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On the projective theory of spinors

by

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Introduction.

It has been recognised by many writers [1]) that there are in projective relativity three fundamental spinors invariant under those spin-transformations which correspond to the rotations of the local world attached to every point of space-time. The object of this paper is to give a new proof of the existence of these spinors and to deduce consequences thereform, by starting not in the usual way in which the theory of spinors is presented [2]), but with the consideration of a 6-dimensional Euclidean space which is in a relation with the 4-dimensional spin-space similar to that of the well-known Plücker-Klein correspondence. This method of approach is proposed by Schouten and Haantjes [3]), who treat the present problem with the aid of orthogonal systems of reference but obtain final results independent of them. Our main result here is a theorem (Theorem III), from which the existence of one ($\Omega^{A}_{.\bar{B}}$ or $\omega_{A\bar{B}}$) of the above-mentioned spinors is an immediate consequence. After the second spinor (r_{AB}) is introduced to adapt our theory into projective relativity, the third spinor ($\Omega_{A\bar{B}}$ or $\omega^{A}_{.\bar{B}}$) arises correspondingly. Defining the concept of special bivectors in the spin-space, we arrive at the hypercomplex numbers of Dirac with all their properties. Some

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[1]) See, for example, O. VEBLEN, Spinors in projective relativity [Proc. Nat. Acad. Sci. **19** (1933), 979—989], where some literature is also given.

[2]) As a representative of this work we cite here W. PAULI, Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten II [Ann. der Physik **18** (1933), 337—372]. In this paper Pauli, with a remark by Haantjes, proves the existence of two of the three spinors but not the third nor a relation between the two obtained. Veblen, in his paper cited above, mentions without proof all the three spinors and their relations, valid for a 6-dimensional Euclidean space with a specified signature.

[3]) J. A. SCHOUTEN & J. HAANTJES, Konforme Feldtheorie II, R_2 und Spinraum [Ann. R. Scuola Norm. Sup. Pisa **4** (1935), 175—189].

known results concerning the isomorphisms between the spin-space and the local space-time are restated in terms of the quantities which we have found.

1. *Spin-space and R_6.* [4])

In an affine space E_4 an affinor-density is called a *spinor* if it is of weight $\frac{1}{4}(r-s)$, where r denotes the number of contravariant suffices and s that of covariant ones. The law of transformation of a (weighted) affinor is uni-modular when and only when the affinor is a spinor. We shall only consider spinors in E_4 and for the sake of simplicity the words vector, bivector, etc. when referred to E_4 shall all mean spinors with their appropriate weights. The E_4 of all spinors is called the *spin-space*. Complex quantities are not excluded in E_4, so a spinor also has a conjugate weight if it carries both suffices of the first kind ($A, B, \ldots = 1, 2, 3, 4$) and of the second kind ($\bar{A}, \bar{B}, \ldots = \bar{1}, \bar{2}, \bar{3}, \bar{4}$). [5])

In E_4 there already exist two numerical skew spinors g^{ABCD} and g_{ABCD}, defined in every reference-system by

$$g^{1234} = \frac{1}{2}, \qquad g_{1234} = \frac{1}{2}.$$

We shall use these to raise and lower the suffices (two at a time) of bivectors v^{AB}, v_{AB} according to the following way

$$v^{AB} = g^{ABCD} v_{CD}, \qquad v_{AB} = g_{ABCD} v^{CD},$$

and identify the two bivectors thus related by giving them the same central letter.

The totality of bivectors v^{AB} in E_4 constitutes a six-dimensional affine manifold E_6, the six components of v^{AB} being the components of a contravariant vector of E_6 referred to a special reference-system. In a general reference-system of E_6 the components of this vector v^{α} ($\alpha, \beta, \ldots = 0, 1, \ldots, 5$) are linear functions of v^{AB}:

$$v^{\alpha} = \frac{1}{2} \chi^{\alpha}_{.AB} v^{AB}$$

where $\chi^{\alpha}_{.AB}$ are in general complex constants skew-symmetrical

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[4]) Those equations in this section which do not involve the consideration of determinants are already known and are to be found in Schouten and Haantjes' paper as cited above.

[5]) Cf. J. A. Schouten & D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie I [Groningen, 1935], p. 8.

in A, B. We shall suppose that the 6-row determinant Det $(\chi^{\alpha}_{.AB})$, where the upper suffix α denotes the rows in the order $(0, 1, \ldots, 5)$ and the lower pair of suffices AB denotes the columns in the order (23, 31, 12, 14, 24, 34), be different from zero. If we form the quantity

$$\chi^{\alpha AB} = g^{ABCD} \chi^{\alpha}_{.CD}$$

then the 6-row determinant of $\chi^{\alpha AB}$, where the suffices α and AB respectively denote the rows and columns in the same orders as mentioned above, is seen to be

$$\text{Det}\,(\chi^{\alpha AB}) = -\,\text{Det}\,(\chi^{\alpha}_{.AB}).$$

A reciprocal quantity $\chi_{\beta}^{.AB}$ may be defined by the relation

$$\chi^{\alpha}_{.AB}\, \chi_{\beta}^{.AB} = 4A^{\alpha}_{\beta} \tag{1.1}$$

where A^{α}_{β} is the unit-affinor of E_6 and where the numerical factor 4 is inserted on the right in order that the Dirac operators derived later shall satisfy an equation with the right numerical factor. Multiplying (1.1) by $\chi^{\beta}_{.CD}$ and summing for β, we get, since Det $(\chi^{\alpha}_{.AB}) \neq 0$,

$$\chi_{\beta}^{.AB}\, \chi^{\beta}_{.CD} = 4\alpha^{A}_{[C}\, \alpha^{B}_{D]} \tag{1.2}$$

where α^{A}_{B} represents the unit-spinor of E_4. The 6-row determinant Det $(\chi_{\beta}^{.AB})$, with its rows and columns arranged in the orders stated above, is from either (1.1) or (1.2) connected with the Det $(\chi^{\alpha}_{.AB})$ by the relation

$$\text{Det}\,(\chi^{\alpha}_{.AB})\,\text{Det}\,(\chi_{\beta}^{.CD}) = 2^{6}.$$

In the same way if we form the quantity

$$\chi_{\beta AB} = g_{ABCD}\, \chi_{\beta}^{.CD},$$

we find

$$\text{Det}\,(\chi_{\beta AB}) = -\,\text{Det}\,(\chi_{\beta}^{.AB}).$$

On account of (1.2) the correspondence from the bivectors v^{AB} of E_4 to the vectors v^{α} of E_6 is reversible:

$$v^{AB} = \frac{1}{2}\chi_{\beta}^{.AB}\, v^{\beta}.$$

We now introduce a *real* non-singular symmetrical tensor $g_{\alpha\beta}$ into E_6 so that E_6 becomes an Euclidean space R_6 [6]). Let the sign of the determinant $\mathfrak{g} = \text{Det}\,(g_{\alpha\beta})$ be ε. Then ε is -1 or $+1$

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[6]) We also use the term „Euclidean" when $g_{\alpha\beta}$ is not positive definite.

according as the *index* of $g_{\alpha\beta}$ (the number of positive units among the orthogonal components of $g_{\alpha\beta}$) is odd or even. $g^{\alpha\beta}$ being the inverse of $g_{\alpha\beta}$, we shall use $g^{\alpha\beta}$ and $g_{\alpha\beta}$ to raise and lower Greek suffices. It is possible to require that $\chi^{\alpha}_{.AB}$ and $\chi_{\beta}^{\cdot CD}$ shall be obtained one from the other by two different processes of raising and lowering suffices, for then the $\chi^{\alpha}_{.AB}$, whose values are as yet unspecified, are by (1.1) simply a solution of the equation

$$\chi^{\alpha}_{.AB}\,\chi^{\beta AB} = 4g^{\alpha\beta}. \tag{1.3}$$

From this equation the Det $(\chi^{\alpha}_{.AB})$ is completely determined but for the sign:

$$\text{Det}\,(\chi^{\alpha}_{.AB}) = \pm\sqrt{-\frac{2^6}{\mathfrak{g}}} = \pm\frac{2^3}{\sqrt{|\mathfrak{g}|}}\sqrt{-\varepsilon}. \tag{1.4}$$

Hence the equation (1.3), when the g's are considered as given and the χ's as unknowns, admits two classes of solutions: *Two solutions* $\chi^{\alpha}_{.AB}$, $'\chi^{\alpha}_{.AB}$ *belong to the same class or to different classes according as* Det $(\chi^{\alpha}_{.AB})$ *and* Det $('\chi^{\alpha}_{.AB})$ *are equal or opposite.*

Since $\chi^{\alpha}_{.[AB}\,\chi^{\beta}_{.CD]}$ must be proportional to g_{ABCD}, we obtain, after having determined the factor of proportionality by the aid of (1.3),

$$\chi^{\alpha}_{.[AB}\chi^{\beta}_{.CD]} = \frac{2}{3}\,g^{\alpha\beta}\,g_{ABCD}\,.$$

Multiplying this relation by g^{EBCD}, the result may be written

$$\boxed{\chi^{(\alpha}_{.BE}\,\chi^{\beta)AE} = g^{\alpha\beta}\,\alpha^{A}_{B}} \tag{1.5}$$

which will be referred to as the *fundamental equation* in the sense that if $\chi^{\alpha}_{.AB}$ is a solution of (1.3), it is also a solution of (1.5) and conversely. Indeed equation (1.5) comprises all the foregoing relations.

2. *Solutions of the fundamental equation.*

Since (1.5) is equivalent to (1.3), the number of independent equations in the set (1.5) is then $\frac{1}{2}\cdot 6\cdot 7 = 21$. As the number of unknowns $\chi^{\alpha}_{.AB}$ is $6^2 = 36$, the general solution of (1.5) therefore involves $36 - 21 = 15$ arbitrary parameters. We now proceed to establish the

THEOREM I. $\underset{0}{\chi}{}^{\alpha}_{.AB}$ *being any particular solution of* (1.5), *the most general solution of the same class is given by*

$$\chi^{\alpha}_{.AB} = \sigma\,\underset{0}{\chi}{}^{\alpha}_{.CD}\,S^{C}_{.A}\,S^{D}_{.B} \qquad (\sigma = \pm 1) \tag{2.1}$$

where $S^A_{.B}$ *is an arbitrary spinor of* (4-*row*) *determinant* 1.[7])

For the proof we multiply the two sides of (2.1) by g^{EFAB} and note that $S^A_{.B}$ is uni-modular, thus obtaining

$$\chi^{\alpha EF} = \sigma \underset{0}{\chi}{}^{\alpha}_{.CD}\, g^{EFAB}\, S^C_{.A}\, S^D_{.B} = \sigma \underset{0}{\chi}{}^{\alpha}_{.CD}\, g^{CDAB}\, \overset{-1}{S}{}^E_{.A}\, \overset{-1}{S}{}^F_{.B}$$

where $\overset{-1}{S}{}^A_{.B}$ is the inverse of $S^A_{.B}$ satisfying the relations

$$\overset{-1}{S}{}^A_{.E}\, S^E_{.B} = \alpha^A_B\,, \quad S^A_{.E}\, \overset{-1}{S}{}^E_{.B} = \alpha^A_B\,.$$

Hence $\chi^{\alpha AB}$ is of the form

$$\chi^{\alpha AB} = \sigma\, \underset{0}{\chi}{}^{\alpha CD}\, \overset{-1}{S}{}^A_{.C}\, \overset{-1}{S}{}^B_{.D}\,.$$

From this and (2.1) it follows at once that

$$\chi^{(\alpha}_{.BE}\, \chi^{\beta)AE} = \sigma^2\, g^{\alpha\beta}\, \alpha^A_B\,.$$

Hence $\chi^{\alpha}_{.AB}$ as given by (2.1) satisfies (1.5) for $\sigma^2 = 1$. This solution is of the same class as $\underset{0}{\chi}{}^{\alpha}_{.AB}$ since if we take the 6-row determinant on both sides of (2.1), regarding the quantity $2S^{[C}_{.A}\, S^{D]}_{.B}$ (which is skew-symmetrical in A, B as well as in C, D) on the right of (2.1) as a 6-row square matrix with CD and AB indicating the rows and columns in the order (23, 31, 12, 14, 24, 34), we get

$$\mathrm{Det}\,(\chi^{\alpha}_{.AB}) = \mathrm{Det}\,(\underset{0}{\chi}{}^{\alpha}_{.AB})$$

on account of the fact that the (6-row) determinant of $2S^{[C}_{.A}\, S^{D]}_{.B}$ is equal to the cube of the (4-row) determinant of $S^A_{.B}$ and is therefore equal to unity.

Conversely if $\chi^{\alpha}_{.AB}$ is any solution of (1.5) of the same class as $\underset{0}{\chi}{}^{\alpha}_{.AB}$, then by (1.2) we have

$$\chi^{\beta}_{.AB}\, \chi_{\beta}^{.CD} = \underset{0}{\chi}{}^{\beta}_{.AB}\, \underset{0}{\chi}{}_{\beta}^{.CD}.$$

Multiplying this by $\chi^{\alpha}_{.CD}$ the result may be written

$$\chi^{\alpha}_{.AB} = f^{\alpha}_{.\beta}\, \underset{0}{\chi}{}^{\beta}_{.AB}\,,$$

where

$$f^{\alpha}_{.\beta} = \frac{1}{4}\chi^{\alpha}_{.CD}\, \underset{0}{\chi}{}_{\beta}^{.CD}$$

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[7]) Thus 15 of the 16 components of $S^A_{.B}$ are arbitrary. It may be remarked here that if in the proof of the theorem we first leave unspecified the value S of the determinant of $S^A_{.B}$, it can be seen later that there is no loss of generality in putting $S = 1$.

can be verified to be a uni-modular orthogonal transformation with respect to $g_{\alpha\beta}$:

$$\text{Det}\,(f^{\alpha}_{.\beta}) = +1\,, \quad g_{\alpha\beta} f^{\alpha}_{.\gamma} f^{\beta}_{.\delta} = g_{\gamma\delta}\,.$$

It follows from the isomorphism between E_4 and R_6 (see (4.2)) that there exists a uni-modular spinor $S^{A}_{.B}$ such that

$$f^{\alpha}_{.\beta} \underset{0}{\chi}{}^{\beta}_{.AB} = \underset{0}{\chi}{}^{\alpha}_{.CD} S^{C}_{.A} S^{D}_{.B}\,.$$

Hence $\chi^{\alpha}_{.AB}$ is expressible in the form (2.1).

We shall next establish the

THEOREM II. *$\underset{0}{\chi}{}^{\alpha}_{.AB}$ being any particular solution of* (1.5), *the most general solution of different class is given by*

$$\chi^{\alpha}_{.AB} = \sigma \underset{0}{\chi}{}^{\alpha CD} S_{CA} S_{DB} \quad (\sigma = \pm 1) \tag{2.2}$$

where S_{AB} is an arbitrary spinor of (4-row) *determinant* 1.

The verification that $\chi^{\alpha}_{.AB}$ as given by (2.2) is a solution of (1.5) runs in very much the same way as we have done for the preceding theorem. That the solution (2.2) belongs to a different class from $\underset{0}{\chi}{}^{\alpha}_{.AB}$ can be seen by taking the 6-row determinant of both sides of (2.2) and using reasoning similar to the above, the result being

$$\text{Det}\,(\chi^{\alpha}_{.AB}) = \text{Det}\,(\underset{0}{\chi}{}^{\alpha AB}) = -\,\text{Det}\,(\underset{0}{\chi}{}^{\alpha}_{.AB})\,.$$

The converse can also be proved in a way analogous to that for Theorem I, but we shall give the following simpler and more elegant proof suggested to me by Dr. J. Haantjes. If $\underset{0}{S}_{AB}$ is any spinor of determinant 1, then, as we have just seen,

$$\underset{1}{\chi}{}^{\alpha}_{.AB} = \underset{0}{\chi}{}^{\alpha CD} \underset{0}{S}_{CA} \underset{0}{S}_{DB}$$

is a solution of (1.5) of different class from $\underset{0}{\chi}{}^{\alpha}_{.AB}$. Then if $\chi^{\alpha}_{.AB}$ is an arbitrary solution of (1.5) of different class from $\underset{0}{\chi}{}^{\alpha}_{.AB}$, the two solutions $\chi^{\alpha}_{.AB}$, $\underset{1}{\chi}{}^{\alpha}_{.AB}$ are of the same class and therefore according to Theorem I there exists a spinor $\underset{1}{S}{}^{A}_{.B}$ of determinant 1 such that

$$\begin{aligned}
\chi^{\alpha}_{.AB} &= \sigma \underset{1}{\chi}{}^{\alpha}_{.CD} \underset{1}{S}{}^{C}_{.A} \underset{1}{S}{}^{D}_{.B} \\
&= \sigma \underset{0}{\chi}{}^{\alpha CD} \underset{0}{S}_{CE} \underset{1}{S}{}^{E}_{.A} \underset{0}{S}_{DF} \underset{1}{S}{}^{F}_{.B} \\
&= \sigma \underset{0}{\chi}{}^{\alpha CD} S_{CA} S_{DB}\,,
\end{aligned}$$

where

$$S_{AB} = \underset{0}{S}_{AE}\, \underset{1}{S}^{E}_{.B}$$

is a spinor of determinant 1. Hence $\chi^{\alpha}_{.AB}$ is expressible in the form (2.2).

Combining the above two theorems we can state the

THEOREM III. *Let* $\underset{0}{\chi}^{\alpha}_{.AB}$, $\chi^{\alpha}_{.AB}$ *be any two solutions of* (1.5). *Then if* Det $(\chi^{\alpha}_{.AB})$ = Det $(\underset{0}{\chi}^{\alpha}_{.AB})$ *there exists a spinor* $S^{A}_{.B}$ *of unit determinant satisfying* (2.1), *and if* Det $(\chi^{\alpha}_{.AB}) = -$ Det $(\underset{0}{\chi}^{\alpha}_{.AB})$ *there exists a spinor* S_{AB} *of unit determinant satisfying* (2.2).

We now proceed to affirm that equation (2.1) for given $\underset{0}{\chi}^{\alpha}_{.AB}$ and $\chi^{\alpha}_{.AB}$ and for a definite choice of the sign σ determines $S^{A}_{.B}$ but for the factor ± 1. For if $\Sigma^{A}_{.B}$ is another spinor satisfying (2.1) with the same sign σ, than

$$\underset{0}{\chi}^{\alpha}_{.CD}\,(\Sigma^{C}_{.A}\,\Sigma^{D}_{.B} - S^{C}_{.A}\,S^{D}_{.B}) = 0$$

or

$$\Sigma^{[C}_{.A}\,\Sigma^{D]}_{.B} = S^{[C}_{.A}\,S^{D]}_{.B}$$

from which we get

$$\Sigma^{A}_{.B} = \pm\, S^{A}_{.B}\,.$$

We shall leave open the choice of this factor. For the other sign of σ the solution $S^{A}_{.B}$ is simply multiplied by $i = \sqrt{-1}$, since in doing so the unity of the determinant of $S^{A}_{.B}$ is unaffected while the right-hand side of (2.1) is affected by the sign -1. Similarly equation (2.2) for given $\underset{0}{\chi}^{\alpha}_{.AB}$, $\chi^{\alpha}_{.AB}$ and for a definite choice of σ determines S_{AB} but for its sign.

3. *The first fundamental spinor* $\Omega^{A}_{.\overline{B}}$ *or* $\omega_{A\overline{B}}$.

The complex conjugate of $\chi^{\alpha}_{.AB}$ is denoted by $\overline{\chi}^{\alpha}_{.\overline{A}\overline{B}}$. From (1.4) we get, since the complex conjugate of $\sqrt{-\varepsilon}$ is $-\varepsilon\sqrt{-\varepsilon}$,

$$\text{(3.1)} \qquad \operatorname{Det}(\overline{\chi}^{\alpha}_{.\overline{A}\overline{B}}) = -\,\varepsilon \operatorname{Det}(\chi^{\alpha}_{.AB}).$$

Now as the right-hand side of (1.5) is real, by taking the complex conjugate of (1.5) we see that $\overline{\chi}^{\alpha}_{.\overline{A}\overline{B}}$ is a solution of (1.5) whenever $\chi^{\alpha}_{.AB}$ is, and hence by Theorem III and (3.1) we have the [8])

[8]) The equations (3.2), (3.3), (3.4) and (3.5) have been obtained in a different way by Schouten and Haantjes, l.c. pp. 180, 182.

THEOREM IV. *According as ε is -1 or $+1$, there exists a spinor $\Omega^{A}_{.\overline{B}}$ or $\omega_{A\overline{B}}$ of unit determinant satisfying the equation*

$$\boxed{\overline{\chi}^{\alpha}_{.\overline{A}\overline{B}} = \sigma\, \chi^{\alpha}_{.CD}\, \Omega^{C}_{.\overline{A}}\, \Omega^{D}_{.\overline{B}}} \tag{3.2}$$

or $(\sigma = \pm 1)$.

$$\boxed{\overline{\chi}^{\alpha}_{.\overline{A}\overline{B}} = \sigma\, \chi^{\alpha CD}\, \omega_{C\overline{A}}\, \omega_{D\overline{B}}} \tag{3.3}$$

For a definite choice of σ the spinor $\Omega^{A}_{.\overline{B}}$ or $\omega_{A\overline{B}}$ is completely determined by (3.2) or (3.3) but for its sign. According to the case, we shall refer to $\Omega^{A}_{.\overline{B}}$ or $\omega_{A\overline{B}}$ as the *first fundamental spinor*. Its complex conjugate will be denoted by $\overline{\Omega}^{\overline{A}}_{.B}$ or $\overline{\omega}_{\overline{A}B}$, and its inverse by $\overset{-1}{\Omega}{}^{\overline{A}}_{.B}$ or $\overset{-1}{\omega}{}^{\overline{A}B}$. The complex conjugate of its inverse, which is also the inverse of its complex conjugate, will be denoted by $\overset{-1}{\overline{\Omega}}{}^{A}_{.\overline{B}}$ or $\overset{-1}{\overline{\omega}}{}^{A\overline{B}}$.

The complex conjugate of (3.2) is

$$\chi^{\alpha}_{.AB} = \sigma\, \overline{\chi}^{\alpha}_{.\overline{C}\overline{D}}\, \overline{\Omega}^{\overline{C}}_{.A}\, \overline{\Omega}^{\overline{D}}_{.B},$$

or, making use of (3.2) itself,

$$\chi^{\alpha}_{.AB} = \chi^{\alpha}_{.EF}\, \Omega^{E}_{.\overline{C}}\, \Omega^{F}_{.\overline{D}}\, \overline{\Omega}^{\overline{C}}_{.A}\, \overline{\Omega}^{\overline{D}}_{.B},$$

which, when we put $Z^{E}_{.A} = \Omega^{E}_{.\overline{C}}\, \overline{\Omega}^{\overline{C}}_{.A}$, may be written in the form

$$\chi^{\alpha}_{.AB} = \chi^{\alpha}_{.EF}\, Z^{E}_{.A}\, Z^{F}_{.B}$$

from which it follows that $Z^{A}_{.B} = \pm\, \alpha^{A}_{B}$, or

$$\boxed{\overline{\Omega}^{\overline{A}}_{.B} = a\, \overset{-1}{\Omega}{}^{\overline{A}}_{.B}} \quad \text{where } a = \pm 1. \tag{3.4}$$

A hermitian spinor [9]) of the type $\Omega^{A}_{.\overline{B}}$ with the property (3.4) is called *positively* or *negatively invertible* according as a is $+1$ or -1.

Similarly if we take the complex conjugate of (3.3) and make use of (3.3) itself we arrive in an analogous manner at the result

$$\boxed{\overline{\omega}_{\overline{A}B} = b\, \omega_{B\overline{A}}} \quad \text{where } b = \pm 1. \tag{3.5}$$

Thus, according as b is $+1$ or -1, the hermitian spinor $\omega_{A\overline{B}}$ is *symmetrical* or *alternating*.

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[9]) A quantity is called „hermitian" if it carries both barred and unbarred suffices. Here, contrary to the common usage, a hermitian matrix has not any property of symmetry if the word „symmetrical" is not inserted.

4. *Isomorphism between E_4 and R_6.*

The problem of isomorphism between E_4 and R_6 has been treated by Cartan for the cases when the index of $g_{\alpha\beta}$ is 3, 4, 6 [10]), and has been further studied by Schouten and Haantjes for any index [11]). In this section we re-establish their results in a simple presentation, in terms of what we have just found.

The equation

$$v^\alpha = \frac{1}{2}\chi^\alpha_{.AB}\, v^{AB} \tag{4.1}$$

establishes a (1-1) correspondence between the ∞^{12} real and complex bivectors v^{AB} of E_4 and the ∞^{12} real and complex vectors v^α of R_6. If v^{AB} is subject to a spin-transformation $T^A_{.B}$:

$$'v^{AB} = v^{CD}\, T^A_{.C}\, T^B_{.D},$$

we find from (4.1) that v^α is subject to the induced transformation

$$'v^\alpha = l^\alpha_{.\beta}\, v^\beta,$$

where

$$l^\alpha_{.\beta} = \frac{1}{4}\chi^\alpha_{.AB}\, T^A_{.C}\, T^B_{.D}\, \chi_\beta^{.CD}. \tag{4.2}$$

From (4.2) it is easily found that

$$\mathrm{Det}\ (l^\alpha_{.\beta}) = [\mathrm{Det}\ (T^A_{.B})]^3\,,$$

$$g^{\gamma\delta}\, l^\alpha_{.\gamma}\, l^\beta_{.\delta} = g^{\alpha\beta}\, \mathrm{Det}\ (T^A_{.B})\,.$$

Hence $l^\alpha_{.\beta}$ is a rotation of R_6 (orthogonal transformation with determinant $+1$) if we assume the condition

$$\mathrm{Det}\ (T^A_{.B}) = 1. \tag{4.3}$$

Subject to this condition, $T^A_{.B}$ is a 15-parameter group of ∞^{30} real and complex spin-transformations. By (4.2) this group is doubly isomorphic with the 15-parameter group of all the ∞^{30} real and complex rotations of R_6.

In order that the vector v^α of R_6 given by (4.1) be real, i.e. $\bar{v}^\alpha = v^\alpha$, the bivector v^{AB} must satisfy the condition

$$\bar{\chi}^\alpha_{.\bar{A}\bar{B}}\, \bar{v}^{\bar{A}\bar{B}} = \chi^\alpha_{.AB} v^{AB}$$

which, when the χ's are eliminated by the aid of (3.2) or (3.3), takes one of the following forms

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[10]) E. Cartan, Les groupes simples, finis et continus [Ann. Ecole norm. sup **31** (1914), 263—355], particularly p. 354.

[11]) Schouten and Haantjes, l.c. pp. 181, 183.

$$v^{AB} = \sigma \bar{v}^{\bar{C}\bar{D}} \Omega^{A}_{.\bar{C}} \Omega^{B}_{.\bar{D}} \quad \text{for } \varepsilon = -1, \tag{4.4}$$

$$v_{AB} = \sigma \bar{v}^{\bar{C}\bar{D}} \omega_{A\bar{C}} \omega_{B\bar{D}} \quad \text{for } \varepsilon = +1. \tag{4.5}$$

Because of (3.4) or (3.5), the complex conjugate of (4.4) or (4.5) is identical with itself. Hence either (4.4) or (4.5) furnishes only 6 independent equations for the restriction of v^{AB}. With this restriction, v^{AB} represents ∞^6 real and complex bivectors of E_4, in (1-1) correspondence with all the ∞^6 real vectors of R_6.

In order that the rotation $l^{\alpha}_{.\beta}$ of R_6 given by (4.2) be real, the right-hand side of (4.2) must be equal to its complex conjugate, a fact requiring that $T^{A}_{.B}$ must satisfy one of the two relations

$$\Omega^{A}_{.\bar{B}} = \pm \overset{-1}{T}{}^{A}_{.C} \Omega^{C}_{.\bar{D}} \overline{T}{}^{\bar{D}}_{.\bar{B}} \quad \text{for } \varepsilon = -1 \tag{4.6}$$

$$\omega_{A\bar{B}} = \pm T^{C}_{.A} \omega_{C\bar{D}} \overline{T}{}^{\bar{D}}_{.\bar{B}} \quad \text{for } \varepsilon = +1 \tag{4.7}$$

which express that $T^{A}_{.B}$ leaves $\Omega^{A}_{.\bar{B}}$ or $\omega_{A\bar{B}}$ invariant but for the factor ± 1. Either (4.6) or (4.7) defines the group property of $T^{A}_{.B}$. The complex conjugate of (4.6) or (4.7) is identical with itself and therefore either (4.6) or (4.7) furnishes only 16 equations for the restriction of $T^{A}_{.B}$. As a consequence of these latter we have from (4.6) that the Det $(T^{A}_{.B})$ is real or from (4.7) that the absolute value of Det $(T^{A}_{.B})$ is 1. This reduces (4.3) to a single equation since the complex conjugate of (4.3) is either unpermitted or superfluous. Thus $T^{A}_{.B}$ is in either case a group of ∞^{15} real and complex spin-transformations doubly isomorphic with the group of all the ∞^{15} real rotations of R_6.

Thus we have shown how the *real* space R_6 and its *real* geometric objects (vectors, rotations, etc.) can be represented by corresponding images in E_4 satisfying invariant conditions. These images in E_4 are in general complex since reality in E_4 is not a property which is invariant. In the following it is understood that R_6 is real and everything in it is real.

5. *The R_5 of projective relativity and the second and third fundamental spinors.*

In projective relativity we have at each point of space-time a complex spin-space E_4 and a real projective space P_4 with a hyperquadric [12]). Analytically the real P_4 with a hyperquadric

[12]) See J. A. Schouten, La théorie projective de la relativité [Ann. Institut H. Poincaré 5 (1935), 51—58].

is equivalent to a real Euclidean space R_5 [13]), and so the geometry of P_4 and E_4 can be studied on a fixed hyperplane R_5 of R_6. Without loss of generality we may suppose that this hyperplane passes through the origin of R_6 and therefore determines a covariant vector r_β of R_6 to within a non-zero factor. As R_5 must be a proper Euclidean space, it is not tangent to the null-cone of R_6; in other words r_β is not a null-vector and we can normalise it such that

$$g^{\alpha\beta} r_\alpha r_\beta = -4 \tag{5.1}$$

after which r_β is determined but for the sign. The normal of R_5 in R_6 is represented by the vector $r^\alpha = g^{\alpha\beta} r_\beta$ which, by the inverse of (4.1),

$$r^{AB} = \frac{1}{2} \chi_\beta^{\cdot AB} r^\beta, \tag{5.2}$$

corresponds to a fixed spin-bivector r^{AB} determined but for the sign. This we call the *second fundamental spinor* [14]). From (5.1) we have

$$r^{AB} r_{AB} = -4 \tag{5.3}$$

from which it follows that *the* 4*-row determinant of r^{AB} or r_{AB} is unity*. Since $r^{[AB} r^{CD]}$ must be proportional to g^{ABCD}, we obtain, after having determined the factor of proportionality by the aid of (5.3),

$$r^{[AB} r^{CD]} = -\frac{2}{3} g^{ABCD}.$$

Multiplying this relation by g_{EBCD}, the result may be written

$$r^{AE} r_{EB} = \alpha_B^A \tag{5.4}$$

showing that r_{AB} $(= g_{ABCD} r^{CD})$ *is at the same time the inverse* of r^{AB}.

As r^α is real, we have from (4.4) or (4.5) that r^{AB} is subject to the condition

$$\boxed{r^{AB} = \sigma \bar{r}^{\bar{C}\bar{D}} \Omega^A_{\cdot \bar{C}} \Omega^B_{\cdot \bar{D}}} \quad \text{for } \varepsilon = -1, \tag{5.5}$$

$$\boxed{r_{AB} = \sigma \bar{r}^{\bar{C}\bar{D}} \omega_{A\bar{C}} \omega_{B\bar{D}}} \quad \text{for } \varepsilon = +1. \tag{5.6}$$

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[13]) The straight lines of R_5 passing through the origin are regarded as the points of P_4 and the origin of R_5 is excluded from this representation.

[14]) Cf. J. A. Schouten, Raumzeit und Spinraum [Zeitschrift für Physik **81** (1933), 405—417], particularly p. 409, where he obtains this spinor by fixing a definite reference-system in R_5.

In order to give an unambiguous relation to the first and second fundamental spinors, we shall hereafter take

(5.7) $$\sigma = +1$$

so that (3.2), (3.3), (5.5), (5.6) become

$$\left.\begin{aligned} &(3.2)' && \bar{\chi}^{\alpha}_{.\bar{A}\bar{B}} = \chi^{\alpha}_{.CD}\,\Omega^{C}_{.\bar{A}}\,\Omega^{D}_{.\bar{B}} \\ &(5.5)' && r^{AB} = \bar{r}^{\bar{C}\bar{D}}\,\Omega^{A}_{.\bar{C}}\,\Omega^{B}_{.\bar{D}} \end{aligned}\right\} \text{ for } \varepsilon = -1,$$

$$\left.\begin{aligned} &(3.3)' && \bar{\chi}^{\alpha}_{.\bar{A}\bar{B}} = \chi^{\alpha CD}\,\omega_{C\bar{A}}\,\omega_{D\bar{B}} \\ &(5.6)' && r_{AB} = \bar{r}^{\bar{C}\bar{D}}\,\omega_{A\bar{C}}\,\omega_{B\bar{D}} \end{aligned}\right\} \text{ for } \varepsilon = +1.$$

In the case $\varepsilon = -1$, if we define the *third fundamental spinor*

(5.8) $$\Omega_{A\bar{B}} = r_{AE}\,\Omega^{E}_{.\bar{B}},$$

then on account of (3.4) the relation (5.5)′ is equivalent to

(5.9) $$\boxed{\bar{\Omega}_{\bar{A}B} = -a\,\Omega_{B\bar{A}}}\,.$$

Hence $\Omega_{A\bar{B}}$ *is hermitian symmetrical or alternating according as* $\Omega^{A}_{.\bar{B}}$ *is negatively or positively invertible.*

In the case $\varepsilon = +1$ the *third fundamental spinor* is defined by

(5.10) $$\omega^{A}_{.\bar{B}} = r^{AE}\,\omega_{E\bar{B}}\,.$$

Then on account of (3.5) the relation (5.6)′ is equivalent to

(5.11) $$\boxed{\bar{\omega}^{\bar{A}}_{.B} = +b\,\overset{-1}{\omega}{}^{\bar{A}}_{.B}}\,.$$

Hence $\omega^{A}_{.\bar{B}}$ *is positively or negatively invertible according as* $\omega_{A\bar{B}}$ *is hermitian symmetrical or alternating.*

6. *Special spin-bivectors.*

In order that the vector v^{α} corresponding to the bivector v^{AB} by (4.1) lies in R_5 ($r_{\beta}v^{\beta} = 0$), it is necessary and sufficient that v^{AB} satisfies the condition

(6.1) $$r_{AB}\,v^{AB} = 0.$$

Those bivectors of E_4, which are in involution with the fixed bivector r^{AB} by (6.1), are called *special* bivectors; these and only these correspond to vectors of R_6 lying in R_5. With the exception of r^{AB}, only special bivectors in E_4 will be considered. Of course these special bivectors are subject to the condition (4.4) or (4.5), for the corresponding vectors in R_5 are real. The complex conjugate of (6.1) is, in consequence of (4.4) and (5.5), or alternatively of (4.5) and (5.6), identical with itself. Thus (6.1) consists of only one equation and hence the v^{AB} subject to all the above-mentioned conditions are ∞^5 real and complex bivectors of E_4, in (1-1) correspondence with all the ∞^5 real vectors of R_5.

We shall use r^{AB} and r_{AB} to raise and lower single spin-suffices, and in consequence of (5.4) always sum with respect to the second suffix of r^{AB}, r_{AB}. Thus if we raise a suffix and then lower it again we get the original suffix with no change of sign. For transvection we have $v^A w_A = -v_A w^A$.

If we multiply the relation (cf. § 5)

$$g_{ABCD} = -\frac{3}{2} r_{[AB} r_{CD]}$$

by v^{CD}, we obtain

$$g_{ABCD}\, v^{CD} = r_{AC} r_{BD} v^{CD} - \frac{1}{2} r_{AB} r_{CD} v^{CD},$$

which reduces to

$$g_{ABCD}\, v^{CD} = r_{AC} r_{BD}\, v^{CD} \tag{6.2}$$

when and only when v^{CD} is special. Thus *the process of lowering (raising) pairs of suffices by* g_{ABCD} (g^{ABCD}) *and that of lowering (raising) single suffices by* r_{AB} (r^{AB}) *are for special bivectors and for special bivectors alone equivalent.*

7. *The fundamental tensor of* R_5.

Imagine in R_5 any system of coordinates $x^\varkappa$ ($\varkappa, \lambda, \ldots = 0, 1, \ldots, 4$), Cartesian or curvilinear. Then along R_5 the Cartesian coordinates X^α of R_6 are functions of $x^\varkappa$ satisfying the equation

$$r_\alpha X^\alpha = 0.$$

Differentiating this relation with respect to x^λ we get, since r_α is constant,

$$r_\alpha B^\alpha_\lambda = 0, \tag{7.1}$$

where $B^\alpha_\lambda = \dfrac{\partial X^\alpha}{\partial x^\lambda}$ is the connection-affinor between R_6 and R_5[15]).

[15]) Cf. Schouten & Struik, l.c. p. 90.

A reciprocal quantity $B^{\varkappa}_{\beta}$ is uniquely defined by the equations

$$B^{\varkappa}_{\beta} B^{\beta}_{\lambda} = B^{\varkappa}_{\lambda}, \quad B^{\varkappa}_{\beta} r^{\beta} = 0 \tag{7.2}$$

where $B^{\varkappa}_{\lambda}$ is the unit-affinor of R_5. By definition the unit-affinor $B^{\varkappa}_{\lambda}$ of R_5 written in the suffices of R_6 becomes

$$B^{\alpha}_{\beta} \equiv B^{\alpha}_{\lambda} B^{\lambda}_{\beta}, \tag{7.3}$$

which is related to the unit-affinor A^{α}_{β} of R_6 by the formula

$$A^{\alpha}_{\beta} = B^{\alpha}_{\beta} - \frac{1}{4} r^{\alpha} r_{\beta} \text{ [16]}. \tag{7.4}$$

By means of B^{α}_{λ} and $B^{\varkappa}_{\beta}$, the fundamental metric tensor $G_{\lambda\varkappa}$ of R_5 and its inverse $G^{\varkappa\lambda}$ are respectively defined by

$$\begin{cases} G_{\lambda\varkappa} = g_{\beta\alpha} B^{\beta}_{\lambda} B^{\alpha}_{\varkappa}, \\ G^{\varkappa\lambda} = g^{\alpha\beta} B^{\varkappa}_{\alpha} B^{\lambda}_{\beta}. \end{cases} \tag{7.5}$$

We shall also use these to lower and raise suffices of R_5. It turns out that $B^{\varkappa}_{\beta}$ [as originally defined by (7.2)] *is merely the quantity obtained from* B^{α}_{λ} *by raising and lowering its suffices*. Thus we may write (7.4) in the form

$$g_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{4} r_{\alpha} r_{\beta}, \tag{7.6}$$

where $G_{\alpha\beta} = G_{\varkappa\lambda} B^{\varkappa}_{\alpha} B^{\lambda}_{\beta}$ is the fundamental tensor of R_5 written in the suffices of R_6. For an orthogonal system of reference in R_5, (7.6) shows that the fundamental tensor of R_6 has the diagonal form

$$g_{\alpha\beta}: \quad \left(\begin{array}{ccc:c} G_{00} & & 0 & \\ & \ddots & & 0 \\ 0 & & G_{44} & \\ \hdashline & 0 & & -1 \end{array}\right),$$

from which it follows that, if η is the sign of Det $(G_{\varkappa\lambda})$, the sign ε of Det $(g_{\alpha\beta})$ is $\varepsilon = -\eta$. In projective relativity there are two possible signatures of $G_{\varkappa\lambda}$, namely $(----+)$ and $(+---+)$ [17]. The first signature $(----+)$ corresponds to $\eta = +1$, $\varepsilon = -1$ and hence we have the three fundamental spinors $\Omega^{A}_{\cdot\bar{B}}$, r_{AB}, $\Omega_{A\bar{B}}$. As the quantum theory naturally prefers $\Omega_{A\bar{B}}$ to be

[16]) For the unit-affinor of R_6 must decompose into two parts $A^{\alpha}_{\beta} = B^{\alpha}_{\beta} + \varrho r^{\alpha} r_{\beta}$, one belonging to R_5 and the other to the normal of R_5. By contraction we have $6 = 5 - 4\varrho$.

[17]) See SCHOUTEN [Footnote [12]) above], p. 66.

hermitian symmetrical instead of alternating [18]), we see from (5.9) that $a = -1$ and hence from (3.4) that $\Omega^{A}_{.\bar{B}}$ is negatively invertible. The second signature $(+ - - - +)$ corresponds to $\eta = -1$, $\varepsilon = +1$ and hence the three fundamental spinors are $\omega_{A\bar{B}}$, r_{AB}, $\omega^{A}_{.\bar{B}}$. When we require that $\omega_{A\bar{B}}$ is hermitian symmetrical, we have from (3.5) that $b = +1$ and hence from (5.11) that $\omega^{A}_{.\bar{B}}$ is positively invertible.

8. *The Dirac operators.*

From the six bivectors $\chi^{\alpha}_{.AB}$ with the upper suffix referred to R_6 we form the five bivectors

(8.1) $$\alpha^{\varkappa}_{.AB} = B^{\varkappa}_{\alpha}\, \chi^{\alpha}_{.AB}$$

with the upper suffix referred to R_5. By the second of (7.2) it follows that the $\alpha^{\varkappa}_{.AB}$ are *special bivectors*:

(8.2) $$\alpha^{\varkappa}_{.AB}\, r^{AB} = 0.$$

Owing to this property a single spin-suffix in $\alpha^{\varkappa}_{.AB}$ can be raised by r^{AB} [19]) and thus we obtain five contra-co-variant spinors $\alpha^{\varkappa A}_{..B}$ each of whose matrices has a zero *spur* ($\alpha^{\varkappa E}_{..E} = 0$) in consequence of (8.2). If we multiply the fundamental equation (1.5) by $B^{\varkappa}_{\alpha}\, B^{\lambda}_{\beta}$ we obtain

(8.3) $$\alpha^{(\varkappa|A|}_{..E}\, \alpha^{\lambda)E}_{..B} = G^{\varkappa\lambda}\, \alpha^{A}_{B}$$

or, in matrix notation,

$$\alpha^{(\varkappa}\, \alpha^{\lambda)} = G^{\varkappa\lambda},$$

showing that the five matrices $\alpha^{\varkappa}$ have all the properties of the Dirac operators.

From (1.1) and (1.2) we easily deduce the following relations

(8.4) $$\alpha^{\varkappa}_{.AB}\, \alpha_{\lambda}^{;AB} = 4 B^{\varkappa}_{\lambda},$$

(8.5) $$\alpha_{\lambda}^{;AB}\, \alpha^{\lambda}_{.CD} = 4 \alpha^{A}_{[C}\, \alpha^{B}_{D]} + r^{AB}\, r_{CD}.$$

Let us form the expression

(8.6) $$W^{\varkappa\lambda}_{..AB} = \alpha^{[\varkappa}_{.AC}\, \alpha^{\lambda]}_{.BD}\, r^{CD}.$$

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[18]) A hermitian symmetrical quantity multiplied by a pure imaginary number becomes hermitian alternating, and vice versa. But a positively invertible quantity cannot be changed into a negatively invertible quantity by multiplication with any number.

[19]) On the other hand the quantities $\chi^{\alpha A}_{..B}$ have no meaning since the six bivectors $\chi^{\alpha}_{.AB}$ are not special.

This quantity is skew in $\varkappa$, λ and symmetrical in A, B and hence its independent components may be regarded as forming a 10-row square matrix. Lowering its upper suffices by $G_{\lambda\varkappa}$ and raising its lower suffices by r^{AB} we obtain

$$W_{\varkappa\lambda}^{\cdot\cdot AB} = - \alpha_{[\varkappa}^{\cdot AC} \alpha_{\lambda]}^{\cdot BD} r_{CD}. \tag{8.7}$$

With the aid of (8.5) it is easy to verify the relation

$$W^{\varkappa\lambda}_{\cdot\cdot CD} W_{\varkappa\lambda}^{\cdot\cdot AB} = 8\, \alpha^A_{(C}\, \alpha^B_{D)} \tag{8.8}$$

from which we see that the 10-row determinants of both $W^{\varkappa\lambda}_{\cdot\cdot AB}$ and $W_{\varkappa\lambda}^{\cdot\cdot AB}$ do not vanish. On account of this property, if we multiply (8.8) by $W_{\mu\nu}^{\cdot\cdot CD}$, the result can be written

$$W^{\varkappa\lambda}_{\cdot\cdot CD}\, W_{\mu\gamma}^{\cdot\cdot CD} = 8\, B^{\varkappa}_{[\mu}\, B^{\lambda}_{\gamma]}\,. \tag{8.9}$$

By means of $W^{\varkappa\lambda}_{\cdot\cdot AB}$ and $W_{\varkappa\lambda}^{\cdot\cdot AB}$ there exists a (1-1) correspondence between the symmetrical spinors ψ^{AB} of E_4 and the bivectors $\psi^{\varkappa\lambda}$ of R_5:

$$\psi^{\varkappa\lambda} = \frac{1}{\sqrt{8}}\, W^{\varkappa\lambda}_{\cdot\cdot AB}\, \psi^{AB}, \tag{8.10}$$

$$\psi^{AB} = \frac{1}{\sqrt{8}}\, W_{\varkappa\lambda}^{\cdot\cdot AB}\, \psi^{\varkappa\lambda}, \tag{8.11}$$

and it can be shown that if in particular $\psi^{AB} = \psi^A \psi^B$, where ψ^A is a fixed spin-vector, then $\psi^{\varkappa\lambda}$ is a *simple* bivector of the form $\psi_1^{[\varkappa}\psi_2^{\lambda]}$, where $\psi_1^{\varkappa}$, $\psi_2^{\varkappa}$ are any two vectors of R_5 lying on a fixed plane P and are such that their alternating product is unvaried. Schouten [20]) has shown that the plane P lies in the null-cone of R_5. Geometrically the simple bivector $\psi^{\varkappa\lambda} = \psi_1^{[\varkappa}\psi_2^{\lambda]}$ is interpreted as a closed region on P with definite area but with no definite boundary, rotating arbitrarily round an unreversed direction. This is probably the counterpart of a geometric interpretation of the wave-function ψ^A appearing in the Dirac equation.

9. *Isomorphism between E_4 and R_5.*

The (1-1) correspondence between the special bivectors of E_4 and the vectors of R_5 is analytically expressed by

$$\left.\begin{aligned} v^{\varkappa} &= \frac{1}{2}\, \alpha^{\varkappa}_{\cdot AB}\, v^{AB} \\ v^{AB} &= \frac{1}{2}\, \alpha_{\lambda}^{\cdot AB}\, v^{\lambda} \end{aligned}\right. \qquad (r_{AB}\, v^{AB} = 0\,). \tag{9.1}$$

[20]) Schouten [Footnote [14])], p. 411.

In order that the spin-transformation $T^A_{.B}$:

$$'v^{AB} = v^{CD}\, T^A_{.C}\, T^B_{.D}$$

changes every special bivector v^{AB} into a special bivector $'v^{AB}$, we must have

$$r^{AB} = \varrho\, r^{CD}\, T^A_{.C}\, T^B_{.D},$$

where ϱ is arbitrary. As the determinant of $T^A_{.B}$ is unity, it follows that $\varrho = \pm 1$ and we have

$$r^{AB} = \pm\, r^{CD}\, T^A_{.C}\, T^B_{.D}. \tag{9.2}$$

In other words $T^A_{.B}$ leaves r^{AB} invariant but for the factor ± 1.

The induced transformation on $v^{\varkappa}$ is

$$l^{\varkappa}_{.\lambda} = \frac{1}{4}\, \alpha^{\varkappa}_{.AB}\, T^A_{.C}\, T^B_{.D}\, \alpha^{.CD}_{\lambda} = B^{\varkappa}_{\alpha}\, e^{\alpha}_{.\beta}\, B^{\beta}_{\lambda}, \tag{9.3}$$

and it can be verified that $l^{\varkappa}_{.\lambda}$ is a rotation with respect to the fundamental tensor $G_{\lambda\varkappa}$ of R_5.

For $l^{\varkappa}_{.\lambda}$ to be real, we have that when $\varepsilon = -1$, $T^A_{.B}$ leaves $\Omega^A_{.\bar{B}}$, r_{AB} and therefore also $\Omega_{A\bar{B}}$ invariant but for the factor ± 1, and that when $\varepsilon = +1$, $T^A_{.B}$ leaves $\omega_{A\bar{B}}$, r_{AB} and therefore also $\omega^A_{.\bar{B}}$ invariant but for the factor ± 1 [21]). In either case the complex conjugate of (9.2) is identical with itself, and hence (9.2) consists of 6 equations of which only 5 are independent since by taking the determinant on both sides of (9.2) we get an identity. Thus $T^A_{.B}$ is a group of $\infty^{15-5} = \infty^{10}$ real and complex spin-transformations doubly isomorphic with the group of all the ∞^{10} real rotations of R_5.

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[21]) Cf. Schouten & Haantjes, l.c. p. 185; Cartan, l.c. p. 354.