$ and $b_1 = b_2$ as we have already remarked, so that $\beta_1(x) = \beta_2(x)$.

§ 6. *The definite form of the asymptotic formula in the case* $\pi_{12} = \pi_{34} \neq 0,\ \pi_{24} = 0,\ \pi_{14}^2 \neq \pi_{12}^2$.

Whereas the case $\pi_{24} \neq 0$ still shows a great resemblance with that treated by Hobson, the case $\pi_{24} = 0$ differs much more from it. We shall restrict ourselves to non-Sturmian boundary conditions, so that $\pi_{12} = \pi_{34} \neq 0$. Furthermore we take $\pi_{14}^2 \neq \pi_{12}^2$ in this paragraph. It is to be observed that $\pi_{14} \neq 0$ because of the identity $\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23} = 0$.

We can suppose $\pi_{12} = \pi_{34} = 1$ (so $\pi_{14}^2 \neq 1$) and we replace the original boundary conditions by their linear combinations

$$(29) \quad \begin{cases} \pi_{14}\, u(0) \qquad\quad + u(\pi) \qquad\qquad\qquad = 0, \\ \qquad\qquad u'(0) + \pi_{13}\, u(\pi) + \pi_{14}\, u'(\pi) = 0, \end{cases}$$

from which result by the substitutions

$$u(0) = \cos\tau \quad , \quad u'(0) = \varrho \sin\tau$$
$$u(\pi) = \cos(\pi\varrho-\tau) + O(\varrho^{-2}), \quad u'(\pi) = -\varrho\sin(\pi\varrho-\tau) + O(\varrho^{-1})$$

the conditions

$$(30)\quad \begin{cases} \pi_{14}\cos\tau + \cos(\pi\varrho-\tau) = O(\varrho^{-2}), \\ \sin\tau - \pi_{14}\sin(\pi\varrho-\tau) = -\varrho^{-1}\pi_{13}\cos(\pi\varrho-\tau) + O(\varrho^{-2}). \end{cases}$$

From

$$\cos(\pi\varrho-\tau) = -\pi_{14}\cos\tau + O(\varrho^{-2}),$$
$$\sin(\pi\varrho-\tau) = \pi_{14}^{-1}\sin\tau + O(\varrho^{-1})$$

follows by taking squares and adding

$$1 = \pi_{14}^2\cos^2\tau + \pi_{14}^{-2}(1-\cos^2\tau) + O(\varrho^{-1}),$$

so, because $-\frac{\pi}{2} < \tau \leqq \frac{\pi}{2}$:

$$(31)\qquad \cos\tau = (1+\pi_{14}^2)^{-\frac{1}{2}} + O(\varrho^{-1}).$$

After this preliminary formula for $\cos\tau$ we shall now try to find an asymptotic formula of the form (15) for ϱ_n. For this purpose we write (30) as

$$\begin{cases} (\pi_{14}+\cos\pi\varrho)\cos\tau + \sin\pi\varrho\sin\tau = O(\varrho^{-2}), \\ -\pi_{14}\sin\pi\varrho\cos\tau + (1+\pi_{14}\cos\pi\varrho)\sin\tau = \varrho^{-1}\pi_{13}\pi_{14}\cos\tau + O(\varrho^{-2}). \end{cases}$$

The determinant D of the coefficients of $\sin\tau$ and $\cos\tau$ on the left hand side is $D \equiv 2\pi_{14} + (1+\pi_{14}^2)\cos\pi\varrho$, so that

$$D\cos\tau = -\varrho^{-1}\pi_{13}\pi_{14}\sin\pi\varrho\cos\tau + O(\varrho^{-2})$$

or

$$D \equiv 2\pi_{14} + (1+\pi_{14}^2)\cos\pi\varrho = -\varrho^{-1}\pi_{13}\pi_{14}\sin\pi\varrho + O(\varrho^{-2})$$

$\left(\frac{O(\varrho^{-2})}{\cos\tau} = O(\varrho^{-2}) \text{ by } (31)\right)$, so

$$\cos\pi\varrho = -2\pi_{14}(1+\pi_{14}^2)^{-1} - \varrho^{-1}\pi_{13}\pi_{14}(1+\pi_{14}^2)^{-1}\sin\pi\varrho + O(\varrho^{-2}).$$

If p $(0 < p < 1)$ is defined by $\cos p\pi = -2\pi_{14}(1+\pi_{14}^2)^{-1}$ and $\bar\varrho = 2n \pm p$ (n a natural number), we have

$$\cos\pi\varrho = \cos\pi\bar\varrho - \varrho^{-1}\pi_{13}\pi_{14}(1+\pi_{14}^2)^{-1}\sin\pi\bar\varrho + O(\varrho^{-2}).$$

As $\cos\pi\bar\varrho \neq \pm 1$, we can use Taylor's formula for arc cos x to obtain $\pi\varrho$ itself. This gives

$$\pi\varrho = \pi\bar\varrho + \frac{1}{-\sin\pi\bar\varrho}\cdot\frac{-\pi_{13}\pi_{14}\sin\pi\bar\varrho}{1+\pi_{14}^2}\cdot\frac{1}{\varrho} + O(\varrho^{-2}),$$
$$\pi\varrho = (2n\pm p)\pi + \varrho^{-1}\pi_{13}\pi_{14}(1+\pi_{14}^2)^{-1} + O(\varrho^{-2}).$$

To fix the index of this ϱ, we again use Birkhoff's results. Calling the eigenvalues of the problem with the boundary conditions

$$u(0) = 0, \quad u(\pi) = 0$$

l_n $(n=0, 1, 2, \ldots)$, we have $l_{n-1} \leqq \lambda_n \leqq l_n$. But $l_n^{\frac{1}{2}} = n+1+O(n^{-1})$ by lemma 5, so

$$(32) \quad \begin{cases} \varrho_{2n-1} = 2n - p + \dfrac{\pi_{13}\pi_{14}}{\pi(1+\pi_{14}^2)} \cdot \dfrac{1}{2n-p} + O(n^{-2}), \\ \varrho_{2n} = 2n + p + \dfrac{\pi_{13}\pi_{14}}{\pi(1+\pi_{14}^2)} \cdot \dfrac{1}{2n+p} + O(n^{-2}). \end{cases}$$

Our next task is to find an asymptotic formula for τ_n. We shall repeat the calculation that led to (31) considering now also the term $O(\varrho^{-1})$. Starting from

$$\begin{aligned} \cos(\pi\varrho-\tau) &= -\pi_{14}\cos\tau + O(\varrho^{-2}), \\ \sin(\pi\varrho-\tau) &= \pi_{14}^{-1}\sin\tau + \varrho^{-1}\pi_{13}\pi_{14}^{-1}\cos(\pi\varrho-\tau) + O(\varrho^{-2}) \\ &= \pi_{14}^{-1}\sin\tau - \varrho^{-1}\pi_{13}\cos\tau + O(\varrho^{-2}), \end{aligned}$$

we obtain by taking squares and adding

$$\begin{aligned} 1 &= \pi_{14}^2\cos^2\tau + \pi_{14}^{-2}(1-\cos^2\tau) - 2\varrho^{-1}\pi_{13}\pi_{14}^{-1}\sin\tau\cos\tau + O(\varrho^{-2}), \\ \pi_{14}^2 - 1 &= (\pi_{14}^4-1)\cos^2\tau - 2\varrho^{-1}\pi_{13}\pi_{14}\sin\tau\cos\tau + O(\varrho^{-2}), \\ 1 &= (1+\pi_{14}^2)\cos^2\tau + 2\varrho^{-1}\pi_{13}\pi_{14}(1-\pi_{14}^2)^{-1}\sin\tau\cos\tau + O(\varrho^{-2}), \\ \cos^2\tau &= (1+\pi_{14}^2)^{-1}[1-2\varrho^{-1}\pi_{13}\pi_{14}(1-\pi_{14}^2)^{-1}\sin\tau\cos\tau + O(\varrho^{-2})], \\ \cos\tau &= (1+\pi_{14}^2)^{-\frac{1}{2}}[1-\varrho^{-1}\pi_{13}\pi_{14}(1-\pi_{14}^2)^{-1}\sin\tau\cos\tau + O(\varrho^{-2})]. \end{aligned}$$

Now calling $\bar{\tau}$ the „principal" part of τ (so $\cos\bar{\tau} = (1+\pi_{14}^2)^{-\frac{1}{2}}$), we have

$$\cos\tau = \cos\bar{\tau} + \varrho^{-1}A\sin\bar{\tau} + O(\varrho^{-2})$$

with $A = -\pi_{13}\pi_{14}(1-\pi_{14}^4)^{-1}$. So at last we obtain

$$(33) \quad \tau_n = \bar{\tau}_n + \frac{1}{-\sin\bar{\tau}_n} \cdot \frac{A\sin\bar{\tau}_n}{\varrho_n} + O(\varrho_n^{-2}) = \bar{\tau}_n + \frac{\pi_{13}\pi_{14}}{1-\pi_{14}^4} \cdot \frac{1}{\varrho_n} + O(\varrho_n^{-2}).$$

To fix $\bar{\tau}_n$ we observe that all $\bar{\tau}_n$ have the same cosine, so $\bar{\tau}_n$ can only take two values τ and $-\tau$. A simple reckoning shows that from the first line of (30) follows

$$\sin\pi\varrho_n\sin\tau_n = \pi_{14}(1-\pi_{14}^2)(1+\pi_{14}^2)^{-\frac{3}{2}} + O(\varrho^{-2}),$$

so

$$\sin\pi\bar{\varrho}_n\sin\bar{\tau}_n = \pi_{14}(1-\pi_{14}^2)(1+\pi_{14}^2)^{-\frac{3}{2}}.$$

As from $\cos \pi\bar{\varrho}_n = -2\pi_{14}(1+\pi_{14}^2)^{-1}$ and (32) results that

$$\begin{cases} \sin \pi\bar{\varrho}_{2n-1} = (1-\pi_{14}^2)(1+\pi_{14}^2)^{-1} \operatorname{sgn} (\pi_{14}^2-1), \\ \sin \pi\bar{\varrho}_{2n} \quad = (1-\pi_{14}^2)(1+\pi_{14}^2)^{-1} \operatorname{sgn} (1-\pi_{14}^2), \end{cases}$$

we find

$$\begin{cases} \sin \bar{\tau}_{2n-1} = \pi_{14}(1+\pi_{14}^2)^{-\frac{1}{2}} \operatorname{sgn} (\pi_{14}^2-1), \\ \sin \bar{\tau}_{2n} \quad = \pi_{14}(1+\pi_{14}^2)^{-\frac{1}{2}} \operatorname{sgn} (1-\pi_{14}^2). \end{cases}$$

So we can put

$$\begin{cases} \bar{\tau}_{2n-1} = \tau, \\ \bar{\tau}_{2n} \quad = -\tau. \end{cases}$$

THEOREM 3.

If $Q(x)$ is of bounded variation in $(0, \pi)$, $\pi_{12} = \pi_{34} = 1$, $\pi_{24} = 0$ and $\pi_{14}^2 \neq 1$, so that the boundary conditions can be written in the form (29), we have for the normalized eigenfunctions of the problem, defined by (2) and (29) the asymptotic formulae

$$(34) \quad \begin{cases} u_{2n-1}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos[(2n-p)x-\tau] + \frac{\beta(x)}{2n-p} \sin[(2n-p)x-\tau] + O(n^{-2}), \\ u_{2n}(x) \quad = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos[(2n+p)x+\tau] + \frac{\beta(x)}{2n+p} \sin[(2n+p)x+\tau] + O(n^{-2}), \end{cases}$$

where p $(0 < p < 1)$ and τ are defined by

$$(35) \quad \cos p\pi = -2\pi_{14}(1+\pi_{14}^2)^{-1}, \quad \begin{cases} \cos \tau = (1+\pi_{14}^2)^{-\frac{1}{2}}, \\ \sin \tau = \pi_{14}(1+\pi_{14}^2)^{-\frac{1}{2}} \operatorname{sgn} (\pi_{14}^2-1), \end{cases}$$

and where

$$\beta(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[-\frac{\pi_{13}\pi_{14}}{\pi(1+\pi_{14}^2)} x + \frac{\pi_{13}\pi_{14}}{1-\pi_{14}^4} - \frac{1}{2} \int_0^x Q(t)dt \right].$$

Proof.

As (32) and (33) have the form (15) with

$$\bar{\varrho}_{2n-1} = 2n - p, \ \bar{\varrho}_{2n} = 2n + p, \ \bar{\tau}_{2n-1} = \tau, \ \bar{\tau}_{2n} = -\tau,$$

substitution in (16) and (17) and normalizing gives the formulae (34).

Remark.

In a certain respect the case $\pi_{24} = 0$ is more general than $\pi_{24} \neq 0$ as the asymptotic formulae obtained also show. By the dominating rôle of $u'(0)$ and $u'(\pi)$ in the boundary conditions the case $\pi_{24} \neq 0$ is to a high degree equivalent with the simple case $u'(0) = 0$, $u'(\pi) = 0$, whereas this equivalence disappears as soon as $\pi_{24} = 0$.

§ 7. *The definite form of the asymptotic formula in the case*

$$\pi_{12} = \pi_{34} \neq 0,\ \pi_{24} = 0,\ \pi_{14}^2 = \pi_{12}^2,\ \pi_{13} \neq 0.$$

While in the preceding paragraph we supposed $\pi_{14}^2 \neq \pi_{12}^2$, we shall now discuss the case $\pi_{14}^2 = \pi_{12}^2$. It will turn out that the influence of π_{13} and $Q(x)$ becomes greater. We restrict ourselves here to $\pi_{13} \neq 0$ and it seems necessary to impose upon $Q(x)$ a heavier condition than in the preceding paragraphs to obtain satisfactory asymptotic formulae. Again we can take $\pi_{12}=\pi_{34}=1$ without loss of generality and we shall prove the following theorems:

THEOREM 4.

If $Q(x)$ has a first derivative which is of bounded variation in $(0, \pi)$, $\pi_{12} = \pi_{34} = \pi_{14} = 1$, $\pi_{24} = 0$, $\pi_{13} \neq 0$, so that the boundary conditions can be written in the form

$$(36)\qquad \begin{cases} u(0) + u(\pi) = 0, \\ u'(0) + \pi_{13}u(\pi) + u'(\pi) = 0, \end{cases}$$

we have for the normalized eigenfunctions of the problem, defined by (2) and (36) the asymptotic formulae

$$(37)\qquad \begin{cases} u_{2n}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\,(2n+1)x + \frac{\beta_2(x)}{2n+1} \sin\,(2n+1)x + O(n^{-2}), \\ u_{2n+1}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin\,(2n+1)x + \frac{\beta_1(x)}{2n+1} \cos\,(2n+1)x + O(n^{-2}), \end{cases}$$

with

$$\beta_2(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[-\frac{\pi_{13}}{\pi} x - \frac{Q(0) - Q(\pi)}{4\pi_{13}} + \frac{1}{2\pi} \int_0^\pi Q(x) \cdot x\, dx - \frac{1}{2} \int_0^x Q(t) dt \right],$$

$$\beta_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[\frac{Q(0) - Q(\pi)}{4\pi_{13}} + \frac{1}{2} \int_0^x Q(t) dt \right].$$

THEOREM 5.

If $Q(x)$ has a first derivative which is of bounded variation in $(0, \pi)$, $\pi_{12} = \pi_{34} = -\pi_{14} = 1$, $\pi_{24} = 0$, $\pi_{13} \neq 0$, so that the boundary conditions can be written in the form

$$(38)\qquad \begin{cases} u(0) - u(\pi) = 0, \\ u'(0) + \pi_{13}\, u(\pi) - u'(\pi) = 0, \end{cases}$$

we have for the normalized eigenfunctions of the problem, defined by (2) and (38) the asymptotic formulae

$$(39)\quad \begin{cases} u_{2n-1}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos 2nx + \frac{\beta_1(x)}{2n} \sin 2nx + O(n^{-2}), \\ u_{2n}(x) \;\; = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin 2nx + \frac{\beta_2(x)}{2n} \cos 2nx + O(n^{-2}), \end{cases}$$

where $\beta_1(x)$ *and* $\beta_2(x)$ *have a similar form as in theorem* 4.

Proof.

We shall give the principal points of the proof of theorem 4, that of theorem 5 will be omitted because it runs on the same lines.

The boundary conditions (36) can be written in the form

$$(40)\quad \begin{cases} \cos(\pi\varrho - \tau) = -\cos\tau + \varrho^{-1}\int_0^\pi Q(t)\,u(t)\sin\varrho(\pi - t)dt, \\ \sin(\pi\varrho - \tau) = \sin\tau - \varrho^{-1}\pi_{13}\cos\tau - \varrho^{-1}\int_0^\pi Q(t)u(t)\cos\varrho(\pi - t)dt, \end{cases}$$

which gives

$$(41)\quad \begin{cases} \cos\pi\varrho = -1 + \varrho^{-1}\pi_{13}\sin\tau\cos\tau + O(\varrho^{-2}), \\ \sin\pi\varrho = -\varrho^{-1}\pi_{13}\cos^2\tau + O(\varrho^{-2}). \end{cases}$$

If however $\sin\pi\varrho = O(\varrho^{-1})$ then $\cos\pi\varrho = -1 + O(\varrho^{-2})$, so that $\sin\tau\cos\tau = O(\varrho^{-1})$, because $\pi_{13} \neq 0$.

Now there are two possibilities:

I. $\cos\tau = O(\varrho^{-1})$.

II. $\sin\tau = O(\varrho^{-1})$.

If we keep to our old agreement $-\frac{\pi}{2} < \tau \leqq \frac{\pi}{2}$, it would be impossible to decide in case I whether $\bar{\tau}_n = -\frac{\pi}{2}$ or $\bar{\tau}_n = \frac{\pi}{2}$. Therefore we shall change the agreement and fix that $\bar{\tau}_n = \frac{\pi}{2}$. In case II we have $\bar{\tau}_n = 0$, so always

$$(42)\qquad \tau_n = \bar{\tau}_n + \varrho_n^{-1} b_n \qquad \text{with } b_n = O(1).$$

From (41) follows

$$\begin{aligned} \sin\pi\varrho &= O(\varrho^{-2}) && \text{in case I ,} \\ \sin\pi\varrho &= -\varrho^{-1}\pi_{13} + O(\varrho^{-2}) && \text{in case II,} \end{aligned}$$

so

$$(43)\quad \begin{cases} \varrho_n = \bar{\varrho}_n + O(\varrho_n^{-2}) & \text{in case I ,} \\ \varrho_n = \bar{\varrho}_n + \bar{\varrho}_n^{-1}\pi_{13}\pi^{-1} + O(\varrho_n^{-2}) & \text{in case II,} \end{cases}$$

where $\bar{\varrho}_n$ is an odd natural number because $\cos\pi\bar{\varrho}_n = -1$. As

(42) and (43) can be written in the form (15) we have

$$u_n(x) = \cos(\bar{\varrho}_n x - \bar{\tau}_n) + \beta_n^*(x)\bar{\varrho}_n^{-1} \sin(\bar{\varrho}_n x - \bar{\tau}_n) + O(\varrho_n^{-2})$$

with

$$\beta_n^*(x) = b_n - \frac{1}{2}\int_0^x Q(t)dt \qquad \text{in case I},$$

$$\beta_n^*(x) = -\frac{\pi_{13}}{\pi}x + b_n - \frac{1}{2}\int_0^x Q(t)dt \qquad \text{in case II},$$

so in both cases $\beta_n^*(x) = b_n + \gamma_n(x)$.

What remains to be done is to see for which n occurs the case I and for which n case II and to find out how b_n depends on n. Just to assure that b_n only depends on I or II occurring, it seems necessary to impose the already mentioned heavier condition upon $Q(x)$.

We have to consider the integrals in (40).

Omitting easy calculations we get

$$I_1 = \int_0^\pi Q(t)u_n(t)\sin\varrho_n(\pi - t)dt = \int_0^\pi Q(t)u_n(t)\sin\bar{\varrho}_n t\,dt + O(\varrho_n^{-2}) =$$

$$= \int_0^\pi Q(t)\cos(\bar{\varrho}_n t - \bar{\tau}_n)\sin\bar{\varrho}_n t\,dt +$$

$$+ \bar{\varrho}_n^{-1}\int_0^\pi Q(t)\beta_n^*(t)\sin(\bar{\varrho}_n t - \bar{\tau}_n)\sin\bar{\varrho}_n t\,dt + O(\varrho_n^{-2}) =$$

$$= I_3 + I_4 + O(\varrho_n^{-2}).$$

Using that $Q(x)$ has a first derivative which is of bounded variation, it follows that

$$I_3 = (4\bar{\varrho}_n)^{-1}[Q(0) - Q(\pi)]\cos\bar{\tau}_n + O(\varrho_n^{-2}),$$

$$I_4 = (2\bar{\varrho}_n)^{-1}\cos\bar{\tau}_n\int_0^\pi Q(t)\,\gamma_n(t)dt + O(\varrho_n^{-2}),$$

so

$$I_1 = \bar{\varrho}_n^{-1}\cos\bar{\tau}_n\left[\frac{1}{4}(Q(0) - Q(\pi)) + \frac{1}{2}\int_0^\pi Q(t)\gamma_n(t)dt\right] + O(\varrho_n^{-2}) =$$

$$= \bar{\varrho}_n^{-1}P_n\cos\bar{\tau}_n + O(\varrho_n^{-2}),$$

where P_n can only take the values P_1 (in case I) and P_2 (in case II). In a similar way we can find

$$I_2 = \int_0^\pi Q(t)u_n(t)\cos\varrho_n(\pi - t)dt =$$

$$= \bar{\varrho}_n^{-1}\sin\bar{\tau}_n\left[\frac{1}{4}(Q(\pi) - Q(0)) + \frac{1}{2}\int_0^\pi Q(t)\gamma_n(t)dt\right] + O(\varrho_n^{-2}) =$$

$$= \bar{\varrho}_n^{-1}Q_n\sin\bar{\tau}_n + O(\varrho_n^{-2}).$$

The boundary conditions (40) can be written now as

$$\begin{cases} \cos(\pi\varrho_n - \tau_n) = -\cos\tau_n + \bar{\varrho}_n^{-2} P_n \cos\bar{\tau}_n + O(\varrho_n^{-3}), \\ \sin(\pi\varrho_n - \tau_n) = \sin\tau_n - \varrho_n^{-1}\pi_{13}\cos\tau_n - \bar{\varrho}_n^{-2} Q_n \sin\bar{\tau}_n + O(\varrho_n^{-3}). \end{cases}$$

Taking squares and adding gives

$$-2\bar{\varrho}_n^{-2} P_n \cos^2\bar{\tau}_n - 2\varrho_n^{-1}\pi_{13}\sin\tau_n\cos\tau_n - 2\bar{\varrho}_n^{-2} Q_n \sin^2\bar{\tau}_n = O(\varrho_n^{-3}).$$

In case I $(\sin\bar{\tau}_n = 1,\ \cos\bar{\tau}_n = 0,\ \sin\tau_n = 1 + O(\varrho_n^{-2}))$ this takes the form

$$\cos\tau_n = -\bar{\varrho}_n^{-1}\pi_{13}^{-1} Q_1 + O(\varrho_n^{-2}),$$

(44) $$\tau_n = \frac{\pi}{2} + \bar{\varrho}_n^{-1}\pi_{13}^{-1} Q_1 + O(\varrho_n^{-2}),$$

where

$$Q_1 = \frac{1}{4}(Q(\pi) - Q(0)) - \frac{1}{4}\int_0^\pi dx\, Q(x)\int_0^x Q(t)dt = \frac{1}{4}(Q(\pi) - Q(0)),$$

for $\int_0^\pi dx \cdot Q(x)\int_0^x Q(t)dt = 0$ because $\int_0^\pi Q(x)dx = 0.$

Substitution of (43) and (44) in (16) and (17) gives

$$u_n(x) = \cos\left(\bar{\varrho}_n x - \frac{\pi}{2}\right) + \beta_1^*(x)\bar{\varrho}_n^{-1}\sin\left(\bar{\varrho}_n x - \frac{\pi}{2}\right) + O(\varrho_n^{-2}) =$$
$$= \sin\bar{\varrho}_n x - \beta_1^*(x)\bar{\varrho}_n^{-1}\cos\bar{\varrho}_n x + O(\varrho_n^{-2})$$

with

$$\beta_1^*(x) = \frac{Q(\pi) - Q(0)}{4\pi_{13}} - \frac{1}{2}\int_0^x Q(t)\,dt.$$

In the same way we find when II occurs

$$u_n(x) = \cos\bar{\varrho}_n x + \beta_2^*(x)\bar{\varrho}_n^{-1}\sin\bar{\varrho}_n x + O(\varrho_n^{-2})$$

with

$$\beta_2^*(x) = -\frac{\pi_{13}}{\pi}x - \frac{Q(0) - Q(\pi)}{4\pi_{13}} + \frac{1}{2\pi}\int_0^\pi Q(x)x\,dx - \frac{1}{2}\int_0^x Q(t)\,dt.$$

From Birkhoff's results we conclude, using again lemma 5, that $\bar{\varrho}_{2n} = \bar{\varrho}_{2n+1} = 2n + 1$. It is impossible that both for $u_{2n}(x)$ and $u_{2n+1}(x)$ case I occurs, for that would be incompatible with their being orthogonal to each other.

The same can be said about case II, so of $u_{2n}(x)$ and $u_{2n+1}(x)$ one has the „principal'' term $\cos(2n+1)x$ and the other one $\sin(2n+1)x$. For that which follows it is of no importance how we distribute the indices $2n$ and $2n+1$ among them. After normalizing them we obtain the formulae (37).

§ 8. *The definite form of the asymptotic formula in the still remaining cases.*

We still have to consider problems with boundary conditions

$$u(0) + u(\pi) = 0, \quad u'(0) + u'(\pi) = 0$$

or

$$u(0) - u(\pi) = 0, \quad u'(0) - u'(\pi) = 0.$$

Here the influence of $Q(x)$ is still greater than in the preceding case. As an example we mention the following theorem that can be proved with methods analogous to those used in § 7.

THEOREM 6.

If $Q(x)$ has a second derivative which is of bounded variation in $(0, \pi)$ and $Q(0) \neq Q(\pi)$, we have for the normalized eigenfunctions of the problem, defined by (2) *and*

$$u(0) + u(\pi) = 0, \quad u'(0) + u'(\pi) = 0,$$

the asymptotic formulae

$$\begin{cases} u_{2n}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\left[(2n+1)x - \frac{\pi}{4}\right] + \frac{\beta_2(x)}{2n+1} \sin\left[(2n+1)x - \frac{\pi}{4}\right] + O(n^{-2}), \\ u_{2n+1}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\left[(2n+1)x + \frac{\pi}{4}\right] + \frac{\beta_1(x)}{2n+1} \sin\left[(2n+1)x + \frac{\pi}{4}\right] + O(n^{-2}), \end{cases}$$

where $\beta_1(x)$ and $\beta_2(x)$ have a similar form as in the cases already treated.

If $Q(0) = Q(\pi)$ we can obtain similar formulae under the assumption that $Q(x)$ has a third derivative which is of bounded variation in $(0, \pi)$ and that $Q'(0) \neq Q'(\pi)$.

Probably it will be possible to find such formulae if $Q^{(i)}(0) = Q^{(i)}(\pi)$ $(i=0, 1, \ldots, n-1)$, $Q^{(n)}(0) \neq Q^{(n)}(\pi)$ and if $Q^{(n+2)}(x)$ is of bounded variation in $(0, \pi)$.

§ 9. *The behaviour of the „Fourierseries" $SL(f)$ and the sets of uniqueness in the case $\pi_{24} \neq 0$.*

As the problems with $\pi_{24} \neq 0$ show so much resemblance with Sturm-Liouville problems it need not amaze us that the equiconvergence theorem of Haar and the theorem of Zygmund about the sets of uniqueness (both mentioned in the Introduction) remain valid without any difference. The proofs undergo only slight changes, so we shall omit them here.

§ 10. *The behaviour of the „Fourierseries" $SL(f)$ for the system* $\left\{\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos\left[(2n \pm p)x \pm \tau\right]\right\}$.

If we consider the simplest possible case of those treated in § 6, we have to take $Q(x) \equiv 0$ and the boundary conditions

$$\begin{cases} \pi_{14}\,u(0) + u(\pi) = 0, \\ u'(0) + \pi_{14}u'(\pi) = 0. \end{cases} \quad (\pi_{14}^2 \neq 1)$$

The system of eigenfunctions is then

$$\left\{v_n(x) \equiv \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\left(\varrho_n x - \tau_n\right)\right\}$$

$$\text{with} \begin{cases} \varrho_{2n-1} = 2n - p, & \tau_{2n-1} = \tau, \\ \varrho_{2n} = 2n + p, & \tau_{2n} = -\tau, \end{cases} \tag{45}$$

where p $(0 < p < 1)$ and τ are defined by (35).

We shall study the behaviour in $(0, \pi)$ of the „Fourierseries" $SL(f) \equiv \Sigma(f, v_n)v_n(x)$ of an integrable function $f(x)$. For this purpose we need some theorems about common Fourierseries. (For the notations we refer to the Introduction.)

Lemma 6. [9])

If $f_1(x) = f_2(x)$ in (a, b), then $S(f_1)$ and $S(f_2)$ are uniformly equiconvergent in $(a+\varepsilon, b-\varepsilon)$.

Lemma 7. [10])

If $\sigma(x)$ has the period π and satisfies a Lipschitz-condition of order 1, then $S(\sigma f)$ and $\sigma(x)S(f)$ are uniformly equiconvergent in $(0, \pi)$.

Lemma 8.

$S(\cos px \cdot f(x))$ and $\cos px\, S(f)$ are uniformly equiconvergent in every interval $(\varepsilon, \pi-\varepsilon)$. The same holds good for $S(\sin px \cdot f(x))$ and $\sin px\, S(f)$.

Proof.

Consider a function $\sigma(x)$ of period π, satisfying a Lipschitz-condition of order 1 and coinciding with $\cos px$ in $\left(\frac{\varepsilon}{2}, \pi-\frac{\varepsilon}{2}\right)$. From lemma 6 follows that $S(\cos px \cdot f(x))$ and $S(\sigma(x)f(x))$ are uniformly equiconvergent in $(\varepsilon, \pi-\varepsilon)$ and from lemma 7 the same for $S(\sigma(x)f(x))$ and $\sigma(x)S(f)$ in $(0, \pi)$. In $(\varepsilon, \pi-\varepsilon)$, $S(\cos px \cdot f(x))$ is therefore uniformly equiconvergent with $\sigma(x)S(f) = \cos px\, S(f)$.

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9) Tr. S. 2.51.

10) Tr. S. 2.531.

The second half of the lemma is proved in the same way.

Calling now $sl_n[f]$ the $(2n+1)$-th partial sum of $SL(f)$ (so the same notation as always used for trigonometrical series), we have

$$sl_n[f] = \frac{2}{\pi}\int_0^{\pi} f(t) \sum_0^{2n} \cos(\varrho_k x - \tau_k)\cos(\varrho_k t - \tau_k)dt =$$
$$= \frac{1}{\pi}\int_0^{\pi} f(t) \sum_0^{2n} \{\cos[\varrho_k(x+t) - 2\tau_k] + \cos\varrho_k(x-t)\}dt.$$

Taking into consideration (45) and adding in $\sum_0^{2n}$ the terms with indices $k = 2l-1$ and $k = 2l$, we obtain

$$sl_n[f] = \frac{2}{\pi}\int_0^{\pi} f(t)\cos[p(x+t)+2\tau]\left[\frac{1}{2} + \sum_1^n \cos 2l(x+t)\right]dt +$$
$$+ \frac{2}{\pi}\int_0^{\pi} f(t)\cos p(x-t)\left[\frac{1}{2} + \sum_1^n \cos 2l(x-t)\right]dt = I_1 + I_2.$$

As we have for the $(2n+1)$-th partial sum of $S(f)$:

$$s_n[f] = \frac{2}{\pi}\int_0^{\pi} f(t)\left[\frac{1}{2} + \sum_1^n \cos 2l(x-t)\right]dt,$$

we see that

$$I_2 = \cos px \cdot s_n[\cos px \cdot f(x)] + \sin px \cdot s_n[\sin px \cdot f(x)].$$

To obtain for I_1 a similar expression we remark that from (35) follows

$$\cos(p\pi+2\tau) = 0, \quad \sin(p\pi+2\tau) = \operatorname{sgn}(1-\pi_{14}^2),$$

so

$$I_1 = \frac{2}{\pi}\int_0^{\pi} f(t)\cos[p\{t-(\pi-x)\} + p\pi + 2\tau]\left[\frac{1}{2} + \sum_1^n \cos 2l\{t-(\pi-x)\}\right]dt =$$
$$= -\operatorname{sgn}(1-\pi_{14}^2)\frac{2}{\pi}\int_0^{\pi} f(t)\sin p\{t-(\pi-x)\}\left[\frac{1}{2} + \sum_1^n \cos 2l\{t-(\pi-x)\}\right]dt =$$
$$= \operatorname{sgn}(\pi_{14}^2-1)\{\cos p(\pi-x)\, s_n[\sin p(\pi-x) f(\pi-x)] -$$
$$- \sin p(\pi-x)\, s_n[\cos p(\pi-x)\cdot f(\pi-x)]\} =$$
$$= \operatorname{sgn}(\pi_{14}^2-1)\{\sin px \cdot s_n[\cos px \cdot f(\pi-x)] - \cos px \cdot s_n[\sin px \cdot f(\pi-x)]\}.$$

So

$$\begin{aligned} sl_n[f] &= \cos px\{s_n[\cos px \cdot f(x)] + \operatorname{sgn}(1-\pi_{14}^2)s_n[\sin px \cdot f(\pi-x)]\} + \\ &+ \sin px\{s_n[\sin px \cdot f(x)] + \operatorname{sgn}(\pi_{14}^2-1)s_n[\cos px \cdot f(\pi-x)]\} = \\ &= \cos px \cdot s_n[\varphi_1(x)] + \sin px \cdot s_n[\varphi_2(x)]. \end{aligned} \tag{46}$$

THEOREM 7.

For an integrable $f(x)$ the series $SL(f)$ and $S(f)$ are uniformly equiconvergent in every interval $(\varepsilon, \pi-\varepsilon)$.

Proof.

From (46) follows by lemma 8 that the difference of the $(2n+1)$-th partial sums of $SL(f)$ and $S(f)$ converges to zero uniformly in $(\varepsilon, \pi-\varepsilon)$. As the coefficients in $SL(f)$ as well as in $S(f)$ converge to zero, we have the same for the difference of the $2n$-th partial sums and the theorem is proved.

There can be no equiconvergence in the whole interval $(0, \pi)$, for all $v_n(x)$ satisfy $\pi_{14} v_n(0) + v_n(\pi) = 0$, so also $\pi_{14} sl_n[f(0)] + sl_n[f(\pi)] = 0$, whereas $s_n[f(0)] = s_n[f(\pi)]$. Should there be equiconvergence e.g. at $x = 0$, then there would be no equiconvergence at $x = \pi$ because $\pi_{14} \neq -1$. We can prove something about the behaviour of $SL(f)$ in the whole interval $(0, \pi)$ if $f(x)$ satisfies, together with some other condition, the boundary condition $\pi_{14} f(0) + f(\pi) = 0$, so one of those satisfied by the eigenfunctions.

THEOREM 8.

If the continuous function $f(x)$, satisfying $\pi_{14} f(0) + f(\pi) = 0$, is of bounded variation in $(0, \pi)$ or satisfies a Lipschitz-condition of positive order there, then $SL(f)$ converges to $f(x)$ uniformly in the whole interval $(0, \pi)$.

Proof.

If $f(x)$ is continuous and of bounded variation in $(0, \pi)$ or satisfies a Lipschitz-condition of positive order there, the same can be said of the functions $\varphi_i(x)$ $(i=1, 2)$ in (46). Now an easy calculation shows that $\varphi_i(0)=\varphi_i(\pi)$ because $\pi_{14} f(0)+f(\pi)=0$. This leads to the result that $s_n[\varphi_i]$ converges to $\varphi_i(x)$ uniformly in $(0, \pi)$ as follows from well-known theorems about Fourier-series. So $sl_n[f] \equiv \cos px \cdot s_n[\varphi_1] + \sin px \cdot s_n[\varphi_2]$ converges to $\cos px \cdot \varphi_1(x) + \sin px \cdot \varphi_2(x) \equiv f(x)$ uniformly in $(0, \pi)$.

THEOREM 9.

The system $\left\{\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\left[(2n \pm p)x \pm \tau\right]\right\}$ *is complete.*

Proof.

If $(f, v_n) = 0$ for all n, the sum of $SL(f)$ is identically zero, so by theorem 7 $S(f)$ converges to zero everywhere in $(0, \pi)$, except perhaps in 0 and π. As the set consisting of the two points 0 and π is a set of uniqueness for trigonometrical series [11]), $f(x)=0$.

[11]) Tr. S. 11.32.

§ 11. *The completeness of the system of eigenfunctions.*

As already remarked in § 2 it seems inevitable to use a theorem about integral equations if we want to prove the completeness of the system of eigenfunctions in the general case.

LEMMA 9.

If $\{u_n(x)\}$ is the system of eigenfunctions of one of the problems treated in the §§ 5, 6, 7, 8, this system is complete for the class of continuous functions.

Proof.

It is no loss of generality to assume that $\lambda = 0$ is not an eigenvalue of the considered problem. Now to this problem is adjoined a function $G(x, t)$ (Green's function), having the properties:

a) If $f(x)$ is continuous in $(0, \pi)$,

$$u(x) = \int_0^\pi G(x, t) f(t)\, dt$$

satisfies

$$u''(x) + Q(x)u(x) = -f(x).$$

b) $G(x, t) = \sum_0^\infty \lambda_n^{-1} u_n(x)\, u_n(t)$. We remark that this series converges uniformly in x and t because $\lambda_n^{-1} = O(n^{-2})$.

If now $(f, u_n) = 0$ for a continuous $f(x)$ and all n, by b)

$$u(x) = \int_0^\pi G(x, t) f(t)\, dt = \sum_0^\infty \lambda_n^{-1} u(x)(f, u_n) = 0,$$

so by a):

$$f(x) = -\left[u''(x) + Q(x)\, u(x)\right] = 0.$$

LEMMA 10.

If $f(x)$ is integrable there exists a $h(x)$, continuous in $(0, \pi)$, with $(f, u_n) = (f+h, v_n)$ for all n, where $v_n(x)$ is the „principal" part of $u_n(x)$.

Proof.

(For the case treated in § 6, the proofs in the other cases running on parallel lines.)

Omitting the indices n, we have

$$(f, u) = (f, v) + r + O(n^{-2}),$$

where

$$r = \int_0^\pi f(x)\,\beta(x)\,\frac{\sin\left[(2n\pm p)\,x \pm \tau\right]}{2n \pm p}\,dx = \int_0^\pi \frac{\sin\left[(2n\pm p)\,x\pm\tau\right]}{2n\pm p}\,d\varphi(x) =$$

$$= \frac{\varphi(\pi)\sin\left[(2n\pm p)\pi\pm\tau\right]}{2n\pm p} - (\varphi, v) = \frac{\pm\,\varphi(\pi)\sin(p\pi+\tau)}{2n} - (\varphi, v) + O(n^{-2}),$$

where $\varphi(x) = \int_0^x f(t)\,\beta(t)dt$ is continuous. Because series with coefficients $O(n^{-2})$ are „Fourierseries" of continuous functions, it remains to be proved that the same holds good for

$$\sum_1^\infty \frac{v_{2n-1}(x) - v_{2n}(x)}{2n}. \tag{47}$$

Defining the continuous function $\psi(x)$ by

$$\psi(x) = \sum_1^\infty \left(\frac{2}{\pi}\right)^{\frac{1}{2}} n^{-1} \sin 2nx \ \text{ for } \ 0 < x < \pi,$$

$$\psi(0) = \psi(0+), \quad \psi(\pi) = \psi(\pi-),$$

it is a simple reckoning (using (46)) to prove that (47) is the Fourierseries of $\sin(px+\tau)\psi(x)$.

THEOREM 10.

If $\{u_n(x)\}$ is the system of eigenfunctions of one of the problems treated in the §§ 5, 6, 7, 8, this system is complete for the class of integrable functions.

Proof.

(For the case treated in § 6.)

If $(f, u_n) = 0$ for an integrable $f(x)$ and all n, by lemma 10 there is a continuous $h(x)$ with $(f, u_n) = (f+h, v_n) = 0$ for all n. Because however $\{v_n(x)\}$ is complete (theorem 9), $f(x) + h(x) = 0$, so $(f, u_n) = (-h, u_n) = 0$. But $h(x)$ is continuous, so by lemma 9 $h(x) = 0$. From $f(x) + h(x) = 0$ follows then that also $f(x) = 0$.

§ 12. *The behaviour of the „Fourierseries" $SL(f)$ in the general case.*

We shall again restrict ourselves to the case treated in § 6 and we shall prove the analogon of Haar's equiconvergence theorem, mentioned in the Introduction.

THEOREM 11.

The series $\Sigma\,(f, u_n)\,u_n(x)$ and $\Sigma\,(f, v_n)\,v_n(x)$ are uniformly equiconvergent in $(0, \pi)$ for every integrable $f(x)$.

Proof.

The proof is analogous to Hobson's proof of Haar's theorem. At first it is shown that the difference of the partial sums of the two considered series converges uniformly to a continuous $h(x)$ and then that $h(x) \equiv 0$.

Writing (34) in the form

$$u_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos(\varrho_n x - \tau_n) + \beta(x)\, \varrho_n^{-1} \sin(\varrho_n x - \tau_n) + O(n^{-2}),$$

we obtain for the difference of the $(2n+1)$-th partial sums (omitting the terms with index 0) an expression of the form $I_1 + I_2 + I_3$, where

$$I_1 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\pi} f(t)\, \beta(x) \sum_1^{2n} \varrho_k^{-1} \sin(\varrho_k x - \tau_k) \cos(\varrho_k t - \tau_k)\, dt,$$

I_2 has a similar form and I_3 is a sum with terms $O(k^{-2})$, converging therefore to a continuous function. In I_1 we change ϱ_k^{-1} for $k = 2l - 1$ and $k = 2l$ into $(2l)^{-1}$, this being allowed because the difference is a series with terms $O(k^{-2})$. Adding then the terms with $k = 2l - 1$ and $k = 2l$ we find an expression which can be split up into terms of the form

$$\varphi(x) \int_0^{\pi} \psi(t) \sum_1^{n} l^{-1} \sin 2l(x \pm t)\, dt =$$

$$= \varphi(x) \left[\sum_1^{n} l^{-1} \sin 2lx \int_0^{\pi} \psi(t) \cos 2lt\, dt \pm \sum_1^{n} l^{-1} \cos 2lx \int_0^{\pi} \psi(t) \sin 2lt\, dt \right].$$

Both sums occurring in the last bracket converge uniformly as term by term integrated partial sums of common Fourierseries, so the same holds good for I_1. In a similar way this is proved for I_2, so

$$\Sigma \left[(f, u_n)\, u_n(x) - (f, v_n)\, v_n(x) \right] = h(x)$$

uniformly in x.

To prove that $h(x) \equiv 0$ we remark that

$$\sum_0^{\infty} (f, v_n)(g, v_n) = (f, g) \tag{48}$$

for $f(x)$ integrable and $g(x)$ of bounded variation. This case of Parseval's theorem asserting that it is allowed to integrate $\Sigma(f, v_n) v_n(x)$ term by term, after having multiplied it with $g(x)$, follows from the fact that this theorem holds good for common Fourierseries [12]) and from (46).

Now using (48) with $f = f(x)$ and $g = u_i(x)$, we have

$$(h, u_i) = \Sigma\left[(f, u_n)(u_n, u_i) - (f, v_n)(v_n, u_i)\right] =$$
$$= (f, u_i) - \Sigma\,(f, v_n)(u_i, v_n) = (f, u_i) - (f, u_i) = 0$$

for every i. From the completeness of $\{u_n(x)\}$ follows that $h(x) \equiv 0$.

The theorem just proved enables us to assert that the theorems 7 and 8 also hold good for the „Fourierseries" $SL(f)$ considered in this paragraph. Because of their importance we shall mention them again.

THEOREM 12.

If $\{u_n(x)\}$ is the system of eigenfunctions of the problem defined by

$$u''(x) + [Q(x) + \lambda]u(x) = 0 \qquad (0 \leqq x \leqq \pi),$$

$$\begin{cases} \pi_{14}\, u(0) \qquad + u(\pi) = 0, \\ \qquad u'(0) + \pi_{13}\, u(\pi) + \pi_{14}\, u'(\pi) = 0, \end{cases} \qquad (\pi_{14}^2 \neq 1)$$

where $Q(x)$ is continuous and of bounded variation, $\{t_n(x)\}$ is the trigonometrical system $\left\{\left(\frac{1}{\pi}\right)^{\frac{1}{2}}, \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos 2nx, \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin 2nx\right\}$ *and $f(x)$ is integrable in $(0, \pi)$, then the „Fourierseries"*

$$\Sigma\,(f, u_n)\, u_n(x) \text{ and } \Sigma\,(f, t_n)\, t_n(x)$$

are uniformly equiconvergent in every interval $(\varepsilon, \pi-\varepsilon)$.

If the continuous function $f(x)$, satisfying $\pi_{14} f(0) + f(\pi) = 0$, is of bounded variation in $(0, \pi)$ or satisfies a Lipschitz-condition of positive order there, $\Sigma\,(f, u_n)\, u_n(x)$ converges to $f(x)$ uniformly in the whole interval $(0, \pi)$.

In the cases treated in the §§ 7, 8, similar theorems hold good and for the system mentioned in theorem 5, the equiconvergence with $\Sigma(f, t_n)t_n(x)$ in $(\varepsilon, \pi-\varepsilon)$ is even replaced by equiconvergence in $(0, \pi)$.

§ 13. *Sets of uniqueness.*

In this paragraph we shall understand by a trigonometrical series a series $\Sigma\,(a_n \cos nx + b_n \sin nx)$ and by $S(f)$ the Fourier-series of $f(x)$ in the usual meaning. Again $\{u_n(x)\}$ is the system of eigenfunctions of the problem treated in § 6.

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12) Tr. S. 4.44.

THEOREM 13.

If $c_n \to 0$ there is a trigonometrical series $\Sigma\,(a_n \cos nx + b_n \sin nx)$, uniformly equiconvergent with $\Sigma\, c_n u_n(x)$ in $(0, \pi)$.

The proof rests on several lemmas.

LEMMA 11. [13])

If a_n and $b_n \to 0$ and $S(g)$ has Fouriercoefficients $O(n^{-3})$, there is a trigonometrical series with coefficients converging to zero, uniformly equiconvergent with $g(x)\,\Sigma\,(a_n \cos nx + b_n \sin nx)$ in $(-\pi, \pi)$.

LEMMA 12.

If $c_n \to 0$ there is a series $\Sigma(a_n \cos nx + b_n \sin nx)$ (a_n and $b_n \to 0$), uniformly equiconvergent with $\Sigma\, c_n v_n(x)$ in $(0, \pi)$, where $v_n(x)$ is the „principal'' part of $u_n(x)$.

Proof.

The series $\Sigma\, c_n v_n(x)$ can be written in the form

$$\Sigma \left\{ d_n \cos\left[(2n-p)\,x - \tau\right] + e_n \cos\left[(2n+p)\,x + \tau\right]\right\} =$$
$$= \cos\,(px+\tau)\,\Sigma\,(d_n+e_n)\cos 2nx + \sin\,(px+\tau)\,\Sigma\,(d_n-e_n)\sin 2nx.$$

Defining $g_1(x)$ and $g_2(x)$ in $(-\pi, \pi)$ in such a way that they coincide with $\cos(px+\tau)$ resp. $\sin(px+\tau)$ in $(0, \pi)$ and have Fouriercoefficients $O(n^{-3})$, the proof follows immediately from lemma 11.

LEMMA 13. [14])

Writing the formulae (34) in the form

$$(49) \qquad u_n(x) = v_n(x) + \beta(x)\,\varrho_n^{-1} \sin\,(\varrho_n x - \tau_n) + n^{-2}\,\alpha_n(x),$$

$\alpha(x) = \sum\limits_1^\infty c_n n^{-2}\,\alpha_n(x)$ satisfies a Lipschitz-condition of positive order.

Proof.

Differentiating (16) and comparing the result with (18) we see that $n^{-2}\,\alpha_n'(x) = O(n^{-1})$, so

$$|\,\alpha(x+h) - \alpha(x)\,| \leqq \left(\sum_1^N + \sum_{N+1}^\infty\right) |\,c_n\,|\, n^{-2}\,|\,\alpha_n(x+h) - \alpha_n(x)\,| \leqq$$
$$\leqq K_1\,|\,h\,| \sum_1^N n^{-1} + K_2 \sum_{N+1}^\infty n^{-2} \leqq K_3\,|\,h\,| \log N + K_4 N^{-1}.$$

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[13]) Tr. S. 11.42.

[14]) See [3]).

Taking $N = [|h|^{-1}]$ we find

$$|\alpha(x+h)-\alpha(x)| < K_5 |h| \log |h|^{-1} < K_\alpha |h|^\alpha$$

for every α in $0 < \alpha < 1$ and $|h| \leqq \frac{1}{2}$.

Proof of Theorem 13.

Applying the notation (49), by lemma 12 there is a trigonometrical series uniformly equiconvergent with $\sum_1^\infty c_n v_n(x)$ in $(0, \pi)$. It is not difficult to see that by the same method we can find a trigonometrical series uniformly equiconvergent with $\sum_1^\infty c_n \beta(x) \varrho_n^{-1} \sin(\varrho_n x - \tau_n)$, while from lemma 13 follows that $C(c_0 u_0(x) + \alpha(x))$ converges uniformly to $c_0 u_0(x) + \alpha(x)$.

THEOREM 14.

Every set of uniqueness for trigonometrical series, lying in $(0, \pi)$, *is also a set of uniqueness for series* $\sum c_n u_n(x)$ $(c_n \to 0)$.

Proof.

Let the set E in $(0, \pi)$ be a set of uniqueness for trigonometrical series and let $\sum c_n u_n(x)$ converge to a finite integrable $f(x)$ on its complement CE. We have to prove that $c_n = (f, u_n)$ for all n.

Calling $\sum (a_n \cos nx + b_n \sin nx)$ (a_n and $b_n \to 0$) the trigonometrical series uniformly equiconvergent with $\sum c_n u_n(x)$ in $(0, \pi)$, this series also converges to $f(x)$ on CE. Then

$$\sum_0^\infty \int_0^x (a_n \cos nt + b_n \sin nt) dt = \int_0^x f(t) dt$$

uniformly in $(0, \pi)$ [15]), so also

$$\sum_0^\infty \int_0^x c_n u_n(t)\, dt = \int_0^x f(t) dt$$

uniformly in $(0, \pi)$. From

$$\lim_{N\to\infty} \int_{x_1}^{x_2} \left[\sum_0^N c_n u_n(t) - f(t) \right] dt = 0$$

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[15]) Tr. S. 11.47. In the last theorem of 11.47 it is allowed to replace the words „an at most enumerable set E of points" by „a set of uniqueness E".

uniformly follows by the second mean-value theorem

$$\lim_{N\to\infty} \int_0^{\pi} \left[\sum_0^N c_n u_n(t) - f(t)\right] u_k(t)\, dt = 0$$

for all k, so $c_k = (f, u_k)$ for all k.

We remark that the theorem of Du Bois-Reymond for the series $\sum c_n u_n(x)$ follows from theorem 14. [16])

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[16]) It is possible to give a direct proof of the theorem of Du Bois-Reymond without having to refer to the deep-lying theorems in Tr. S. 11.42 and 11.47.