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by

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Introduction.

A ring S which satisfies the double-chain-condition (or the equivalent maximum and minimum condition) for the right-hand (in short: r.h.) ideals, possesses accoording to E. Artin [1]) a nilpotent radical R, and the quotient ring S/R is semi-simple. This fact, as well as the results obtained by Artin concerning the „primary" and the „completely primary" rings attached to S are valid for a wider class of rings. *In the present note it is shown that the maximum condition can be omitted without affecting the results achieved by Artin.*

The method used in the present note is partly an improvement of one used by the author in a previous paper [2]). On the other hand, the results obtained presently yield a generalisation of the principal theorem proved in L, which can be stated now as follows: ***Each nil-subring of a ring which satisfies the minimum condition for the r.h. ideals, is nilpotent.*** This statement is in particular a solution of a problem raised by G. Köthe [3]), whether or not there exist potent nil-rings which satisfy the maximum or the *minimum* condition for the r.h. ideals.

The importance of the nil-rings was first emphasized by Köthe [3]), who considers rings of a more general type but of similar structure, and which actually might contain potent (i.e. non-nilpotent) nil-subrings, and even nil-ideals.

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[1]) E. ARTIN, Zur Theorie der hyperkomplexen Zahlen [Hamb. Abh. **5** (1927), 251—260], referred to as A.

[2]) J. LEVITZKI, Über nilpotente Unterringe [Math. Ann. **105** (1931), 620—627], referred to as L.

[3]) G. KÖTHE, Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist [Math. Zeitschr. **32** (1930), 161—186], referred to as K.

I. Notations and preliminary remarks.

1. As it is well known, a ring S is said to satisfy the minimum (maximum) condition for the r.h. ideals, if each non empty set of r.h. ideals contains a r.h. ideal which is not a proper subset of any other r.h. ideal of the set (respectively, which does not contain any other r.h. ideal of the set as a proper subset). This is equivalent with the condition that for each infinite „descending chain" of r.h. ideals $\mathfrak{R}_1 \supseteqq \mathfrak{R}_2 \supseteqq \ldots$ (resp. for each „ascending chain" $\mathfrak{R}_1 \subseteqq \mathfrak{R}_2 \subseteqq \ldots$) an index m exists such that $\mathfrak{R}_i = \mathfrak{R}_k$ if $m < i$, $m < k$. We say in short: S is a r.h.m.c.-ring.

2. If A is a finite or an infinite subset of a ring S, then let $\overline{S}$ be the set containing A and all the finite sums and differences of all the finite products which can be derived from the elements of A. Then $\overline{S}$ is evidently a subring of S, and we say: $\overline{S}$ is generated by A. Evidently $\overline{S}$ is the minimal subring (in the usual sense) of S containing A.

3. If A and B are subsets of a ring S, then we denote by AB the set of all the finite sums of elements which have the form ab, where $a \epsilon A$, $b \epsilon B$. Thus if B is a r.h. ideal or A is a l.h. ideal, then also AB is a r.h. ideal, or respectivley a l.h. ideal.

4. If $\mathfrak{R}$ is a r.h. ideal of a ring S, and $\overline{S}$ is a subset of S, then we denote by $(\overline{S}, \overline{S}\mathfrak{R})$ the minimal r.h. ideal containing $\overline{S}$ as well as $\overline{S}\mathfrak{R}$, i.e. the set of all elements of the form $\Sigma \pm \lambda_i s_i + \Sigma s'_\varrho r_\varrho$ where the λ_i are positive integers, s_i, $s'_\varrho \epsilon \overline{S}$ and $r_\varrho \epsilon \mathfrak{R}$.

5. If $\mathfrak{R}_1, \ldots, \mathfrak{R}_m$ are r.h. ideals of a ring S, then their sum, i.e. the set of all elements of the form $\sum_{i=1}^{m} r_i$, $r_i \epsilon \mathfrak{R}_i$ is denoted by $(\mathfrak{R}_1, \ldots, \mathfrak{R}_m)$.

6. If S' and S'' are subsets of a ring S, then we denote by $[S', S'']$ the set of all elements of S which belong to S' as well as to S''. If S'' is a r.h. ideal in S and S' is a subring of S, then evidently $[S', S'']$ is a r.h. ideal in S'.

7. An element a of a ring S is called nilpotent of index n if $a^n = 0$, $a^{n-1} \neq 0$.

A nilring is a ring which contains nilpotent elements only.

A ring N is called nilpotent of index n, if $N^n = 0$, $N^{n-1} \neq 0$.

An element a of a ring S is called properly nilpotent, if the r.h. ideal aS is nilpotent.

8. A r.h. ideal $\mathfrak{R}$ of a ring S is called primitive, if $\mathfrak{R}$ is potent (i.e. not nilpotent) and does not contain a potent r.h. ideal as a proper subset. It follows easily, that if S is a r.h.m.c.-ring, then each potent r.h. ideal of S contains a primitive r.h. ideal.

II. On subrings generated by finite subsets.

LEMMA 1. If S is a r.h.m.c.-ring (see I, 1), and if a finite set of elements $a_1, \ldots, a_m$ of S generates a potent ring N (see I, 2), then for a certain positive integer λ the r.h. ideal $\mathfrak{R} = N^\lambda S$ which is evidently different from zero (since N is potent), satisfies the relations

$$(1) \qquad (a_1\mathfrak{R}, \ldots, a_m\mathfrak{R}) = \mathfrak{R}; \quad \mathfrak{R} = N^\lambda S = N^{\lambda+1}S = \ldots$$

Proof. Since evidently $NS \supseteqq N^2S \supseteqq N^3S \supseteqq \ldots$, it follows by I, 1 that for a certain positive integer λ the relations $N^\lambda S = N^{\lambda+1}S = \ldots$ are true. From the definition of N it follows further that $N^2 = (a_1N, \ldots, a_mN)$, and hence $N^\lambda S = N^{\lambda+2}S = N^2(N^\lambda S) = (a_1N, \ldots, a_mN)N^\lambda S = (a_1N^{\lambda+1}S, \ldots, a_nN^{\lambda+1}S) = (a_1N^\lambda S, \ldots, a_nN^\lambda S)$; by setting $\mathfrak{R} = N^\lambda S$ we have $\mathfrak{R} = (a_1\mathfrak{R}, \ldots, a_n\mathfrak{R})$, which completes the proof.

LEMMA 2 [4]). If S is a r.h.m.c.-ring, and if for a certain r.h. ideal $\mathfrak{R}$, $\mathfrak{R} \neq 0$, and a finite set of elements $a_1, \ldots, a_m$ of S the relation $\mathfrak{R} = (a_1\mathfrak{R}, \ldots, a_m\mathfrak{R})$ holds, then there exists an infinite sequence $b_1, b_2, \ldots$ each b_λ being a certain a_j, such that the relation $0 \subset b_\lambda b_{\lambda+1} \cdots b_{\lambda+\sigma}\mathfrak{R} \subseteqq b_\lambda\mathfrak{R}$ is satisfied for arbitrary positive integers λ and σ.

Proof. From $\mathfrak{R} \neq 0$ it follows that not all the $a_i\mathfrak{R}$ are zero. Let i_1 be the minimal index such that $a_{i_1}\mathfrak{R} \neq 0$. Since $a_{i_1}\mathfrak{R} = (a_{i_1}a_1\mathfrak{R}, \ldots, a_{i_1}a_m\mathfrak{R}) \neq 0$, it follows that not all the $a_{i_1}a_\sigma\mathfrak{R}$ are zero, and again let i_2 be the minimal index such that $a_{i_1}a_{i_2}\mathfrak{R} \neq 0$. This process can be infinitely repeated, and setting $b_j = a_{i_j}$ we obtain (by induction) the required infinite sequence. The lemma follows now from

$$0 \subset b_\lambda b_{\lambda+1} \cdots b_{\lambda+\sigma}\mathfrak{R} \subseteqq b_\lambda \cdots b_{\lambda+\sigma-1}\mathfrak{R} \subseteqq \ldots \subseteqq b_\lambda\mathfrak{R}.$$

THEOREM 1. If S is a r.h.m.c.-ring and N is a subring of S which is generated by a finite subset $a_1, a_2, \ldots, a_m$, then N is potent if and only if S contains a r.h. ideal $\mathfrak{R}$ such that $\mathfrak{R} \neq 0$ and $(a_1\mathfrak{R}, \ldots, a_m\mathfrak{R}) = \mathfrak{R}$.

Proof. This is an immediate consequence of lemmas 1 and 2.

THEOREM 2. Each nil-subring of a r.h.m.c.-ring S, which is generated by a finite subset of S is nilpotent. Moreover: Let N be a potent subring of S which is generated by the finite set

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[4]) The proof of this lemma is included in the proof of theorem 2 in L.

$d_1, d_2, \ldots, d_m$; then N contains potent elements of the form $c_1 c_2 \cdots c_j$, where each c_i is a certain d_j.

Proof. Let $a_1, \ldots, a_s$ be a subset of the d_j so that the ring N which is generated by the a_i is still potent and s is of the least possible value. By lemma 1 follows the existence of a r.h. ideal $\mathfrak{R}$ so that $\mathfrak{R} \neq 0$, and the relations (1) are satisfied. By lemma 2 we deduce the existence of an infinite sequence $b_1, b_2, \ldots$ such that each b_j is a certain a_i and the relation

$$(2) \qquad 0 \subset b_\lambda b_{\lambda+i} \cdots b_{\lambda+\varrho}\mathfrak{R} \subseteqq b_\lambda \mathfrak{R}$$

is satisfied for each λ and ϱ. If now $s = 1$ then obviousely a_1 is a potent element and the theorem is proved. If $s > 1$ then by assumption each proper subset of the a_i generates a nilpotent ring, in particular each a_i is nilpotent. In this case let t denote the index of the nilpotent element a_1 and let u be the index of the nilpotent ring which is generated by the finite set $a_2, \ldots, a_s$. It follows easily that in each product of the form $p_{\lambda,\varrho} = b_{\lambda+1} b_{\lambda+\varrho} \cdots b_{\lambda+\varrho}$ where $\varrho > t$, $\varrho > u$ at least one of the factors must be equal to a_1 and at least one of them must be different from a_1, hence, for a certain positive integer k (which can be chosen so that $k \leqq u$) the elements of the infinite sequence $b_k, b_{k+1}, \ldots$ can be joined to finite subsequences $b_k, b_{k+1}, \ldots b_{k+\varrho_1}$; $b_{k+\varrho_1+1}, b_{k+\varrho_2+2}, \ldots, b_{k+\varrho_1+\varrho_2}$; $\ldots$ which have the following properties: The product of the elements of each subsequence, the factors being taken in the written order, has the form $a_1^\sigma a_{i_1}, a_{i_2} \cdots a_{i_\varrho}$, where $i_\lambda \neq 1$, $\lambda = 1, \ldots, \varrho$, hence

$$(3) \qquad \sigma + \varrho < t + u.$$

We denote by p_λ the product which belongs to the λ^{th} subsequence, and thus obtain the infinite sequence

$$(4) \qquad p_1, p_2, \ldots$$

Since the p_λ are of the form $a_1^\sigma a_{i_1} \cdots a_{i_\varrho}$, where the a_{i_λ} are taken from the finite set $a_2, \ldots, a_s$, it follows by (3) that also p_λ belong to a certain finite set which we denote by $\bar{d}_1, \ldots, \bar{d}_m$. Since, further, from the definition of the p_λ it follows easily that each product of the form $p_\lambda p_{\lambda+1} \cdots p_{\lambda+\varrho}$ is equal to a certain $p_{i,k}$, we have

$$(5) \qquad 0 \subset p_\lambda p_{\lambda+1} \cdots p_{\lambda+\varrho}\mathfrak{R} \subseteqq P_\lambda \mathfrak{R} \subseteqq a_1 \mathfrak{R}$$

for arbitrary positive λ and ϱ. The ring generated by the $\bar{d}_\lambda$ is in virtue of (5) potent and evidently a subring of N. Again let

$\bar{a}_1, \ldots, \bar{a}_{\bar{s}}$ be a subset of the $\bar{d}_i$ such that the ring $\bar{N}$ which is generated by the $\bar{a}_i$ — and is therefore a subring of N — is still potent, while s' is of the least possible value. As before follows the existence of a r.h. ideal $\bar{\mathfrak{R}}$ such that the relations (1), in which λ, $\mathfrak{R}$ and N are replaced respectively by λ' $\bar{\mathfrak{R}}$ and $\bar{N}$ where λ' is a suitably chosen integer, are again satisfied. Since $\bar{N} \subseteqq N$ we have $\bar{\mathfrak{R}} = \bar{N}^{\lambda'} S = \bar{N}^{\lambda+\lambda'} S \subseteqq N^{\lambda+\lambda'} S = N^{\lambda} S = \mathfrak{R}$, hence $\bar{a}_i \bar{\mathfrak{R}} \subseteqq \bar{a}_i \mathfrak{R}$; on the other hand, from the definition of the $\bar{a}_i$ follows by (5) that $\bar{a}_i \mathfrak{R} \subseteqq a_1 \mathfrak{R}$, and hence we have

$$(6) \qquad \bar{\mathfrak{R}} = (\bar{a}_1 \bar{\mathfrak{R}}, \ldots, \bar{a}_{\bar{s}} \bar{\mathfrak{R}}) \subseteqq (\bar{a}_1 \mathfrak{R}, \ldots, \bar{a}_{\bar{s}} \mathfrak{R}) \subseteqq a_1 \mathfrak{R}.$$

Since evidently $a_1 \mathfrak{R} \subset \mathfrak{R}$ (otherwise $a_1 \mathfrak{R} = \mathfrak{R}$, i.e. the element a_1 is potent, which contradicts $s > 1$) it follows by (6) that

$$(7) \qquad \bar{\mathfrak{R}} \subset \mathfrak{R}.$$

If now $\bar{s} = 1$, then $\bar{a}_1 \bar{\mathfrak{R}} = \bar{\mathfrak{R}}$, i.e. the element $\bar{a}_1$ is potent and has according to its definition the required form, hence in this case the theorem is proved. In case $\bar{s} > 1$, the process which applied on the r.h. ideal $\mathfrak{R}$ leads to the construction of the r.h. ideal $\mathfrak{R}$ such that $\bar{\mathfrak{R}} \subset \mathfrak{R}$, can be now repeatedly applied on $\bar{\mathfrak{R}}$ and thus in a similar way the r.h. ideal $\bar{\bar{\mathfrak{R}}}$ can be found so that

$$(8) \qquad \mathfrak{R} \supset \bar{\mathfrak{R}} \supset \bar{\bar{\mathfrak{R}}}.$$

Thus each new step either provides a potent element which has the required form, or adds a further r.h. ideal which is a proper subset of the preceding. By the minimum condition it follows therefore that after a finite number of steps a potent element can be found which has the required form.

III. On the divisors of zero in a r.h.m.c.-ring.

Theorem 3. Let S be a r.h.m.c.-ring, S' a subring of S and $\mathfrak{R}$ a r.h. ideal in S; let further T denote the maximal subset of S' satisfying the relation $S'T \subseteqq \mathfrak{R}$ [5]). Then a finite subset $\bar{S}$ of S' can be found so that T is also the maximal subset of S' satisfying the relation $\bar{S}T \subseteqq \mathfrak{R}$.

Proof. Let S_0 be an arbitrary finite subset of S' and T_0 the maximal subset of S' satisfying the relation $S_0 T_0 \subseteqq \mathfrak{R}$. We consider now the set of all r.h. ideals of the form $\mathfrak{R}_0 = (T_0, T_0 S)$,

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[5]) i.e. T is the set of all elements t of S which satisfy the relative $St \subseteqq \mathfrak{R}$.

then in virtue of the minimum condition follows the existence of a certain finite subset $\overline{S} = (a_1, \ldots, a_\lambda)$ of S so that if $\overline{T}$ is the maximal subset of S' satisfying the relation $\overline{S}\overline{T} \subseteqq \mathfrak{R}$, the r.h. ideal $\overline{\mathfrak{R}} = (\overline{T}, \overline{T}S)$ is minimal (i.e. for any r.h. ideal $\mathfrak{R}_0$ of the set we have $\overline{\mathfrak{R}} \not\supset \mathfrak{R}_0$). From $S'T \subseteqq \mathfrak{R}$ follows in particular $\overline{S}T \subseteqq \mathfrak{R}$, i.e. $T \subseteqq \overline{T}$; the theorem will be evidently proved if we show that $T = \overline{T}$. To this end it is evidently sufficient to prove that each element a of S' satisfies the relation $a\overline{T} \subseteqq \mathfrak{R}$. In fact, setting $\overline{\overline{S}} = (a_1, \ldots, a_\lambda, a)$ and denoting by $\overline{\overline{T}}$ the maximal subset of S' satisfying the relation $\overline{\overline{S}}\overline{\overline{T}} \subseteqq \mathfrak{R}$, it follows from $\overline{\overline{S}} \supseteqq \overline{S}$ that $\overline{\overline{T}} \subseteqq \overline{T}$; hence by setting $\overline{\overline{\mathfrak{R}}} = (\overline{\overline{T}}, \overline{\overline{T}}S)$ we have $\overline{\overline{\mathfrak{R}}} \subseteqq \overline{\mathfrak{R}}$ which in view of the minimality of $\overline{\mathfrak{R}}$ implies $\overline{\overline{\mathfrak{R}}} = \overline{\mathfrak{R}}$. Since $\overline{\mathfrak{R}} \supseteqq \overline{T}$, $\overline{\overline{\mathfrak{R}}} \supseteqq \overline{\overline{T}}$, and evidently $\overline{S}\,\overline{\mathfrak{R}} \subseteqq \mathfrak{R}$, $\overline{\overline{S}}\,\overline{\overline{\mathfrak{R}}} \subseteqq \mathfrak{R}$ it follows that $\overline{T} = [\overline{\mathfrak{R}}, S']$, $\overline{\overline{T}} = [\overline{\overline{\mathfrak{R}}}, S']$ which implies $\overline{T} = \overline{\overline{T}}$, i.e. $a\overline{T} \subseteqq \mathfrak{R}$, q.e.d.

Corollary. Evidently the set $\overline{S}$ can be replaced by any finite subset of S' containing $\overline{S}$; in case $S' \neq 0$ it may be therefore assumed that also $\overline{S} \neq 0$.

Theorem 4. Let S be a r.h.m.c.-ring, $S' \neq 0$ a nil-subring of S and T the maximal subset of S' satisfying the relation $S'T = 0$; then $T \neq 0$.

Proof. Applying theorem 3 to the special case $\mathfrak{R} = 0$ we deduce the existence of a finite subset $\overline{S}$ of S' such that the maximal subset T of S' which satisfies the relation $S'T = 0$ is also the maximal subset of S' satisfying the relation $\overline{S}T = 0$ and furthermore (corollary to theorem 3) $\overline{S} \neq 0$. If now N denotes the nil-subring of S which is generated by $\overline{S}$, then (by theorem 2) N is nilpotent and hence denoting by m the index of N we have $N^{m-1} \neq 0$ and $0 = N^m = NN^{m-1} \supseteqq \overline{S}N$, i.e. $N^{m-1} \subseteqq T$ which implies $T \neq 0$, q.e.d.

Theorem 5. Let S be a r.h.m.c.-ring, S' a nil-subring of S, T the maximal subset of S' satisfying the relation $S'T = 0$ and $\overline{T}$ the maximal subset of S' satisfying the relation $S'\overline{T} \subseteqq (T, TS)$. Then if $S' \supset T$ also $\overline{T} \supset T$.

Proof. Applying theorem 3 to the special case $\mathfrak{R} = (T, TS)$ we deduce the existence of a finite subset $\overline{S} = (a_1, \ldots, a_m)$ of S' such that $\overline{T}$ is the maximal subset of S' satisfying the relation $\overline{S}\overline{T} \subseteqq T$. Since for $S' = 0$ the theorem is self evident, we may assume $S' \neq 0$ and hence (by the corollary to theorem 3) also $\overline{S} \neq 0$. From $S' \supset T$ follows the existence of an element a of S

such that $a \in S'$ but $a \notin T$; by the corollary to theorem 3 we may now replace $\overline{S}$ by the set $S^* = (a_1, \ldots, a_m, a)$. Let now N^* be the ring generated by S^* then (by theorem 2) N^* is nilpotent, and if λ is the index of N^*, it follows from $N^{*\lambda} = 0$ that $S^* N^{*\lambda-1} = 0 \subseteqq (T, TS)$, and hence $N^{*\lambda-1} \subseteqq \overline{T}$. Let now r be the smallest positive integer for which $N^{*r} \subseteqq \overline{T}$; then in case $r = 1$ the theorem is true, since then $N^* \subseteqq \overline{T}$ but N^* not $\subseteqq T$ in virtue of $a \in N$, $a \notin T$; in case $r > 1$ we prove that N^r not $\subseteqq T$ and thus complete the proof of the theorem. In fact, from $N^r \subseteqq T$ would follow $NN^{r-1} \subseteqq T$ and hence $N^{r-1} \subseteqq \overline{T}$ in contradiction to the minimality of r.

Corollary. If S' is a nil-subring of S such that $S'^2 = S'$, then $S' = 0$.

Proof. In fact, from $S'\overline{T} \subseteqq (T, TS)$ follows $S'^2\, T \subseteqq (S'T, S'TS) = 0$. Further in virtue of $S'^2 = S'$ we have $S'\overline{T} = 0$, which implies $\overline{T} \subseteqq T$. Since in general $T \subseteqq \overline{T}$, we obtain $T = \overline{T}$, and hence (by the theorem just proved) $S' = T$. From $S'\overline{T} = 0$ follows $S'^2 = 0$, q.e.d.

IV. On the structure of a r.h.m.c.-ring.

The proof of the theorems concerning the structure of a r.h.m.c.-ring which were announced in the introduction follow now easily from the results obtained in the previous sections.

Theorem 6. Each r.h.m.c.-ring S possesses a nilpotent radical.

Proof. In fact, let $\mathfrak{R}$ be the radical of S, i.e. the set of all properly nilpotent elements (see I, 7) of S, then, as it is well known, $\mathfrak{R}$ is a nil-subring (moreover: a nil-ideal) of S. Applying the minimum condition we deduce from $\mathfrak{R} \supseteqq \mathfrak{R}^2 \supseteqq \mathfrak{R}^3 \supseteqq \ldots$ the existence of a positive integer λ such that $\mathfrak{R}^\lambda = \mathfrak{R}^{\lambda+1}$, and hence $\mathfrak{R}^\lambda = (\mathfrak{R}^\lambda)^2$ which by the corollary to theorem 5 implies $\mathfrak{R}^\lambda = 0$, i.e. $\mathfrak{R}$ is nilpotent.

Theorem 7. Each potent r.h. ideal $\mathfrak{R}$ of a r.h.m.c.-ring S possesses a potent element.

Proof. By the minimum condition follows from $\mathfrak{R} \supseteqq \mathfrak{R}^2 \supseteqq \mathfrak{R}^3 \supseteqq \ldots$ as in the proof of theorem 6 the existence of a positive integer λ which satisfies the relation $(\mathfrak{R}^\lambda)^2 = \mathfrak{R}^\lambda$; suppose $\mathfrak{R}^*$, and hence also $\mathfrak{R}^\lambda$ were nil-rings, then by the corollary to theorem 5, $\mathfrak{R}^\lambda$ and hence also $\mathfrak{R}$ were nilpotent, which contradicts the assumption of the theorem.

THEOREM 8. Each potent r.h. ideal $\mathfrak{R}$ of a r.h.m.c.-ring S possesses an idempotent element e (i.e. $e^2=e$, $e\neq 0$).

Proof. By the minimum condition follows the existence of a primitive r.h. ideal $\mathfrak{R}'$ such that $\mathfrak{R}' \subseteqq \mathfrak{R}$. Let r be a potent element of $\mathfrak{R}'$ (theorem 7), then evidently $r\mathfrak{R}' = \mathfrak{R}'$. The argument which now leads to the proof of the theorem is exactly the same as that of theorem 7 in **K**, we merely have to replace the regular r.h. ideals of **K** by the potent r.h. ideals of the present note.

THEOREM 9. If S is a r.h.m.c.-ring, then S is either nilpotent, or it can be represented as a direct sum of primitive r.h. ideals and a nilpotent r.h. ideal.

Proof. If S is nilpotent then the theorem is self evident. If S is potent, then let R_1 be a primitive r.h. ideal of S and e_1 an idempotent element of R_1 (theorem 8); then evidently $R_1 = e_1S$. If $S = \mathfrak{R}_1$ then the theorem is proved. If $S \neq \mathfrak{R}_1$ then the set of all elements of the form $s - e_1s$, where $s \in S$, is a r.h. ideal S_1 which is different from zero and we obtain the representation $S = S_1 + \mathfrak{R}_1$ (direct sum) where $S_1 \subseteqq S$. If S_1 is nilpotent then the theorem is proved: if S_1 is potent, then by similar argument we obtain $S_1 = S_2 + \mathfrak{R}_2$, where $\mathfrak{R}_2$ is a primitive r.h. ideal, S_2 is a potent or a nilpotent r.h. ideal and $S = S_2 + \mathfrak{R}_1 + \mathfrak{R}_2$, $S \supseteqq S_1 \supseteqq S_2$. Hence by the minimum condition we finally obtain the desired representation.

The usual methods lead now from theorems 6 and 9 to

THEOREM 10. Let S be a r.h.m.c.-ring and $\mathfrak{R}$ the nilpotent radical of S, then the quotient-ring $S/\mathfrak{R}$ is semi-simple.

The further theorems describing the structural type of a r.h.m.c.-ring can be now obtained exactly as by Artin, and may be ommitted here.

As a generalisation of the principal theorem of **L** we finally state

THEOREM 12. Each nil-subring S' of a r.h.m.c.-ring S is nilpotent. If further m is the index of the radical R of S and l is the length of the semi-simple ring S/R, then $m + l + 1$ is an upper bound for the indices of the nilpotent subrings of S.

Proof. In fact, let S^* be the homomorphic image of S' in S/R, then S^* is evidently also a nilring. In **L** it was proved that each nil-subring of a ring S which satisfies the minimum and the maximum condition is nilpotent, and if l is the length of S then $l + 1$ is an upper bound for the indices of the nilpotent subrings of S. Since S/R is semi-simple, we may aply the theorem just stated and find that $S^{*^{l+1}}$ is zero in S/R, i.e. $S'^{l+1} \subseteqq R$ and hence $(S'^{l+1})^m = S'^{(l+1)m} = 0$, q.e.d. [6])

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[6]) The present note was sent for publication in October 1938. In December 1938 a note by Charles Hopkins was published on „Nilrings with minimal condition for admissible left ideals" (Duke Math. Journ. 4 (1938), 664—667) in which some of the main results of the present note are proved by a different method. Nevertheless I trust that the present note might be still of some interest since the method used here can be applied also to other interesting classes of rings as I hope to show in a following communication.