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Almost hamiltonian groups [1])

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The elements in a group which transform every subgroup into itself form the norm of the group. The theory of the groups with cyclic norm quotient group has been developed completely in a previous paper [2]). This theory establishes pretty well the relation between the norm of a given group and any element of this group with two essential exceptions: this theory does not give any information, if the norm and the element in question generate together either an abelian group or a hamiltonian group. It is the object of this note to deal with the second of these alternatives under the additional hypothesis that the norm quotient group is abelian. These apparently rather weak assumptions turn out to be very restrictive; and this makes it possible to give a fairly complete theory of this class of groups.

1. The *central* $Z(G)$ consists of all those elements z in a group G which satisfy: $zx = xz$ for every element x in G; and similarly the *norm* $N(G)$ of a group G consists of all those elements g in G which satisfy: $gS = Sg$ for every subgroup S of G.

$Z(G)$ and $N(G)$ are both characteristic (and therefore normal or self-conjugate or invariant) subgroups of the group G and $Z(G)$ is a subgroup of $N(G)$. If norm and central of the group G are different, and if the norm quotient group of G is abelian, then it has been proved [3]) that G is the direct product of its primary components, that [4]) $N(G)$ is the direct product of the

[1]) Presented to the American Math. Soc. November 25/26, 1938.

[2]) R. Baer, Der Kern, eine charakteristische Untergruppe [Comp. Math. 1 (1934), 254—283].

[3]) R. Baer, Gruppen mit vom Zentrum wesentlich verschiedenem Kern und abelscher Faktorgruppe nach dem Kern [Comp. Math. 4 (1936), 1—77], Satz 1, p. 3.

[4]) R. Baer [Comp. Math. 1 (1934)], Satz 4, p. 260.

norms of the primary components, and that consequently it is no loss of generality to assume that the group G is a p-group.

Throughout this paper we shall denote by $\{\ldots\}$ the group which is generated by the enclosed elements and element-sets; and we shall put

$$[x, y] = xyx^{-1}y^{-1}.$$

2. If the group G is a p-group, and if the subgroup N of G is contained in the norm of G and is a normal subgroup of G, then the elements of G may be divided into three classes according to their relation to the distinguished subgroup N.

There are first those elements of G which permute with every element of N. They form the *centralizer* $Z(N < G)$ of N in G. If N is abelian, then these elements z may be characterized by the fact that $\{N, z\}$ is an abelian group.

There are secondly those elements h in G which generate together with N a hamiltonian group. It is known that hamiltonian p-groups are direct products of a quaternion group and of any number of (cyclic) groups of order 2, a quaternion group being generated by two elements u and v which are subject to the relations:

$$u^2 = v^2 = c,\ c^2 = 1,\ uvu^{-1}v^{-1} = c.$$

The elements h so that $\{N, h\}$ is a hamiltonian group form a subset $H(N < G)$ of G which may be void. It will be one of our most fundamental hypotheses to assume that $H(N < G)$ is not vacuous.

If N is an abelian group, then the elements h in $H(N < G)$ have the property: $h^{-1}xh = x^{-1}$ for every x in N. The elements with this last property are said to *invert* N. Since the product of an element in $Z(N < G)$ and of an element which inverts N is itself an element that inverts N, and since the product of two elements which both invert N is an element in $Z(N < G)$, it follows that *the elements in* G *which either invert* N *or belong to* $Z(N < G)$ *form a subgroup* $J(N < G)$ *of* G.

Since we are going to assume that $H(N < G)$ is not vacuous, we may assume without loss of generality that G is a 2-group, i.e. a group all of whose elements are of order a power of 2.

It is known [5]) that N is abelian, if G contains elements which belong neither to $Z(N < G)$ nor to $H(N < G)$. If x is such an

[5]) R. Baer, Gruppen mit hamiltonschen Kern [Comp. Math. 2 (1935), 241—246], Zusatz 3, p. 246.

element, then denote by $2^{n(x)}$ its order and by $2^{a(x)} = 2^{a(N;x)}$ the order of the automorphism which x induces in N. The following properties of such an element x are known: [6])

(2.1) $\{N, x^{2^{a(x)}}\}$ is an abelian group and $x^{2^{a(x)}}$ is an element of maximum order in this group.

(2.2) If g is an element in N, then $x^{-1}gx = g\, x^{2^{n(x)-a(x)}h(g,x)}$ and there exist elements e in N with $h(e, g) = 1$.

(2.3) $1 < n(x) - a(x),\ a(x) \leqq n(x) - a(x)$.

(2.4) $x^{2^{n(x)-a(x)}}$ is an element in N.

3. It will be convenient to introduce the following concept. The subgroup N of the 2-group G is said to be *in hamiltonian situation,* if N and G satisfy the following conditions:

(3.a) N is a normal subgroup of G.

(3.b) N and G/N are both abelian.

(3.c) $H(N < G)$ is not vacuous.

(3.d) $N \leqq N(G)$.

If e.g. G itself is a hamiltonian 2-group, then G contains subgroups which are a direct product of one cyclic group of order 4 and of any (finite or infinite) number of (cyclic) groups of order 2. If N is any such subgroup of G, then N is in hamiltonian situation in G. It is for this very reason that groups which contain subgroups in hamiltonian situation may be called *almost hamiltonian groups.*

It has been proved elsewhere [7]) that the norm of a 2-group is hamiltonian if, and only if, the group is hamiltonian. The norm of a hamiltonian 2-group is therefore not in hamiltonian situation. We shall see later on [8]) that the hamiltonian 2-groups are the only almost hamiltonian groups whose norm is not in hamiltonian situation.

4. It is the object of this section to determine the structure of the subgroups $C(N < G)$ and $J(N < G)$ for subgroups N which are in hamiltonian situation.

LEMMA 4.1: *If the subgroup N of the 2-group G is in hamiltonian situation in G, then*

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[6]) R. BAER [Comp. Math. 1 (1934)], Satz 7, p. 267/268.

[7]) Cp. footnote [5]).

[8]) Theorem 4.4 below.

(I) $C(N<G)$ *is abelian;*
(II) $J(N<G)/C(N<G)$ *is of order* 2;
(III) $J[C(N<G)<G] = J(N<G)$;
(IV) $H(N<G)$ *is the set of all those elements in* $J(N<G)$ *which are not contained in* $C(N<G)$;
(V) $C(N<G)^2 \leqq N$;
(VI) $H(N<G)^2 = N^2$ *is a cyclic group of order* 2 *and* N *is consequently a direct product of one cyclic group of order* 4 *by any (finite or infinite) number of (cyclic) groups of order* 2.

REMARK: There exists — by (VI) — one and only one element different from the group-unit in N^2 and this uniquely determined element will be denoted throughout by c.

PROOF: Since N is in hamiltonian situation in G, N is abelian and $H(N<G)$ is not empty. N is consequently a direct product of one cyclic group of order 4 and of groups of order 2 whose number may be 0, positive finite or infinite. Let u be an element of order 4 in N and v an element in $H(N<G)$. The elements u and v generate a quaternion group and satisfy the relations:

$$u^2 = v^2 = uvu^{-1}v^{-1} = c, \ c^2 = 1$$

where c is the uniquely determined element $\neq 1$ in N^2. Note in particular that v induces an inversion in N.

If w is some element in $C(N<G)$, then v and wv induce the same automorphism in N, namely an inversion. wv is therefore not an element in $C(N<G)$. Since $N \leqq N(G)$, this involves that there are only two possibilities:

Either $\{N, wv\}$ is hamiltonian. This is the case if, and only if, $(wv)^2 = c$.

Or else $\{N, wv\}$ is neither an abelian nor a hamiltonian group. Since the commutators of v and wv with elements in N form the group, generated by c, it follows in this case from (2.4) and (2.3) that there exists a positive integer i so that

$$(wv)^{2^{i+1}} = c.$$

We are going to prove that this second case is void, i.e. that wv belongs always to $H(N<G)$.

To prove this, note firstly that for every element y in $C(N<G)$ there exists a positive number j so that $(yv)^{2^j} = c$ and that therefore in particular $(yv)^2 \neq 1$. Note secondly that for elements y in $C(N<G)$ and not negative integers j we have

$$vy^{j+1} = vyv^{-1}y^{-1}\ yvy^j = [v, y]yvy^j.$$

Since $[v, y]$ is an element in N, it follows that y and $[v, y]$ permute with each other and therefore it follows by complete induction that

$$vy^{j+1} = [v, y]^{j+1} y^{j+1} v.$$

If finally w is an element in $C(N < G)$ and i a positive integer so that

$$(wv)^{2^{i+1}} = c,$$

then

$$c = (wvwv)^{2^i} = (w[v, w]wv^2)^{2^i} = (w[v, w]wc)^{2^i} = \\ = ([v, w]w^2c)^{2^i} = [v, w]^{2^i} w^{2^{i+1}},$$

and since w^{2^i} is an element in $C(N < G)$,

$$1 \neq (w^{2^i} v)^2 = w^{2^i} v w^{2^i} v = w^{2^i} [v, w]^{2^i} w^{2^i} v^2 = \\ = [v, w]^{2^i} w^{2^{i+1}} c = c^2 = 1,$$

and this is impossible. Thus we have proved:

(4.1.1) If v is an element in $H(N < G)$ and w is an element in $C(N < G)$ then $c = (wv)^2$.

This last result is easily transformed into a more convenient form. For, if v is an element in $H(N < G)$ and w an element in $C(N < G)$, then

$$c = (wv)^2 = wvwv = w[v, w]\, wv^2 = w[v, w]\, wc$$

or $w[v, w]\, w = 1$ and consequently $[v, w] = w^{-2}$ or $vwv^{-1} = w^{-1}$. Thus we have proved, since G/N is abelian:

(4.1.2) If v is an element in $H(N < G)$ and w is an element in $C(N < G)$, then wv is an element in $H(N < G)$, $w^2 = [w, v]$ is an element in N and $vwv^{-1} = w^{-1}$.

This last fact shows that v induces in $C(N < G)$ an automorphism and that this automorphism is the inversion. Since an inversion is an automorphism if, and only if, the inverted group is abelian, it follows that $C(N < G)$ is abelian. This proves (I), and (II) is a consequence of the definition of $J(N < G)$ and of the fact the elements in $H(N < G)$ — which exist by our hypothesis — are in $J(N < G)$ but not in $C(N < G)$.

(III) is a consequence of (I) and (4.1.2).

That the elements in $H(N < G)$ are contained in $J(N < G)$, but not in $C(N < G)$, has been remarked before. If conversely the element g in G induces an inversion in N, then g and any element v in $H(N<G)$ induce the same automorphism in N. Hence there exists an element w in $C(N<G)$ so that $g = wv$ and it follows from (4.1.1) that $g^2 = (wv)^2 = c^2$ and that therefore g is an element in $H(N<G)$. This proves (IV).

(V) is a consequence of (4.1.2) and (VI) is a consequence of (4.1.1) and this completes the proof of the Lemma.

Corollary 4.2: *Assume that the subgroup N of the 2-group G is in hamiltonian situation in G.*

(a) *If $N(G)$ is hamiltonian, then $G = N(G) = J(N<G)$.*

(b) *If $N(G)$ is abelian, then $C(N<G) = C[N(G) < G]$ and $J(N<G) = J(N(G) < G)$.*

Proof: If $N(G)$ is hamiltonian, then it is known [9]) that $G = N(G)$ is hamiltonian, since G is a 2-group, and this proves (a).

If $N(G)$ is abelian, then $N(G) \leqq C(N<G)$, since $N \leqq N(G)$ and $C(N<G)$ is the centralizer of N in G. Consequently $N(G) \leqq C[N(G) < G] \leqq C(N<G)$ and this implies $C(N<G) = C[N(G)<G]$, since $C(N<G)$ is abelian — by Lemma 4.1, (I). Since by Lemma 4.1, (III) we have $J(N<G) = J[C(N<G)<G]$, it follows from $N \leqq N(G) \leqq C(N<G)$ that $J(N<G) = J(N(G)<G)$.

Corollary 4.3: *Suppose that the subgroup N of the 2-group G is in hamiltonian situation in G. Then $J(N<G)$ is hamiltonian if, and only if, $C(N<G)^2$ is a cyclic group of order 2.*

Proof: If $C(N<G)^2$ is a cyclic group of order 2, then it follows from Lemma 4.1, (I) that $C(N<G)$ is a direct product of one cyclic group of order 4 and of cyclic groups of order 2, and $J(N<G)$ is hamiltonian as a consequence of Lemma 4.1, (II) and of the fact that $J(N<G)^2 = C(N<G)^2$ by Lemma 4.1, (VI). — That $C(N<G)^2$ is a cyclic group of order 2, if $J(N<G)$ is hamiltonian, is a consequence of Lemma 4.1 and the general structure properties of hamiltonian groups.

Theorem 4.4: *The 2-group G is almost hamiltonian if, and only if, either G is hamiltonian or $N(G)$ is in hamiltonian situation in G.*

Proof: From previous remarks it suffices to show that the norm of an almost hamiltonian, but not hamiltonian 2-group is itself in hamiltonian situation. Since the 2-group G is not hamiltonian, it is known [10]) that its norm $N(G)$ is abelian. Since G

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[9]) Cp. footnote [5]).

[10]) Cp. footnote [5]) and the fact that the norm is either abelian or hamiltonian.

is almost hamiltonian, there exists a normal subgroup N of G so that $N \leqq N(G)$, G/N is abelian and $H(N<G)$ is not vacuous. This implies that $G/N(G)$ is abelian and it follows from Corollary 4.2 that

$$N \leqq N(G) \leqq C[N(G)<G] = \\ = C(N<G) \leqq J[N(G)<G] = J(N<G) \leqq G.$$

If g is any element in $N(G)$ and v an element in $H(N<G)$, then it follows from these inequalities and from Lemma 4.1 that

$$g^{-1}vg = g^{-1}vgv^{-1}v = g^{-2}v$$

and as g is a norm-element, this implies that g^2 is a power of v. Since g is in $C(N<G)$ and since the least power of v, contained in $C(N<G)$, is the second one — by Lemma 4.1 — it follows that g^2 is a power of $v^2 = c$, i.e. that

$$N^2 \leqq N(G)^2 \leqq H(N<G)^2 = N^2 \quad \text{— by Lemma 4.1 —}$$

or $N(G)^2 = N^2 = H(N<G)^2$ is a cyclic group of order 2. Now it follows that $\{N(G), v\}$ for v in $H(N<G)$ is a hamiltonian group and this shows that $N(G)$ is in hamiltonian situation in G.

5. The results of section 4 enable us to give a survey of the almost hamiltonian groups with $J[N(G)<G] = G$.

Theorem 5.1: *Assume that the subgroup N of the 2-group G is in hamiltonian situation in G. Then G is hamiltonian if, and only if,*

(a) $J[N(G)<G] = G$;

(b) $C(N<G)^2$ *is a cyclic group of order* 2.

Proof: The necessity of the conditions is a consequence of Corollary 4.3. — If (b) is satisfied, then it follows from Corollary 4.3 that $J(N<G)$ is hamiltonian. It follows from Corollary 4.2 that $J(N<G) = J[N(G)<G]$ and now it follows from (a) that G is hamiltonian.

Theorem 5.2: *The group C is the subgroup $C[N(G)<G]$ of a suitable 2-group G which is almost hamiltonian, though not hamiltonian and which satisfies $G = J[N(G)<G]$ if, and only if,*

(a) C *is abelian*;

(b) C^4 *contains at most* 2 *elements*;

(c) C^2 *contains at least four elements.*

Proof: Suppose first that the 2-group G is not hamiltonian, though almost hamiltonian and that $G = J[N(G) < G]$. If the subgroup N of G is in hamiltonian situation, then it follows from Corollary 4.2 that $C(N < G) = C[N(G) < G]$ and that $J(N < G) = J[N(G) < G] = G$. It is now a consequence of Lemma 4.1 that $C[N(G) < G]$ is abelian, that $C[N(G) < G]^2 \leqq N$ which proves the necessity of (b); and the necessity of condition (c) is now a consequence of Theorem 5.1.

If conversely the group C satisfies the conditions (a) to (c), then there are two possibilities:

either C is the direct product of a cyclic group Z of order 8 and of an abelian group F so that $F^4 = 1$; or else

C is the direct product of two cyclic groups Z' and Z'' of order 4 and of an abelian group F so that $F^4 = 1$.

In both cases C is an abelian 2-group which contains a subgroup N with the following properties:

(I) N is a direct product of one cyclic group of order 4 and of a group whose elements are of order 1 or 2;

(II) $C^2 \leqq N$.

The uniquely determined element $\neq 1$ in N^2 may be denoted by c.

Let G be the group which is generated by adjoining to C an element v, subject to the relations:

$$v^2 = c,\ vxv^{-1} = x^{-1} \text{ for every } x \text{ in } C.$$

That such a group G exists is a consequence of the facts that C is abelian and that $c = c^{-1}$.

G is not hamiltonian, since G contains either elements of order 8 or more than 2 squares $\neq 1$.

$G = J(N < G)$ is a consequence of the fact that C is abelian and of the definition of G.

The group $\{N, v\}$ is hamiltonian, since it is the direct product of a group whose elements are of order 1 or 2 and of a quaternion group, generated by v and any element u of order 4 in N.

Finally it is possible to represent every element in G in the form xv^i where x is an element in C and $i = 0$ or 1. If y is any element in N, then $y\,xv^i\,y^{-1} = xv^i\,y^{(-1)^i-1}$, since both x and y are elements in the abelian group C and since v induces an inversion in C. If either $i = 0$, or $i = 1$ and $y^2 = 1$, the above formula implies:

$$y\,xv^i\,y^{-1} = xv^i.$$

If $i = 1$ and $y^2 \neq 1$, then $y^2 = c = v^2$ and consequently we find

$$y\,xv\,y^{-1} = xv\,v^{-2} = xv^{-1} = v^{-1}x^{-1} = (xv)^{-1},$$

and this proves that $N \leqq N(G)$. That N is a normal subgroup of G, is obvious, and that G/N is abelian, follows from the fact that $C^2 \leqq N$, and that C^2 is the commutator subgroup of G. Thus it has been proved that N is in hamiltonian situation, and this shows finally that G meets all the requirements of the theorem.

Theorem 5.3: *Suppose that the 2-group G is not hamiltonian, though almost hamiltonian, and that $G = J[N(G) < G]$. Then G and the group H are isomorphic if, and only if,*

(a) *H is not hamiltonian, though almost hamiltonian and satisfies:*

(b) *$C[N(G) < G]$ and $C[N(H) < H]$ are isomorphic,*

(c) *$H = J[N(H) < H]$.*

Proof: It suffices to prove the sufficiency of these conditions. This is a consequence of the fact that G may be generated in adjoining to $C[N(G) < G]$ an element v, subject to the relations:

$vxv^{-1} = x^{-1}$ for every x in $C[N(G) < G]$, v^2 is an element of order 2 in $C[N(G) < G]^2$, and that v^2 is the only element $\neq 1$ in $C[N(G) < G]^4$, if there are such elements;

that H may be generated in adjoining to $C[N(H) < H]$ an element z which satisfies analogous relations as v;

and that there exists by (b) an isomorphism of $C[N(G) < G]$ upon $C[N(H) < H]$, mapping v^2 upon z^2. This isomorphism may clearly be extended to an isomorphism of G upon H, mapping v upon z.

6. The almost hamiltonian 2-groups G which satisfy $G = J[N(G) < G]$ have been completely discussed in the preceding section, and thus we shall assume that $G \neq J[N(G) < G]$. The situation of those elements in G which are not contained in $J[N(G) < G]$ is determined by the following

Lemma 6.1: *If the subgroup N of the 2-group G is in hamiltonian situation in G, if $G \neq J(N < G)$, and if u is an element in N, v an element in $C(N < G)$, w an element in $H(N < G)$ and z an element in G, though not in $J(N < G)$, then*

(I) $G^2 \leqq C(N < G)$, $C(N < G)^4 = 1$;

(II) $z^4 = c$ *(is the uniquely determined element $\neq 1$ in N^2)*;

(III) $[z, u]$ *is in* N^2;

(IV) $[z, v]^2 = 1$, $z\,[z, v] = [z, v]\,z$;

(V) $[z, w]^2 = 1$, $z\,[z, w]\,z^{-1} = [z, w]\,c$.

Proof: It is a consequence of Lemma 4.1 that w induces an inversion in $C(N<G)$ and that $w^2 = c$. Hence

$$czc^{-1} = w\,wzw^{-1}\,w^{-1} = w[w, z]\,zw^{-1} =$$
$$= [z, w]\,wzw^{-1} = [z, w]\,[w, z]\,z = z\,.$$

If f is an element of order 4 in N, then $f^2 = c$ by Lemma 4.1, and therefore

$$z = czc^{-1} = f\,fzf^{-1}f^{-1} = f\,[f, z]\,zf^{-1} = [f, z]\,fzf^{-1} = [f, z]^2\,z,$$

since both f and $[f, z]$ are elements in the abelian group N. This implies

$$[f, z]^2 = 1.$$

If t is an element of order 2 in N, then ztz^{-1} is an element of order 2, and as N is abelian, this implies

$$[u, z]^2 = 1 \text{ for every } u \text{ in } N,$$

since by Lemma 4.1 the orders of the elements in N divide 4. Since z is not contained in $J(N<G)$, and since $N \leqq N(G)$, it follows therefore from (2.2) that z induces in N an automorphism of the exact order 2, and this implies

$$G^2 \leqq C(N<G),$$

as follows from Lemma 4.1.

Since the maximum order of the elements in N is 4, and since the order of the automorphism, induced by z in N, is exactly 2, it follows from (2.1) that the order of z^2 is divisible by 4 and the order of z is therefore divisible by 8.

As $w^2 = c$ and z^2 are both elements in the abelian group $C(N<G)$, and as w induces an inversion in $C(N<G)$ — by Lemma 4.1 — it follows that

$$wz\,[w, z](wz)^{-1} = wzwzw^{-1}z^{-1}z^{-1}w^{-1} = w\,[z, w]\,wz^2w^{-1}z^{-2}w^{-1} =$$
$$= [w, z]\,w^2z^4w^{-2} = [w, z]\,z^4.$$

Since wz is not contained in $J(N<G)$, everything that has been proved for z, may be applied on wz, and consequently we have

$$1 = \left[[z, w], wz\right]^2 = z^8$$

so that the order of z is exactly 8.

This last result implies in particular that the order of wz is 8. Since the order of the automorphism, induced by wz in N, is 2, it follows from (2.2) that the group, generated by all the commutators $[wz, x]$ for x in N, is just the cyclic group of order 2 which is generated by $(wz)^4$. As $z^4 = [wz, [w, z]]$ is an element of order 2 in this group, it follows that

$$z^4 = (wz)^4.$$

Since $[w, z]$, $(zwz^{-1})^2$, z^2 are in the abelian group $C(N < G)$, and since both w and zwz^{-1} are of order 4, it follows that

$$z^4 = (wzwz)^2 = \left([w, z]\, zw^2z\right)^2 = \left([w, z](zwz^{-1})^2 z^2\right)^2 = \\ = [w, z]^2 (zwz^{-1})^4 z^4 = [w, z]^2 z^4$$

or $[w, z]^2 = 1$. Since the commutator of wz and $[w, z]$ is $\neq 1$, $[w, z] \neq 1$ and this shows that the order of $[w, z]$ is exactly 2.

If x is some element in N, then $[wz, x]$ is a power of $(wz)^4$ and $[z, x]$ is a power of z^4, as has been pointed out before. Since $z^4 = (wz)^4$, there exists therefore an integer r so that

$$z^{4r} = [wz, x][z, x] = wzxz^{-1}w^{-1}x^{-1}zxz^{-1}x^{-1} = \\ = w[z, x]xw^{-1}x^{-1}[z, x] = [x, z][w, x][z, x] = [w, x].$$

As N contains elements of order 4, we may in particular choose x as an element of order 4 in N. Since w induces an inversion in N, and since $w^2 = c = x^2$ for elements x of order 4 in N, this implies $[w, x] = c$; and since c as well as z^4 is an element of order 2, it follows now that

$$z^4 = c.$$

This completes the proof of (II) and (V), since z, $[z, w]$ $z[z, w]z^{-1}$ are of order 2, and inversions leave elements of order 2 invariant.

(III) is a consequence of (II) and (2.2).

As v is an element in $C(N < G)$, the elements z and zv induce the same automorphism in N. Since $C(N < G)$ is abelian, the elements z and zv induce even the same automorphism in $C(N < G)$. Thus both z and zv are not contained in $J(N < G)$ and all the previous results may be applied on zv too. Thus

$$c = (zv)^4 = (zvzv)^2 = \left([z, v]\, vz^2v\right)^2 = \left([z, v]\, v^2z^2\right)^2, \quad \text{since } z^2 \text{ is in } C(N < G),$$
$$= [z, v]^2 v^4 z^2 = [z, v]^2 v^4 c \text{ or}$$
$$v^4 = [v, z]^2.$$

Since both z and zw are not contained in $J(N < G)$, this last result may be applied on zw too. Hence

$$v^4 = [v, zw]^2 = (vzwv^{-1}w^{-1}z^{-1})^2 = \left([v, z]\, zvwv^{-1}w^{-1}z^{-1}\right)^2 =$$
$$= \left([v, z]\, zv^2z^{-1}\right)^2, \text{ since } w \text{ inverts the elements } v \text{ in } C(N < G),$$
$$= [v, z]^2\, zv^4z^{-1} = v^4\, zv^4z^{-1}.$$

Hence $1 = zv^4z^{-1}$ and consequently

$$1 = v^4 = [v, z]^2.$$

This completes the proof of (I). To complete the proof of (IV), consider

$$z\,[z, v]\,z^{-1} = z^2vz^{-1}v^{-1}z^{-1} = vzv^{-1}z^{-1} = [v, z] = [z, v],$$

since z^2 is an element of the abelian group $C(N < G)$ and since $[z, v]^2 = 1$. Thus the proof of the lemma is complete.

COROLLARY 6.2: *If G is an almost hamiltonian 2-group, $G \neq J(N(G) < G)$, and if the element z in G is not contained in $J[N(G) < G]$, then*

(I) *z permutes with every element in $C[N(G) < G]^2$;*

(II) *$N(G)$ is the direct product of the subgroup of those elements which permute with z and of the cyclic group of order 2 which is generated by $[w, z]$ for any w in $H(N < G)$.*

REMARK: Throughout this statement it is possible to substitute for $N(G)$ any subgroup N of G which is in hamiltonian situation in G.

PROOF: If v is any element in $C(N < G)$, then

$$zv^2 = zv\,v = [z, v]\,vzv = [z, v]\,v\,[z, v]\,vz = [z, v]^2\,v^2z = v^2z,$$

since $C(N < G)$ is abelian, and Lemma 6.1, (IV) may be applied. This proves the first of our statements and the second one is a consequence of (2.2) and Lemma 6.1, (V) and Corollary 6.2, (I).

COROLLARY 6.3: *If G is an almost hamiltonian 2-group, $G \neq J[N(G) < G]$, and if the element z in G is not contained in $J[N(G) < G]$, then z^2 is an element in the norm of the subgroup $\{J[N(G) < G], z\}$.*

PROOF: Since z^2 is an element in the abelian subgroup $C[N(G) < G]$, it permutes with every element in $\{C[N(G) < G], z\}$. Consequently we need but consider the effect of transformation with z^2 on elements not in $\{C[N(G) < G], z\}$. Such an element is either an element w in $H[N(G) < G]$ or it has the form wz. But

$$z^2wz^{-2} = z^4w = cw = w^3$$

and therefore

$$z^2 wz z^{-2} = z^4 wz = (wz)^4 wz = (wz)^5,$$

as follows from Lemma 6.1.

7. If G is a not-hamiltonian, almost hamiltonian 2-group, then denote by $\mathbf{A}(G)$ the group of those automorphisms which are induced in $C[N(G)<G]$ by the elements of G. It is a consequence of Lemma 4.1, (I), (III), Corollary 4.2 and of Theorem 4.4 that $\mathbf{A}(G)$ is essentially the same as $G/C[N(G)<G]$. The group $\mathbf{A}(G)$ contains always the inversion, and in section 5 those groups G have been surveyed where $\mathbf{A}(G)$ is generated by the inversion. The results of section 6 will permit us to determine all those groups G for which $\mathbf{A}(G)$ is of order 4 or equivalent: $J(N(G)<G)$ is of index 2 in G.

THEOREM 7.1: *Suppose that* $\mathbf{A}$ *is a group of four automorphisms of the group* C. *Then there exists an almost hamiltonian* 2-*group* G *so that*

$$C[N(G)<G] = C \text{ and } \mathbf{A}(G) = \mathbf{A}$$

if, and only if,

(a.1) $\mathbf{A}^2 = 1$;

(a.2) $\mathbf{A}$ *contains the inversion (so that* C *is abelian)*;

(a.3) $1 = (x^{1-\mathfrak{g}})^2 = (x^{1-\mathfrak{g}})^{1-\mathfrak{g}}$ *for every* x *in* C *and every* $\mathfrak{g}$ *in* $\mathbf{A}$;

(b) C *contains an element* f *of order* 4 *and an element* t *of order* 2 *so that* $f^{\mathfrak{g}} = f_{\pm 1}$ *for every* $\mathfrak{g}$ *in* $\mathbf{A}$ *and so that* $t^{1-\mathfrak{g}} = f^2$ *for every* $\mathfrak{g}$ *in* $\mathbf{A}$ *which is different from* 1 *and from the inversion*;

(c) $C^4 = 1$.

PROOF: The necessity of the conditions (a) and (c) is a consequence of Lemma 6.1, (I), (II) and (IV) and the necessity of condition (b) follows from Lemma 6.1, (V) in putting $f = z^2$, $t = [z, w]$ for some w in $H[N(G)<G]$ and some z in G which is not contained in $J[N(G)<G.]$.

If the conditions (a) to (c) are satisfied, then choose an automorphism $\mathbf{k}$ in $\mathbf{A}$ which is neither 1 nor the inversion, and elements f and t in C so that $f^{\mathbf{k}} = f$, $t^{1-\mathbf{k}} = f^2$ and so that f is of order 4 and t of order 2. That this is possible is essentially a consequence of condition (b) and (a.2).

Denote now by G the group which is generated by adjoining to C two elements d and e, subject to the relations:

$$d^2 = f^2,\ e^2 = f,$$
$$d^{-1}xd = x^{-1},\ e^{-1}xe = x^{\mathbf{k}} \text{ for } x \text{ in } C,$$
$$[d, e] = t.$$

That this is in fact an extension of the abelian group C by the abelian group $\mathbf{A}$ which realizes the automorphisms in $\mathbf{A}$, is a consequence of a known theorem [11]) on extensions of abelian groups by abelian groups and of the equations:

$$d^{-1}f^2d = f^2, \text{ since } f^2 \text{ is of order } 2,$$
$$e^{-1}fe = f^{\mathbf{k}} = f \text{ by the choice of } f \text{ and } \mathbf{k},$$
$$f^2ef^{-2}e^{-1} = 1 = [d, e][e, d] = [d, e]\,d\,[d, e]\,d^{-1},$$
$$fdf^{-1}d^{-1} = f^2 = t^{1-\mathbf{k}} = tete^{-1} = [e, d]\,e\,[e, d]\,e^{-1}$$
$$\text{by the choice of } t \text{ and } \mathbf{k}.$$

An element of order 2 in C is left invariant by all the automorphisms in $\mathbf{A}$ if, and only if, it is left invariant by $\mathbf{k}$, since the inversion has exactly the elements of order 2 as fixed elements. Denote by K the subgroup of those elements in C which are of order (1 or) 2 and which are fixed elements for $\mathbf{k}$. It is a consequence of condition (a) that

$$C^2 \leqq K$$

and that K contains therefore all the elements $x^{1-\mathbf{g}}$ for $\mathbf{g}$ in $\mathbf{A}$.

Denote now by N the subgroup of C, generated by K, f and t. This subgroup N of C is a normal subgroup of G which contains all the commutators of elements in G so that G/N is abelian. Since d and f together generate a quaternion group, it follows that $\{N, d\}$ is a hamiltonian subgroup of G.

Since $N = \{K, f, t\}$, it is sufficient for the proof of $N \leqq N(G)$ to show that f, t and every element x in K transform every element in G into a power of itself. The elements in G are of the form yd^ie^j for y in C, i and j each either 0 or 1. Since the elements in K are left invariant by every automorphism of $\mathbf{A}$, K is contained in the central of G and therefore in the norm of G. Furthermore

$$f\,yd^ie^j f^{-1} = yfd^if^{-1}e^j = yd^if^{(-1)^i-1}e^j.$$

This is equal yd^ie^j, if $i = 0$. If $i = 1$, then it is equal to $ydf^{-2}e^j$. This is equal to $(dy)^{-1}$ for $j = 0$, and for $j = 1$ it becomes

[11]) Cp. e.g. R. Baer, Erweiterung von Gruppen und ihren Isomorphismen [Math. Zeitschr. 38 (1934), 375—416], Zusatz, S. 407.

$yde^{-1} = yde^5$. But

$$(yde)^4 = (ydeyde)^2 = (y^{1-\mathbf{k}}dede)^2 = (y^{1-\mathbf{k}}d\,[e, d]\,de^2)^2 = \\ = \left(y^{1-\mathbf{k}}\,[d, e]\,d^2e^2\right)^2 = \left(y^{1-\mathbf{k}}tf^2f\right)^2 = f^2 = e^4$$

as follows from our conditions, and yde^{-3} is therefore $(yde)^5$. Thus f is contained in the norm of G. Finally

$$t\,yd^ie^j\,t^{-1} = yd^ite^jt^{-1} = yd^ie^j\,f^{2j} = yd^ie^{5j}.$$

This is equal yd^i for $j = 0$ and for $j = 1$ it follows again that $e^4 = (yd^ie)^4$ and that our expression equals therefore $(yd^ie)^5$. This completes the proof of $N \leqq N(G)$ and of the fact that N is in hamiltonian situation in G. Hence G meets all the requirements.

For future reference the following fact which has been derived during the proof of the theorem may be stated separately.

COROLLARY 7.2: *If C and $\mathbf{A}$ satisfy the conditions* (a) *to* (c) *of Theorem* 7.1, *if the automorphism* $\mathbf{k}$ *in* $\mathbf{A}$ *is different from the identity and from the inversion, and if f ant t are elements in C so that f is of order* 4, *t of order* 2, $f = f^{\mathbf{k}}$, $t^{1-\mathbf{k}} = f^2$, *then there exists an extension G of C by* $\mathbf{A}$ *which realizes* $\mathbf{A}$ *and which is generated in adjoining to C two elements d and e which are subject to the relations*:

$$d^2 = f^2,\ e^2 = f,$$
$$d^{-1}xd = x^{-1},\ e^{-1}xe = x^{\mathbf{k}} \text{ for } x \text{ in } C,$$
$$[d, e] = t.$$

If K consists of those elements of order 2 *in C which are left invariant by* $\mathbf{k}$ (*and therefore by* $\mathbf{A}$), *then* $N = \{K, f, t\} \leqq N(G)$ *is in hamiltonian situation in* G, $C = C(N < G) = C[N(G) < G]$, $\{C, d\} = J(N < G) = J[N(G) < G]$.

THEOREM 7.3: *Suppose that the* 2*-group G is almost hamiltonian and that $J[N(G) < G]$ is of index* 2 *in G. Then G and the* 2*-group G' are isomorphic if, and only if,*

(a) *G' is almost hamiltonian and $J[N(G') < G']$ is of index* 2 *in G'*;

(b) *there exists an isomorphism of $C[N(G) < G]$ upon $C[N(G') < G']$ which transforms* $\mathbf{A}(G)$ *into* $\mathbf{A}(G')$.

PROOF: The necessity of the conditions being obvious, let us assume therefore that they are satisfied, and that in particular $\mathbf{p}$ is an isomorphism of $C[N(G) < G]$ upon $C[N(G') < G']$ which transforms $\mathbf{A}(G)$ into $\mathbf{A}(G')$.

Let w be some element in $H[N(G) < G]$ and w' an element in

$H[N(G')<G']$. Denote by z some element in G which is not contained in $J[N(G)<G]$ and put $zxz^{-1}=x^{\mathbf{k}}$ for x in $C[N(G)<G]$. Then $\mathbf{k}'=\mathbf{p}^{-1}\mathbf{k}\mathbf{p}$ is an automorphism of $C[N(G')<G']$ which is contained in $\mathbf{A}(G')$ and there exists therefore an element z' in G' so that $z'yz'^{-1}=y^{\mathbf{k}'}$ for y in $C[N(G')<G']$. As $\mathbf{k}$ is different from the identity and from the inversion, the same holds true for $\mathbf{k}'$ and z' is consequently an element which is not contained in $J[N(G')<G']$. Since by condition (a)

$$\begin{array}{ll} G=\{C[N(G)<G],\, w,\, z\} & G'=\{C[N(G')<G'],\, w'\, z'\} \\ w^2, z^2 \text{ in } C[N(G)<G] & w'^2, z'^2 \text{ in } C[N(G')<G'] \\ wxw^{-1}=x^{-1},\ zxz^{-1}=x^{\mathbf{k}} & w'yw'^{-1}=y^{-1},\ z'yz'^{-1}=y^{\mathbf{k}'} \\ \text{for } x \text{ in } C[N(G)<G] & \text{for } y \text{ in } C[N(G')<G'] \\ [w, z] \text{ in } C[N(G)<G] & [w', z'] \text{ in } C[N(G')<G'], \end{array}$$

and since G and G' are completely determined by the above relations, it will be sufficient to prove the following statement:
(7.3.1) *There exists an automorphism* $\mathbf{q}$ *of* $C'=C[N(G')<G']$ *which maps* $w''=w^{2\mathbf{p}}$ *upon* w'^2, $z''=z^{2\mathbf{p}}$ *upon* z'^2, $t''=[w,z]^{\mathbf{p}}$ *upon* $t'=[w',z']$ *and satisfies* $\mathbf{q}\mathbf{k}'=\mathbf{k}'\mathbf{q}$.

For if such an automorphism $\mathbf{q}$ exists, then $\mathbf{pq}$ is an isomorphism of $C[N(G)<G]$ upon $C[N(G')<G']$ which transforms $\mathbf{k}$ into $\mathbf{k}'$ and therefore $\mathbf{A}(G)$ into $\mathbf{A}(G')$ and which maps w^2 upon w'^2, z^2 upon z'^2 and $[w,z]$ upon $[w',z']$ so that it is possible to extend $\mathbf{pq}$ to an isomorphism of G upon G' which maps w upon w' and z upon z'.

Since $w^2=z^4=[w,z]^{1-\mathbf{k}}$ by Lemma 6.1, it follows that

$$w''=z''^2=t''^{1-\mathbf{k}'}$$

and this fact will be used during the proof of (7.3.1).

Another consequence of Lemma 6.1 is that every $y^{1-\mathbf{k}'}$ for y in C' is an element of order 2 or 1 which is left invariant by $\mathbf{k}'$. Since z^2 is left invariant by $\mathbf{k}$, it follows that z'' is left invariant by $\mathbf{k}'$. z'^2 is a fixed-element under $\mathbf{k}'$ and $[w',z']^{1-\mathbf{k}'}=w'^2=z'^4=c'$.

Since both c' and w'' are elements of order 2, they are either equal or independent and we have to distinguish two cases accordingly.

Case 1: $c'=w''$.

Since c' is an element of order 2, it follows that the subgroup of the elements of order 2 in C' is the direct product of $\{c'\}$ and of a suitable group L. The set of all those elements x in C' so that $x^{1-\mathbf{k}'}$ is an element in L is a subgroup K of C'. If y is any element in C', then $y^{1-\mathbf{k}'}$ is an element of order **2** and has therefore the form:

$$y^{1-\mathbf{k}'} = c'^i r \text{ where } i = 0{,}1 \text{ and } r \text{ is in } L.$$

Hence $(yt''^{-i})^{1-\mathbf{k}'} = (y[w', z']^{-i})^{1-\mathbf{k}'} = r$ is an element in L. Since furthermore c' is not contained in L, it follows that neither t'' nor $[w', z']$ is contained in K, and since these elements are of order 2, it follows finally that C' is both the direct product of K and $\{t''\}$ as the direct product of K and of $\{t'\} = \{[w', z']\}$, i.e.

$$C' = \{t''\} \times K = \{t'\} \times K.$$

Both the elements z'' and z'^2 are invariant under $\mathbf{k}'$ and are therefore contained in K. Since they are elements of order four, satisfying $c' = z''^2 = (z'^2)^2$, and since — by Lemma 6.1 — $C'^4 = K^4 = 1$, it follows that there exists a subgroup M of K so that K is the direct product of M and of $\{z''\}$ and so that K is the direct product of M and of $\{z'^2\}$. Thus the following direct decomposition of C' has been derived:

$$C' = \{t''\} \times \{z''\} \times M = \{t'\} \times \{z'^2\} \times M,$$

where $x^{1-\mathbf{k}'}$ is for x in M an element of order 2 in L and where consequently $x^{1-\mathbf{k}'} \neq c'$ for x in M.

Since both t'' and t' are of order 2, and since both z'' and z'^2 are of order 4, there exists therefore a uniquely determined automorphism $\mathbf{q}$ of C' so that

$$t''^{\mathbf{q}} = t',\ z''^{\mathbf{q}} = z'^2,\ x^{\mathbf{q}} = x \text{ for } x \text{ in } M.$$

This automorphism satisfies:

$$w''^{\mathbf{q}} = c'^{\mathbf{q}} = (z''^2)^{\mathbf{q}} = (z'^2)^2 = c' = w'^2$$

and

$$z''^{\mathbf{k}'\mathbf{q}} = z''^{\mathbf{q}} = z'^2 = z'^{2\mathbf{k}'} = z''^{\mathbf{q}\mathbf{k}'},$$

$$t''^{\mathbf{k}'\mathbf{q}} = (c't'')^{\mathbf{q}} = c't' = t'^{\mathbf{k}'} = t''^{\mathbf{q}\mathbf{k}'}.$$

If x is an element in M, then $x^{1-\mathbf{k}'}$ is an element of order 2 and has therefore the form:

$$x^{1-\mathbf{k}'} = t''^i z''^{2j} y = t''^i c'^j y$$

where both i and j are **0** or **1** and where y is in M. Since $x^{1-\mathbf{k}'}$ is invariant under $\mathbf{k}'$, it follows that

$$t''^i c'^j y = x^{1-\mathbf{k}'} = (x^{1-\mathbf{k}'})^{\mathbf{k}'} = t''^i c'^i c'^j y^{\mathbf{k}'} = t''^i c'^j y\, c'^i y^{\mathbf{k}'-1}$$

and that therefore $c'^1 = y^{1-\mathbf{k}'}$. Since y is an element in M, $y^{1-\mathbf{k}'}$ is an element in L and this implies

$$1 = c'^i = y^{1-\mathbf{k}'},\ i = 0.$$

Thus we find finally:

$$x^{\mathbf{k}'\mathbf{q}} = (x\,x^{\mathbf{k}'-1})^{\mathbf{q}} = (x\,c'^j y)^{\mathbf{q}} = xc'^j y = x\,x^{\mathbf{k}'-1} = x^{\mathbf{k}'} = x^{\mathbf{q}\mathbf{k}'},$$

since x, y are in M, since therefore x, y and c' are invariant under $\mathbf{q}$, and since c', y are of order 2. Hence $\mathbf{k}'\mathbf{q} = \mathbf{q}\mathbf{k}'$ and the automorphism $\mathbf{q}$ meets all the requirements of (7.3.1).

Case 2: $c' \neq w''$.

Since $c' = t'^{1-\mathbf{k}'}$ and $w'' = t''^{1-\mathbf{k}'}$ are two independent elements in $C'^{1-\mathbf{k}'}$, and since all the elements in $C'^{1-\mathbf{k}'}$ are of order 2, there exists a subgroup L so that $C'^{1-\mathbf{k}'}$ is the direct product of $\{c'\}$, $\{w''\}$ and L. The set K of all the elements x so that $x^{1-\mathbf{k}'}$ is contained in L forms a subgroup of C'. If y is any element in C', then $y^{1-\mathbf{k}'}$ has the form $c'^i w''^j s$ where i and j are 0 or 1 and where s is an element in L. Consequently

$$(yt'^{-i}t''^{-j})^{1-\mathbf{k}'} = s$$

is an element in L and $yt'^{-i}t''^{-j}$ is an element in K. This proves that

$$C' = \{t', t'', K\}.$$

If $1 = t'^i t''^j s$ where s is an element in K and i and j are 0 or 1, then $1 = (t'^i t''^j s)^{1-\mathbf{k}'} = c'^i w''^j s^{1-\mathbf{k}'}$ where $s^{1-\mathbf{k}'}$ is in L. Hence $i = j = 0$ and $s^{1-\mathbf{k}'} = 1$ [$= s$] and this proves that C' is the direct product of $\{t'\}$, $\{t''\}$ and K.

Since $x^{1-\mathbf{k}'} = 1$ for every element x that is invariant under $\mathbf{k}'$, all the fixed elements of $\mathbf{k}'$ are contained in K. Thus K contains in particular z'', z'^2 and all the elements in L, since $L \leqq C'^{1-\mathbf{k}'}$ and since all the commutators $x^{1-\mathbf{k}'}$ are invariant under $\mathbf{k}'$. Since $z''^2 = w''$, $z'^4 = c'$ are independent elements of order 2, and since the pair w'', c' is independent of L and $L^2 = 1$, there exists a subgroup M of K which contains L so that K is the direct product of $\{z'^2\}$, $\{z''\}$ and M. Thus the following direct decomposition of C' has been derived:

$$C' = \{t'\} \times \{z'^2\} \times \{t''\} \times \{z''\} \times M.$$

Since t' and t'' are of the same order 2, and since z'^2 and z'' are of the same order 4, there exists a uniquely determined automorphism $\mathbf{q}$ of C' so that

$$t'^{\mathbf{q}} = t'',\ t''^{\mathbf{q}} = t',\ (z'^2)^{\mathbf{q}} = z'',\ z''^{\mathbf{q}} = z'^2,\ x^{\mathbf{q}} = x \text{ for } x \text{ in } M.$$

This automorphism $\mathbf{q}$ satisfies:

$$w''^{\mathbf{q}} = (z''^2)^{\mathbf{q}} = (z'^2)^2 = c', \; c'^{\mathbf{q}} = w'',$$

and

$$z''^{\mathbf{k}'\mathbf{q}} = z''^{\mathbf{q}} = z'^2 = (z'^2)^{\mathbf{k}'} = z''^{\mathbf{q}\mathbf{k}'}, \; (z'^2)^{\mathbf{k}'\mathbf{q}} = (z'^2)^{\mathbf{q}\mathbf{k}'}$$
$$t''^{\mathbf{k}'\mathbf{q}} = (t''w'')^{\mathbf{q}} = t'c' = t'^{\mathbf{k}'} = t''^{\mathbf{q}\mathbf{k}'}, \; t'^{\mathbf{k}'\mathbf{q}} = t'^{\mathbf{q}\mathbf{k}'}.$$

If x is any element in M, then $x^{1-\mathbf{k}'}$ is an element in L, since $M \leqq K$; and since $L \leqq M$, this implies that $M^{1-\mathbf{k}'} \leqq M$. Hence we find for elements x in M

$$x^{\mathbf{k}'\mathbf{q}} = (x\,x^{1-\mathbf{k}'})^{\mathbf{q}} = x\,x^{1-\mathbf{k}'} = x^{\mathbf{k}'} = x^{\mathbf{q}\mathbf{k}'}$$

and this shows finally that $\mathbf{k}'\mathbf{q} = \mathbf{q}\mathbf{k}'$ so that $\mathbf{q}$ meets all the requirements of (7.3.1). After what has been remarked before, this completes the proof of the theorem.

8. If G is an almost hamiltonian 2-group, then it has been proved that the elements $\neq 1$ in $G/J[N(G)<G]$ are of order 2 and those groups G where $G/J[N(G)<G]$ is of order 1 or 2 have been discussed completely in the sections 5 and 7.

Lemma 8.1: *If G is an almost hamiltonian 2-group, and if the index of $J[N(G)<G]$ in G is greater than 2, then*

(a) *the order of $G/J[N(G)<G]$ is 4;*

(b) *there exists a „normal'' basis z', z'' of G mod $J[N(G)<G]$ with the following properties:*

(b.I) $z'z''^2 = z''^2z', \; z''z'^2 = z'^2z''$;

(b.II) $[z', z'']^2 = c$ — *where c is the uniquely determined element $\neq 1$ in $N(G)^2$* —;

(b.III) *there exists an element w in $H[N(G)<G]$ so that* $z'[w, z''] = [w, z'']z', \; z''[w, z']z''^{-1} = [w, z']c$;

(b.IV) $z'[z', z'']z'^{-1} = z''[z', z'']z''^{-1} = [z', z'']c$.

Proof: Suppose that w is some element in $H[N(G)<G]$ and that the elements r and s are independent mod $J[N(G)<G]$. Then none of the three elements r, s and rs is contained in $J[N(G)<G]$, and it follows therefore from Lemma 6.1, (II) that

$$c = r^4 = s^4 = (rs)^4 = (rsrs)^2 = \left([r, s]\,sr^2s\right)^2 = \left([r, s]\,sr^2\,s^{-1}\,s^2\right)^2;$$

and since all the three factors in the parenthesis are elements in the abelian group $C[N(G)<G]$, it follows that

$$c = [r, s]^2\,sr^4s^{-1}\,s^4 = [r, s]^2\,s^8 = [r, s]^2$$

and we note:

(8.1.1) If r and s are independent mod $J[N(G)<G]$, then $c = [r, s]^2$.

Assume now that the three elements r, s, t are independent mod $J[N(G)<G]$. Then r and st are independent mod $J[N(G)<G]$ and it follows from (8.1.1) that

$$c = [r, st]^2 = (rstr^{-1}t^{-1}s^{-1})^2 = \left(rsr^{-1}[r, t]\, s^{-1}\right)^2 =$$
$$= \left([r, s]\, s\, [r, t]\, s^{-1}\right)^2 = [r, s]^2 s\, [r, t]^2 s^{-1} = c^2 = 1$$

and this being impossible, (a) is proved.

As r and s are independent mod $J[N(G)<G]$, and as w is an element in $H[N(G)<G]$, the elements w, r, s form a basis of G mod $C[N(G)<G]$. Thus G is an extension of the abelian group $C[N(G)<G]$ by the direct product of the three cyclic groups of order 2 which are generated mod $C[N(G)<G]$ by w, r and s respectively and this extension realizes just the automorphisms of $C[N(G)<G]$ which are induced by w, r and s. Thus it follows from (8.1.1) and from the theorems on extensions of groups [12]) that

$$1 = [w, r]\, s\, [r, w]\, s^{-1},\ [r, s]\, w\, [s, r]\, w^{-1}\, [s, w]\, r\, [w, s]\, r^{-1} =$$
$$= \left[[w, r], s\right] [r, s]^2 \left[[s, w], r\right] = \left[[w, r], s\right] c\left[[s, w], r\right]$$

or

(8.1.2) $c = \left[[w, r], s\right]\left[[s, w], r\right].$

For the same reasons

$$r^2sr^{-2}s^{-1} = [r, s]\, r\, [r, s]\, r^{-1} = [r, s]\, rc\, [s, r]\, r^{-1} = c\, [r, s]\, r\, [s, r]\, r^{-1}$$

or

(8.1.3) $[r^2, s] = c\left[[r, s], r\right];\ [s^2, r] = c\left[[s, r], s\right].$

It is a consequence of Lemma 4.1 that every commutator $\left[[x, y], z\right]$ is a power of c. Hence it follows from (8.1.2) that the notations may be chosen in such a way that

(8.1.2′) $1 = \left[[w, s], r\right];\ c = \left[[w, r], s\right].$

Since $r^4 = s^4 = c$, and since $[r^2, s]$ and $[s^2, r]$ are by (8.1.3) powers of c, it follows that

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[12]) Cp. footnote [11]).

$$(wr)^2 = wrwr = [w, r]\, rw^2r = [w, r]\, rcr = [w, r]\, c\, r^2$$
$$\begin{aligned}[(wr)^2, s] &= (wr)^2 s (wr)^{-2} s^{-1} = [w, r]\, cr^2 s r^{-2} c\, [r, w]\, s^{-1} = \\ &= [w, r]\, c\, [r^2, s]\, sc\, [r, w]\, s^{-1} = [r^2, s]\, [[w, r], s] = \\ &= c\, [r^2, s];\end{aligned}$$
$$[r^2, ws] = r^2 w s r^{-2} s^{-1} w^{-1} = r^2 s r^2 s^{-1} = r^2 s c r^{-2} s^{-1} = c\, [r^2, s];$$

and using these two results we find:

$$[(wr)^2, ws] = c\, [r^2, ws] = c^2\, [r^2, s] = [r^2, s].$$

These formulae are not quite symmetric in r and s, since (8.1.2′) has been used. But following the lines of the above argument, one finds successively:

$$(ws)^2 = [w, s]\, cs^2,$$
$$[(ws)^2, r] = [s^2, r]\, [[w, s], r] = [s^2, r];$$
$$[s^2, wr] = c\, [s^2, r];$$
$$[(ws)^2, wr] = [s^2, wr] = c\, [s^2, r].$$

Considering now the following bases of G mod $C[N(G) < G]$

$$w, r, s; \quad w, wr, s; \quad w, r, ws; \quad w, wr, ws$$

one finds from the above formulae that exactly one of these four bases meets the requirements (b.I) and (b.III) of a normal basis w, z', z'', and it follows from (8.1.1) and (8.1.3) that this basis meets also the requirements (b.II) and (b.IV) of a normal basis. This completes the proof of the lemma.

COROLLARY 8.2: *If G is an almost hamiltonian 2-group so that the index of $J[N(G) < G]$ in G is greater than 2, then the commutator group of G is in hamiltonian situation in G.*

This is a consequence of Lemma 8.1, (b.II).

REMARK 8.3: Suppose that G is an almost hamiltonian 2-group so that the index of $J[N(G) < G]$ in G is greater than 2 and therefore 4. If $\mathbf{p}$ and $\mathbf{q}$ are any two automorphisms in $\mathbf{A}(G)$, then $\mathbf{pq} = \mathbf{qp}$ and therefore

$$x^{(1-\mathbf{p})(1-\mathbf{q})} = x^{1-\mathbf{p}-\mathbf{q}-\mathbf{pq}} = x^{1-\mathbf{q}-\mathbf{p}-\mathbf{qp}} = x^{(1-\mathbf{q})(1-\mathbf{p})}$$

for every x in the abelian group $C[N(G) < G]$. Since all these elements are powers of c, there are exactly two possibilities:

A. $C[N(G) < G]^{(1-\mathbf{p})(1-\mathbf{q})} = 1$ for every pair $\mathbf{p}, \mathbf{q}$ in $\mathbf{A}(G)$ or

B. $C[N(G) < G]^{(1-\mathbf{p})(1-\mathbf{q})} = N(G)^2 = \{c\}$ for a pair $\mathbf{p}, \mathbf{q}$ of elements in $\mathbf{A}(G)$ which forms together with the inversion a basis of $\mathbf{A}(G)$.

$G/C[N(G)<G]$ and $\mathbf{A}(G)$ are essentially the same. A basis of $\mathbf{A}(G)$ may be called admissible, if it contains the inversion; and a basis of $\mathbf{A}(G)$ may be called normal, if it may be represented by a normal basis of $G \bmod C[N(G)<G]$. Every normal basis of $\mathbf{A}(G)$ is admissible and there are 24 admissible bases of $\mathbf{A}(G)$. If G is a group of the above type B, then each admissible basis is normal. But if G is of the above type A, then there are exactly three normal bases of $\mathbf{A}(G)$ and every basis of G mod $J[N(G)<G]$ gives rise to exactly one normal basis of $\mathbf{A}(G)$. The proofs of these facts may be omitted. They will be found by a checking procedure similar to the one, used in the proof of the Lemma 8.1.

If finally $\mathbf{p}$ and $\mathbf{q}$ form a basis of $\mathbf{A}(G)$ mod the inversion, then there exists a direct decomposition

$$N(G) = F \times P \times Q$$

where P and Q are cyclic groups of order 2, $P \times Q$ is a direct factor of $C[N(G)<G]$; and where $\mathbf{p}$ leaves all the elements in $F \times P$ invariant, $\mathbf{q}$ leaves all the elements in $F \times Q$ invariant; and where

$$Q^{1-\mathbf{p}} = P^{1-\mathbf{q}} = N(G)^2 = \{c\}.$$

The proof of this fact is readily derived from the results in section 6. and from the statements in the lemma.

Theorem 8.4: *If G is an almost hamiltonian 2-group so that $G \neq J[N(G)<G]$, then*

(1) $G^4 = N(G)^2 = \{c\}$ *is a cyclic group of order* 2;

(2) $N(G)$ *is the greatest subgroup S of $C[N(G) < G]$ so that $S^{1-\mathbf{k}} \leqq N(G)^2 = G^4$ for every $\mathbf{k}$ in $\mathbf{A}(G)$*;

(3) $N(G)/G^4$ *is the central of G/G^4*;

(4) $G^2 \leqq N(G)$.

Proof: (1) is a consequence of Lemma 6.1 and Lemma 4.1. — The norm of G is abelian and therefore a subgroup of $C[N(G)<G]$. (2) follows from the fact that $w^2 = z^4 = c$ for w in $H[N(G)<G]$ and for z in G, though not in $J[N(G)<G]$. (3) follows from (2) and the fact that $[w, z]$ is not an element in $N(G)^2$, if w is in $H[N(G)<G]$ and z is not in $J[N(G)<G]$, since under these assumptions it follows from Lemma 6.1 that $[z, [w, z]] = c$. — (4) is a consequence of Corollary 6.3 and of (8.1.3).

9. In this section a construction scheme is presented for those almost hamiltonian 2-groups G for which the index of $J[N(G)<G]$ in G is greater than 2 and is therefore 4.

THEOREM 9.1: *Suppose that* $\mathbf{A}$ *is a group of automorphisms of the group* C *and that* s, t', t'', q', q'' *are elements in* C. *Then there exists an almost hamiltonian* 2-*group* G *so that*

(a) *the index of* $J[N(G)<G]$ *in* G *is greater than* 2;

(b) $C[N(G)<G]=C$ *and* $\mathbf{A}(G)=\mathbf{A}$;

(c) *there exists a normal basis* w. z', z'' *of* G *mod* $C[N(G)<G]$ *which satisfies*:

(c′) w *is in* $H[N(G)<G]$;

(c″) $z'^2=q'$, $z''^2=q''$, $[z',z'']=s$, $[w,z']=t'$, $[w,z'']=t''$; *if, and only if,*

(I) $\mathbf{A}$ *is a group of order* 8 *all of whose elements are of order* 2 (*so that* $\mathbf{A}$ *is abelian*) *and the elements* $x^{1-\mathbf{k}}$ *for* $\mathbf{k}$ *in* $\mathbf{A}$ *are of order* 2 *and invariant under* $\mathbf{k}$;

(II) $\mathbf{A}$ *contains the inversion* (*so that* C *is an abelian group*);

(III) $C^4=1$;

(IV) $q'^2=q''^2=s^2$ *is an element* c *of order* 2 *and* t' *and* t'' *are of order* 2;

(V) *there exists a basis* $\mathbf{k}'$, $\mathbf{k}''$ *of* $\mathbf{A}$ *mod the inversion so that*

(V.1) *the elements* q', q'' *are invariant under both* $\mathbf{k}'$ *and* $\mathbf{k}''$,

(V.2) $c=s^{1-\mathbf{k}'}=s^{1-\mathbf{k}''}=t'^{1-\mathbf{k}'}=t''^{1-\mathbf{k}'}=t''^{1-\mathbf{k}''}$, $t'^{\mathbf{k}''}=t'$.

PROOF: The necessity of the conditions follows from Lemma 6.1 and Lemma 8.1, if one chooses as $\mathbf{k}'$ and $\mathbf{k}''$ those automorphisms of $C[N(G)<G]$ which are induced by z' and z'' respectively.

If the conditions (I) to (V) are satisfied, then denote by G the group which is generated in adjoining to C elements w, z', z'' which are subject to the following relations:

$$(\mathrm{R})\quad \begin{cases} wxw^{-1}=x^{-1},\ z'xz'^{-1}=x^{\mathbf{k}'},\ z''xz''^{-1}=x^{\mathbf{k}''} \text{ for } x \text{ in } C,\\ [z',z'']=s,\ [w,z']=t',\ [w',z'']=t'',\\ w^2=c,\ z'^2=q',\ z''^2=q''. \end{cases}$$

We note first that the group G' which is generated in adjoining w and z' only, as well as the group G'' which is generated in adjoining w and z'' only — both subject to the relevant relations in the system (R) — are of the type discussed in Corollary 7.2.

That G is an extension of the abelian group C by the abelian group **A** which realizes the automorphisms of **A**, is a consequence of the theorems on extensions of groups [13]), Corollary 7.2 and of the following equations:

$$[w, z']\,z''\,[z', w]\,z''^{-1}\,[z', z'']\,w\,[z'', z']\,w^{-1}\,[z'', w]\,z'\,[w, z'']\,z'^{-1} = $$
$$= t'\,t'^{\mathbf{k}''}s^2\,t''\,t''^{\mathbf{k}'} = t'^2\,c\,t''^2 c = c^2 = 1 \text{ and}$$
$$[z', z'']\,z'\,[z', z'']\,z'^{-1} = s\,s^{\mathbf{k}'} = s^2 c = c^2 = 1 =$$
$$= q'\,q'^{-\mathbf{k}''} = z'^2 z'' z'^{-2} z''^{-1}$$
$$[z'', z']\,z''\,[z'', z']\,z''^{-1} = 1 = z''^2\,z'\,z''^{-2}\,z'^{-1}.$$

Denote now by K' the set of all the fixed elements of order 2 of $\mathbf{k}'$ and by K'' the set of all the fixed elements of order 2 of $\mathbf{k}''$. The intersection K of K' and K'' contains all the elements of order 2 in C which are fixed elements for every $\mathbf{k}$ in **A**. K contains c and t' is contained in K'', $t't''$ in K'. Hence

$$\{K, t', t''\} = \{K', t'\} = \{K'', t''\}.$$

st'' is invariant under both $\mathbf{k}'$ and $\mathbf{k}''$ and $q'q''$, sq' and sq'' are elements of order 2. Thus $st''q'$, $st''q''$ and $q'q''$ are elements in K. Hence

$$\{K', t', q'\} = \{K'', t'', q''\} = N$$

contains s and N contains every element in C^2, since $C^2 \leqq K$, every element in $C^{1-\mathbf{k}'} \leqq K'$, every element in $C^{1-\mathbf{k}''} \leqq K''$. This implies that G/N is abelian.

$N^2 = \{c\}$ is a cyclic group of order 2, since $c = q'^2 = q''^2$. Consequently

$$\{N, w\} \text{ is a hamiltonian group,}$$

since $\{q', w\}$ is a quaternion group.

Each element of G is contained in at least one of the subgroups G', G'' and $G''' = \{C, z'z'', w\}$ of G. It is a consequence of Corollary 7.2 that $N \leqq N(G')$, $N \leqq N(G'')$. In order to prove $N \leqq N(G)$, it suffices therefore to prove:

every element in N transforms every element of the forms

$$xz'\,z'' \text{ and } xwz'\,z'' \text{ for } x \text{ in } C$$

into a power of itself.

To prove this statement, we note:

$$N^{1 \pm \mathbf{k}'\mathbf{k}''} = \{c\}$$

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[13]) Cp. footnote [11]).

and

$$\begin{aligned}(xz'z'')^4 &= (xz'z''xz'z'')^2 = (x^{1+\mathbf{k}'\mathbf{k}''}z'z''z'z'')^2 = \\ &= (x^{1+\mathbf{k}'\mathbf{k}''}[z', z'']z''z'^2z'')^2 = (x^{1+\mathbf{k}'\mathbf{k}''}[z', z'']z'^2z''^2)^2 = \\ &= s^2q'^2q''^2 = c,\end{aligned}$$

$$\begin{aligned}(xwz'z'')^4 &= (wxz'z''xwz'z'')^2 = (x^{1-\mathbf{k}'\mathbf{k}''}wz'z''wz'z'')^2 = \\ &= (x^{1-\mathbf{k}'\mathbf{k}''}[w, z']z'wz''wz'z'')^2 = \\ &= (x^{1-\mathbf{k}'\mathbf{k}''}[w, z']z'[w, z'']z''w^2z'z'')^2 = \\ &= (x^{1-\mathbf{k}'\mathbf{k}''}[w, z'][w, z'']cz'z''cz'z'')^2 = \\ &= (x^{1-\mathbf{k}'\mathbf{k}''}[w, z'][w, z'']z'z''z'z'')^2 = \\ &= (x^{1-\mathbf{k}'\mathbf{k}''}[w, z'][w, z''][z', z'']z'^2z''^2)^2 = \\ &= t'^2t''^2s^2q'^2q''^2 = c.\end{aligned}$$

If now y is any element in N, then

$$y\,xz'z''\,y^{-1} = y^{1-\mathbf{k}'\mathbf{k}''}xz'z'' = \begin{cases} xz'z'', \text{ if } y^{1-\mathbf{k}'\mathbf{k}''} = 1, \\ (xz'z'')^5, \text{ if } y^{1-\mathbf{k}'\mathbf{k}''} = c, \end{cases}$$

and

$$y\,xwz'z''\,y^{-1} = y^{1+\mathbf{k}'\mathbf{k}''}xwz'z'' = \begin{cases} xwz'z'', \text{ if } y^{1+\mathbf{k}'\mathbf{k}''} = 1, \\ (xwz'z'')^5, \text{ if } y^{1+\mathbf{k}'\mathbf{k}''} = c. \end{cases}$$

This proves the above statement and N is consequently in hamiltonian situation in G. Since $C = C(N < G)$, it follows from Corollary 4.2 that $C = C[N(G) < G]$, since G is certainly not hamiltonian, and now it is readily verified that G meets all the requirements of the theorem.

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