COMPOSITIO MATHEMATICA

WILFRED KAPLAN

The topological structure of a variety defined by an equation

Compositio Mathematica, tome 5 (1938), p. 327-346

<http://www.numdam.org/item?id=CM_1938__5__327_0>

The topological structure of a variety defined by an equation [1])

by

Wilfred Kaplan

Scituate Ctr., Mass.

§ 1. Introduction.

1. We consider the following problem:

Let there be given an explicit equation

$$f(x_1, x_2, \ldots, x_n) = 0 \tag{1}$$

determining a certain configuration S in Euclidean n-dimensional space; in what cases and by what methods can one pass from the equation to the topological structure of S?

For example, consider the equation

$$x_1 \cos x_2 + x_2^7 \sqrt{1 + x_3^5} - \log(1 + x_4^2) + x_5^5 = 0 . \tag{2}$$

How can one discover practically the topology, connectivities, Betti numbers etc. of the configuration thereby determined?

More generally we can consider the case of a system of such equations, e.g.:

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1 = 0 \\ x_1 x_5 + x_2 x_6 + x_3 x_7 + x_4 x_8 = 0 \\ x_5^2 + x_6^2 + x_7^2 + x_8^2 - 1 = 0 . \end{cases} \tag{3}$$

Again the question is the same, of a *practical* analysis of the topology of the locus.

It is possible to consider certain special cases of the above where a geometric interpretation is possible. Thus, the system (3) can be regarded as representing the space of all directed tangents to the 3-dimensional sphere-surface

$$\Sigma_{i=1}^4 x_i^2 - 1 = 0 .$$

[1]) I should like to acknowledge my appreciation to Professor H. Hopf for his friendly advice and encouragement in this work.

Then various geometrical tools can be brought into play to find the topology. The recent work of Ehresmann is in this direction [2]).

There was also a considerable interest in this problem in the early days of topology. Dyck [3]) made investigations concerning the characteristic of certain manifolds defined by equations, though his methods are complicated and his results difficult to understand. Kronecker's work [4]) on systems of equations may also be considered as related. Poincaré, in his first approach to topology [5]), seems to have exactly our problem in mind. In his later developments [6]) of the theory, he returns again to specific equations and seems to regard their topological analysis as a fundamental goal of the combinatorial theory he had developed.

2. In the following paragraphs we shall explain a natural and simple method by which, for a certain large class of examples, the problem can be fully solved. Moreover this method can be generalized to a much larger body of cases; but in the present note, which we regard as a first attempt at the problem, we restrict the exposition to the most simple examples. We plan, at some future date, to present the generalizations in similar detail; at present we content ourselves with a sketch of them at the end of this paper (§ 3).

Specifically, we give in full detail here the method in the case of equations of form

$$\varphi_1(x_1) + \varphi_2(x_2) + \ldots + \varphi_n(x_n) = 0.$$

The essential step is the „linearization" of the functions φ_i,

---

[2]) C. Ehresmann, Sur la topologie de certains espaces homogènes [Ann. of Math. (2) **35** (1934), 396—443]; Sur la topologie de certaines variétés algébriques réelles [Journ. Math. Pures Appl. (9) **1** (1937), 69—100].

[3]) Beiträge zur Analysis Situs. I [Math. Ann. **32** (1888), 457—512]; II [Math. Ann. **37** (1890), 273—316].

[4]) Über Systeme von Funktionen mehrerer Variabeln [Monatsberichte Berl. Akad. **1869**, 159—193, 688—698].

[5]) Analysis Situs [Journ. Ecole Polytechn. (2) **1** (1895), 1—121]. Note especially the approach to the theory in the first few sections.

[6]) 3e Complément à l'Analysis Situs [Bull. Soc. Math. France **30** (1902), 49—70]. Note especially the problem stated on page 49.

4e Complément [Journ. Math. Pures Appl. (5) **8** (1902), 169—214].

5e Complément [Rend. Circ. Mat. Palermo **18** (1904), 45—110]. Note especially p. 50, the study of the topological structure of the n-dimensional cone. See below in this paper, Theorem I, 3; also 4, where the derived non-homogeneous equation is treated.

by replacing them by piecewise linear functions with corners at the extrema of the φ_i. We are then able to picture the configuration as composed of hyperplanar pieces (which are convex cells), whose incidences can be directly determined. From this picture it is then easy to give a representation as a finite or infinite cell-complex whose incidence matrices are fully known. We next give some applications to illustrate the method and also point out its significance with regard to singularities of the locus (see 6, Theorem IV).

In the following section (§ 3) we consider without detail the generalization of the method to equations of form

$$\varphi_1(x_1, x_2) + \varphi_2(x_3, x_4) + \dots + \varphi_n(x_{2n-1}, x_{2n}) = 0$$

though the limitations to effective use of the method become very great. Here the essential step is again a „linearization'' of the φ_i, to piecewise planar functions. The topological structure is then given as before by hyperplanar pieces (convex cells) whose incidences are known.

Finally we consider briefly the general case of an equation (1) and point out the profound difficulties. We consider also the case of a system of equations, which we reduce to the previous case of a single equation.

§ 2. Separation of variables.

3. We consider first a single equation of the form

$$\varphi_1(x_1) + \varphi_2(x_2) + \dots + \varphi_n(x_n) = 0 \tag{4}$$

where the φ_i are real functions of real x_i. The equation is of „separated'' type (by analogy with differential equations), a special case, but of importance.

We assume the φ_i satisfy the conditions:

(α) They are defined and continuous for $-\infty < x_i < +\infty$.

(β) They are piecewise strictly monotone or constant; i.e. the x_i-axis can be divided into intervals in each of which the function φ_i is either strictly monotone or else a constant.

There are of course several possibilities in (β) as to the number of intervals. We choose the lengths as large as possible. There may be then a finite number of intervals, corresponding to division points $-\infty, a_1, \dots, a_s, +\infty$; or there may be infinitely many in one direction, e.g. with division points $-\infty, a_1, \dots, a_s, \dots$; or there may be infinitely many in both directions, with division points $\ldots, a_{-s}, \ldots, a_0, \ldots, a_t, \ldots$ We shall consider only the last case, the others being easily derived from it. (We leave the derivation to the reader.)

Suppose further that we know the functions φ_i *well*, in the following sense:

(γ) We know the exact subdivision points a_j of the x_i axis as in (β); we know the exact values of $\varphi_i(x_i)$ at these subdivision points.

THEOREM I. *Let* $\varphi_i(x_i)$, $(i = 1, 2, \ldots, n)$, *be real functions of real* x_i *satisfying the conditions* (α), (β), (γ). *Then the full topological structure of the variety* V *defined by*

$$\varphi_1(x_1) + \varphi_2(x_2) + \ldots + \varphi_n(x_n) = 0 \tag{4}$$

can be determined.

By "can be determined" we mean a representation as described in **2**; i.e. we give a homeomorphic representation of V as a linear cell-complex in n-dimensional Euclidean space; we then give the full incidence matrices.

Proof of Theorem I. The following two facts are evident:

Firstly, let

$$x_i' = f_i(x_i) \qquad (i = 1, 2, \ldots, n)$$

be a topological transformation, for each i, of the x_i-axis onto the x_i'-axis; then the transformation

$$x_i' = f_i(x_i) \qquad (i = 1, 2, \ldots, n)$$

of R^n onto R'^n is topological and V is transformed topologically onto V'.

Secondly, let I $(a \leqq x_i \leqq b)$ be an interval of the given subdivision of the x_i-axis; let I' $(a' \leqq x_i' \leqq b')$ be an interval of the x_i'-axis; then the interval I can be mapped topologically onto the interval I' in such a way that $\varphi_i(x_i(x_i')) \equiv \psi_i(x_i')$ is linear in the interval I'. If $\varphi_i(x_i)$ is strictly monotone in I, the transformation is given explicitly by

$$\frac{x_i' - a'}{b' - a'} = \frac{\varphi_i(x_i) - \varphi_i(a)}{\varphi_i(b) - \varphi_i(a)}.$$

If $\varphi_i(x_i)$ is constant in I, the transformation becomes

$$\frac{x_i' - a'}{b' - a'} = \frac{x_i - a}{b - a}.$$

(In the case of an infinite interval, such as $a \leqq x < +\infty$ the transformation is simply

$$x_i' - a' = \varphi_i(x_i) - \varphi_i(a).)$$

Now suppose that, corresponding to the given subdivision of the x_i-axis, we have made a similar subdivision, arbitrarily chosen, of the x_i'-axis. We can then, by means of the above transformation, map the intervals I of the x_i-axis onto the corresponding intervals I' of the x_i'-axis. We clearly obtain thereby a topological transformation of the x_i-axis onto the x_i'-axis, representable by $x_i' = f_i(x_i)$, $(i = 1, 2, \ldots, n)$. According to our first remark above, we now have a topological transformation of V onto V'.

Let us number the intervals of the x_i-axis by the index $p_i^{j_i}$ and the corresponding ones of the x_i'-axis by the index $q_i^{j_i}$. Thus let $p_i^{j_i}$ represent $b_i^{j_i} \leqq x_i \leqq b_i^{j_i+1}$ and $q_i^{j_i}$ represent $a_i^{j_i} \leqq x_i' \leqq a_i^{j_i+1}$ $(j_i = 0, \pm 1, \pm 2, \ldots)$. We then have an enumeration $(q_1^{j_1}, q_2^{j_2}, \ldots, q_n^{j_n})$ of the n-dimensional intervals of a subdivision of $(x_1', \ldots, x_n')$ space.

But in this space V' is given by the equation

$$\psi_1(x_1') + \psi_2(x_2') + \ldots + \psi_n(x_n') = 0 \tag{4'}$$

where $\psi_i(x_i') = l_i^{j_i} x_i' + m_i^{j_i}$, $l_i^{j_i}$ and $m_i^{j_i}$ constants, in the interval $q_i^{j_i}$. That is, V' is given by

$$l_1^{j_1} x_1' + m_1^{j_1} + l_2^{j_2} x_2' + m_2^{j_2} + \ldots + l_n^{j_n} x_n' + m_n^{j_n} = 0 \tag{5}$$

in the n-dimensional interval $(q_1^{j_1}, \ldots, q_n^{j_n})$. V' is thus composed of hyperplanar pieces of $\leqq (n-1)$ dimensions. (Exceptionally, if all $l_i^{j_i} = 0$, the piece may be the whole interval, i.e. n-dimensional.) A hyperplane (5) may of course intersect an interval $(q_1^{j_1}, \ldots, q_n^{j_n})$ in a variety of ways, some of them degenerate.

Next let us remark that each such hyperplanar piece is a *convex cell.* (The convexity follows from that of the cube; that it is a cell follows from the fact that it is bounded by the $\leqq (n-1)$-dimensional planar pieces in which the plane (5) meets the boundary of the cube.) Its faces are precisely the intersections of (5) with the faces of the cube. If we show that any two such cells meet in a common face, then we know that V' is a cell-complex.

If two cells meet, they must clearly lie in cubes which have a k-dimensional face in common, $0 \leqq k \leqq (n-1)$. Thus suppose (5) meets the locus in the cube

$$(q_1^{j_1}, \ldots, q_k^{j_k}, q_{k+1}^{j_{k+1}+1}, \ldots, q_n^{j_n+1}).$$

In this cube the locus is

$$l_1^{j_1} x_1' + m_1^{j_1} + l_2^{j_2} x_2' + m_2^{j_2} + \ldots + l_k^{j_k} x_k' + m_k^{j_k} + l_{k+1}^{j_{k+1}+1} x_{k+1}' + m_{k+1}^{j_{k+1}+1} + \\ + \ldots + l_n^{j_n+1} x_n' + m_n^{j_n+1} = 0 .$$

This meets the common face in the locus

$$(6) \quad l_1^{j_1} x_1' + m_1^{j_1} + l_2^{j_2} x_2' + m_2^{j_2} + \ldots + l_k^{j_k} x_k' + m_k^{j_k} + l_{k+1}^{j_{k+1}+1} a_{k+1}^{j_{k+1}+1} + \\ + m_{k+1}^{j_{k+1}+1} + \ldots + l_n^{j_n+1} a_n^{j_n+1} + m_n^{j_n+1} = 0 .$$

But the continuity of $\psi_i(x_i')$ gives

$$l_i^{j_i} a_i^{j_i+1} + m_i^{j_i} = l_i^{j_i+1} a_i^{j_i+1} + m_i^{j_i+1}.$$

Hence (6) can be rewritten as

$$l_1^{j_1} x_1' + m_1^{j_1} + l_2^{j_2} x_2' + m_2^{j_2} + \ldots + l_k^{j_k} x_k' + m_k^{j_k} + l_{k+1}^{j_{k+1}} a_{k+1}^{j_{k+1}+1} + \\ + m_{k+1}^{j_{k+1}} + \ldots + l_n^{j_n} a_n^{j_n+1} + m_n^{j_n} = 0$$

which is precisely the intersection of (5) with the common face. That is, the loci in two adjacent cubes meet precisely in their common face in the common face of the two cubes. Hence V' is a cell-complex.

Finally, our assumption (γ) that we know the functions φ_i well means clearly that we know explicitly all the coefficients $l_i^{j_i}$ and $m_i^{j_i}$ of the function $\psi_i(x_i')$ (after the subdivision $q_i^{j_i}$ has been chosen). This implies further that we can actually list the hyperplanar pieces (5) of V' in their different intervals. We can then compute directly all incidences between the cells, as in the preceding paragraph. Thus the full incidence matrices of V' can be formed. This concludes the proof of the theorem.

Remark **I.** We have left a certain freedom above in the choice of the intervals $q_i^{j_i}$. They can of course be chosen simply as the $p_i^{j_i}$, in which case the transformation can be pictured as a simple joining of the successive extrema of the graph of $\varphi_i(x_i)$ by straight lines. The graph is thus flattened out to a broken line. We can also take the $q_i^{j_i}$ as the intervals determined by the integers: $x_i' = 0, \pm 1, \pm 2, \ldots$ This offers a certain convenience for notation.

Remark **II.** Obviously the method applies equally well when some of the φ_i are defined only in a part of the x_i-axis. In this case we obtain a part of such a complex as V'.

4. Let us now apply our method to some simple cases. Consider first an equation

$$(7) \quad a_1 x_1^{n_1} + a_2 x_2^{n_2} + \ldots + a_r x_r^{n_r} + a_{r+1} = 0$$

where $a_1, \ldots, a_{r+1}$ are constants, none equal to zero, and $n_1, \ldots, n_r$ are positive integers or even positive rational numbers of the form $\frac{s}{2t+1}$, where s and t are integers.

Case I. Suppose some n_i, n_1 for example, is either an odd number or the ratio of two odd numbers.

Then we simply rewrite (7) as

$$x_1 = \sqrt[n_1]{\frac{-a_2 x_2^{n_2} - \ldots - a_r x_r^{n_r} - a_{r+1}}{a_1}}.$$

Since n_1 is of the given form, the n_1^{th} root always exists, and we simply have an equation

$$x_1 = \chi(x_2, \ldots, x_r)$$

where χ is defined and continuous everywhere. This means clearly that the locus is equivalent to $(x_2, \ldots, x_r)$ space, i.e. to R^{r-1}. The analysis is therefore trivial.

Case II. No n_i is an odd integer or the ratio of two odd integers.

In this case we cannot extract the root as above. A glance at the functions φ_i involved shows us at once that the topology is equivalent to that of

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_r x_r^2 + \lambda_{r+1} = 0, \quad \lambda_i = \frac{a_i}{|a_i|} = \pm 1.$$

(This results from the fact that $a_i x_i^{n_i}$ and $\lambda_i x_i^2$ increase and decrease together; the intervals of subdivision for both are $-\infty < x_i \leqq 0$ and $0 \leqq x_i < +\infty$.)

The intervals $(p_1^{j_1}, \ldots, p_r^{j_r})$ in r-dimensional space are now precisely the 2^r regions determined by the $(r-1)$-dimensional coordinate hyperplanes $x_i = 0$, $(i = 1, \ldots, r)$. Take the $q_i^{j_i}$ as the $p_i^{j_i}$. Then in each of the 2^r regions in $(x_1', \ldots, x_r')$ space, the locus is given by

$$\lambda_1 |x_1| + \lambda_2 |x_2| + \ldots + \lambda_r |x_r| + \lambda_{r+1} = 0.$$

If $\lambda_1 = \lambda_2 = \ldots = \lambda_{r+1}$ then the locus is degenerate, reducing to the point $(0, \ldots, 0)$. Otherwise there is a locus in every region, composed of part of an $(r-1)$-dimensional hyperplane. The only remaining question is that of identifications. When they are listed, a simplicial subdivision of the locus can then be given if desired. We do not go further with the computation here, though this would be interesting in itself. Indeed a systematic treatise on the topology of the n-dimensional „conic"

$$x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_{n+1}^2 + 1 = 0$$

would prove of value in many branches of mathematics (see above, footnote 5).

Remark. It is worth noting that Case I can be generalized as follows:

THEOREM II. *Let $\varphi(x_i)$ be continuous and strictly monotone for $-\infty < x_1 < \infty$ and $\varphi(-\infty) = -\varphi(+\infty) = \pm\infty$; let $f(x_2, \ldots, x_n)$ be defined and continuous everywhere; then the variety determined by the equation*

$$\varphi(x_1) + f(x_2, \ldots, x_n) = 0$$

is topologically equivalent to $(n-1)$-dimensional Euclidean space.

For under the hypotheses, $\varphi(x_1) = y$ has an inverse $x_1 = \psi(y)$ for all y, and the equation is equivalent to

$$x_1 = \psi(-f(x_2, \ldots, x_n))$$

whence the above result. (Example: 1, equation (2).)

5. Consider now, as a second example, an equation in two variables:

$$\varphi_1(x_1) + \varphi_2(x_2) = 0. \tag{8}$$

Here we suppose that the φ_i satisfy, besides (α), (β), (γ), the further condition:

(δ) The φ_i are constant in no interval, and hence alternately increase and decrease in successive intervals.

We now carry out the linearization, choosing as intervals $q_i^{j_i}$ those determined by the integers, $j_i \leqq x_i' \leqq j_i + 1$. The functions $\psi_i(x_i')$ are then simply infinite polygons of form

$$\psi_i(x_i') \equiv a_i^{j_i} x_i' + b_i^{j_i}.$$

The continuity of $\psi_i(x_i')$ gives

$$a_i^{j_i}(j_i+1) + b_i^{j_i} = a_i^{j_i+1}(j_i+1) + b_i^{j_i+1}. \tag{9}$$

Our locus V' is now given by

$$a_1^{j_1} x_1' + b_1^{j_1} + a_2^{j_2} x_2' + b_2^{j_2} = 0 \quad \text{in } (q_1^{j_1}, q_2^{j_2}). \tag{10}$$

It is thus a graph composed of line segments lying in the different squares $(q_1^{j_1}, q_2^{j_2})$. By condition (δ) no $a_i^{j_i} = 0$ and the line segment (10) is not parallel to the x_1'- or x_2'-axis.

Suppose that in the square $(q_1^{j_1}, q_2^{j_2})$ there is a locus, i.e. that the line (10) meets its square. The line cannot coincide with any side, for then it would be parallel to a coordinate axis. Suppose then that (10) meets the side $x_1' = j_1 + 1$, $j_2 \leqq x_2' \leqq j_2 + 1$. There are then three possibilities:

a) It meets the side in a point (j_1+1, x_2') with $j_2 < x_2' < j_2 + 1$. The locus must then necessarily pass into the interior of the square $(q_1^{j_1}, q_2^{j_2})$.

b) It meets the side in a vertex, say (j_1+1, j_2+1) and passes into the interior of the square.

c) It meets the side in a vertex, say (j_1+1, j_2+1) and fails to pass into the interior of the square. The vertex is then the only point of intersection of the line with the square.

Consider the case a. The value of x' at the intersection is given by

$$a_1^{j_1}(j_1+1) + b_1^{j_1} + a_2^{j_2} x_2' + b_2^{j_2} = 0 \,. \tag{11}$$

But the locus in the adjacent square $(q_1^{j_1+1}, q_2^{j_2})$ is given by

$$a_1^{j_1+1} x_1' + b_1^{j_1+1} + a_2^{j_2} x_2' + b_2^{j_2} = 0 \tag{12}$$

which meets the line $x_1' = j_1 + 1$ in a point x_2' satisfying

$$a_1^{j_1+1}(j_1+1) + b_1^{j_1+1} + a_2^{j_2} x_2' + b_2^{j_2} = 0 \,. \tag{13}$$

It follows from (9) that (11) and (13) are identical, i.e. that the locus in $(q_1^{j_1+1}, q_2^{j_2})$ also meets the side $x_1' = j_1 + 1$, $j_2 \leqq x_2' \leqq j_2 + 1$ in the point (j_1+1, x_2'), which is not a vertex of the square $(q_1^{j_1+1}, q_2^{j_2})$. The line (12) must then also pass into the interior of its square. The same reasoning holds for the other sides of the square $(q_1^{j_1}, q_2^{j_2})$. Hence, *if the locus line meets a side of its square in a point not a vertex, the locus is continued into the square adjacent to that side.* (See Fig. 1.)

We note also that, by condition (δ), $a_i^{j_i}$ and $a_i^{j_i+1}$ have opposite signs, which means that the sign of the slope of (12) is opposite to that of (10); in general the slopes alternate in successive squares (as on a checkerboard).

Now consider case b. We find, as in case a, that the locus line in the adjacent square $(q_1^{j_1+1}, q_2^{j_2})$ also passes through the vertex (j_1+1, j_2+1). Moreover the condition that (10) passes into the interior of its square means that the slope of (10) is positive. Hence the slope of (12) is negative, and the line (12) which passes through the vertex, must also pass into the interior of its square $(q_1^{j_1+1}, q^{j_2})$. But we can apply the same reasoning to the other two squares meeting at the vertex (j_1+1, j_2+1). Thus in $(q_1^{j_1+1}, q_2^{j_2+1})$ the slope is positive and the locus passes through

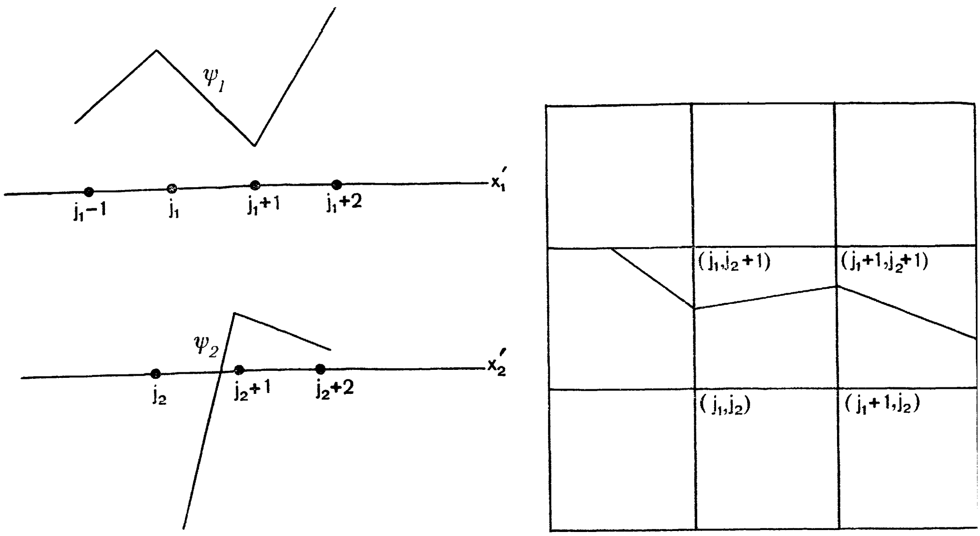

Fig. 1.

the vertex (j_1+1, j_2+1); hence it must pass into the interior of $(q_1^{j_1+1}, q_2^{j_2+1})$. The same holds for the fourth square $(q_1^{j_1}, q_2^{j_2+1})$.

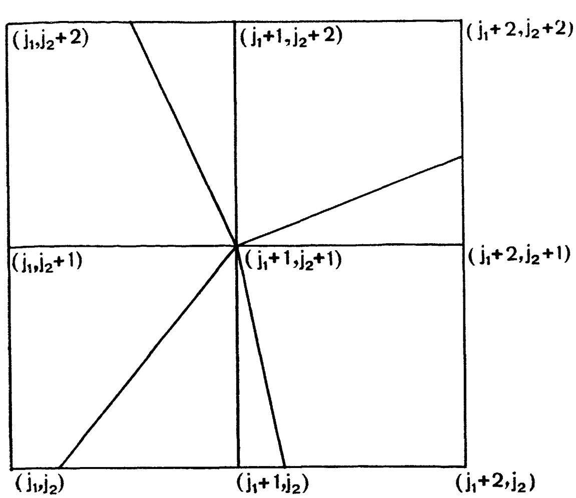

Fig. 2.

In all then, *the locus in the four squares consists of four segments meeting at the common vertex.* (See Fig. 2.)

Finally we have case c. Here the slope of (10) must be negative. Accordingly that of (12) is positive. But, as in case a, (12) must go through the vertex (j_1+1, j_2+1) and hence can meet the square $(q_1^{j_1+1}, q_2^{j_2})$ only in the vertex. The same reasoning applies to the other two squares meeting at the vertex. That is, *the vertex is an isolated point of the locus.*

We have now classified all the local possibilities. It is then easy to see what happens in the large. If we start in a square in which there is a locus segment, we can follow the locus on a broken line in both directions indefinitely unless we come to a singular point, i.e. a vertex of a square. Otherwise, if it meets no vertex, the broken line must form a simple closed path or go to infinity in both directions. Spirals are obviously excluded. We summarize our results as follows:

The locus is a graph composed of line segments lying in the different squares $(q_1^{j_1}, q_2^{j_2})$ *and a certain set of isolated points lying at vertices of the squares;*

the only singularities are the isolated points and branch-points, at vertices, at which four segments meet; through all other points of the locus exactly one broken line path passes; there are no „free ends";

the locus consists of broken lines from singular branch point to singular branch point, or from singular branch point to infinity, or from infinity to infinity, or else forming a closed path.

It is worth studying the singular points a bit further. They occur only at vertices of the squares, hence only at points x_1', x_2' where $\psi_1(x_1')$, $\psi_2(x_2')$ respectively have corners. *The singular point is an isolated point if these corners are extrema of same type of* $\psi_1(x_1')$, $\psi_2(x_2')$, (i.e., both maximum or both minimum). For if we take the vertex as at (j_1+1, j_2+1) and refer to the above discussion, we see that the slope of (10) in $(q_1^{j_1}, q_2^{j_2})$ must then be negative; i.e., the locus meets only the vertex of the square. *Otherwise the singular point is a branch point.* For if the extrema are of opposite type then the slope of (10) is positive and the locus passes into the interior of the square, which implies a branch-point. We can now state these results in terms of our original non-linear equation (7).

THEOREM III. *Let* $\varphi_1(x_1)$, $\varphi_2(x_2)$ *be real functions of real* x_i *satisfying conditions* (α), (β), (γ), (δ). *Then the equation*

$$\varphi_1(x_1) + \varphi_2(x_2) = 0 \tag{8}$$

determines a locus in the (x_1, x_2) *plane satisfying the following conditions:*

(*) *The locus is formed of simple Jordan curve-arcs and a certain set of isolated points.*

(**) *The only topological singularities are the isolated points and branch points at which four curve-arcs meet, having the singularity as common end-point; through all other points of the locus exactly one curve passes (so that the point is interior to the curve); there are no „free ends".*

(***) *The singularities occur at the points* (x_1, x_2) *satisfying* (8) *and such that* φ_1, φ_2 *both have extrema at* x_1, x_2 *respectively; if the extrema are of opposite types (one maximum, one minimum), then four branches of the locus meet at the point; if the extrema are of the same type then the singular point is an isolated point of the locus.*

(****) *The locus consists of simple Jordan curve-arcs from branch-point to singular branch-point, or from singular branch-point to infinity, or from infinity to infinity, or forming closed paths. It is topologically a one-dimensional curved complex.*

6. We can now carry through a similar analysis for the generalization to n dimensions, i.e., to

$$\varphi_1(x_1) + \varphi_2(x_2) + \ldots + \varphi_n(x_n) = 0 \tag{14}$$

where we make the same additional assumption (δ) on the φ_i.

We first linearize our functions φ_i to the functions ψ_i with corners at the integral points $x_i' = j_i$ $(j_i = 0, \pm 1, \pm 2, \ldots)$ of the x_i'-axis. We obtain the new equation

$$\psi_1(x_1') + \psi_2(x_2') + \ldots + \psi_n(x_n') = 0 \tag{15}$$

where $\psi_i(x_i') = a_i^{j_i} x_i' + b_i^{j_i}$ in $q_i^{j_i} : j_i \leqq x_i' \leqq j_i + 1$.

The locus is therefore given by

$$a_1^{j_1} x_1' + b_1^{j_1} + a_2^{j_2} x_2' + b_2^{j_2} + \ldots + a_n^{j_n} x_n' + b_n^{j_n} = 0 \tag{16}$$

in $(q_1^{j_1}, \ldots, q_n^{j_n})$.

The assumption (δ) implies that $a_i^{j_i} \neq 0$ $(i = 1, \ldots, n)$ and that $a_i^{j_i}$ and $a_i^{j_i+1}$ have opposite signs.

The continuity of $\psi_i(x_i')$ gives

$$a_i^{j_i}(j_i+1) + b_i^{j_i} = a_i^{j_i+1}(j_i+1) + b_i^{j_i+1}. \tag{17}$$

We have thus to consider a locus composed of various hyperplanar pieces lying in the cubes $(q_1^{j_1}, \ldots, q_n^{j_n})$. The pieces are convex cells and are incident in common faces. It remains to discuss how each cell lies in its cube and how exactly it is incident with other cells.

I. Consider the first question. *Suppose the plane* (16) *actually meets its cube at a point* P *not a vertex.* If P is interior to the cube, then the cell is $(n-1)$-dimensional. Let us show that:

Even if P *is on the boundary of the cube, then the plane* (16) *must pass into the interior and thus determine an* $(n-1)$*-dimensional convex cell.*

Thus, we assume P lies on a face F: $x_1' = j_1 + 1$, $j_i \leqq x_i' \leqq j_i + 1$ $(i = 2, \ldots, n)$, and wish to determine a second point S of (16) in the cube with $j_i < x_i' < j_i + 1$.

For $n = 2$ we have seen the proposition to be true; i.e., if the line (16) meets its square in a point other than a vertex, then it passes into the interior. Let us assume this true for dimension $(n-1)$; hence we can assume for purpose of induction that the point P itself is interior to the face F (relative to the plane $x_1' = j_1 + 1$).

The plane (16) cannot be parallel to the plane $x_1' = j_1$, by condition (δ), hence meets it. Let Q $(x_1' = j_1,\ x_i' = \bar{x}_i')$ be a point of the intersection. The line $\overline{QP}$ then lies wholly in the plane (16). But on the line $\overline{QP}$ x_1' passes continuously and monotonely from the value j_1 to $j_1 + 1$, x_i' from $\bar{x}_i'$ to values x_i' at P satisfying $j_i < x_i' < j_i + 1$. At some point S on $\overline{QP}$ we must then clearly have $j_i < x_i' < j_i + 1$ $(i = 1, \ldots, n)$. Hence (16) must pass into the interior of the cube. Hence by induction the proposition is proved.

II. But the proposition implies more. For let P be any point not a vertex on a k-dimensional face $(0 < k \leqq n-1)$ of the cell R determined by (16) in its cube C; that is, we take R to be $(n-1)$-dimensional and P on its intersection with some k-face G of C. Let D be any other cube of which G is a face. We have seen that the locus in D meets that in C on G, precisely in the intersection of R with G. Hence P is also part of the locus in D. Therefore by the above proposition the locus in D is also a convex cell of $(n-1)$ dimensions.

III. Let us now go further and consider all the convex cells meeting in P (which we choose as in II, not a vertex). They are evidently all $(n-1)$-dimensional, lying one in each cube which has G as face, and all have a common $(k-1)$-dimensional face H in G $(0<k\leqq n-1)$. What is the structure of this sub-complex of cells about H? We shall establish that *at the point P the locus cells fit together to form an $(n-1)$-dimensional Euclidean neighborhood of the point P.*

For simplicity, take G as

$$0\leqq x_s'\leqq 1 \quad (s=1,\ldots,k); \quad x_t'=0 \quad (t=k+1,\ldots,n).$$

The cubes with G as face are then given by

$$0\leqq x_s'\leqq 1 \qquad (s=1,\ldots,k)$$

$$\begin{cases} 0\leqq x_t'\leqq 1 \\ \quad\text{or} \\ -1\leqq x_t'\leqq 0 \end{cases} \qquad (t=k+1,\ldots,n),$$

where for each t we choose one of the inequalities. P is then a point $(x_1'^0,\ldots,x_k'^0,0,\ldots,0)$, where, since P is not a vertex, we can assume for example

$$0<x_1'^0<1.$$

Consider one of the cubes meeting at G, for example:

$$0\leqq x_i'\leqq 1 \qquad (i=1,\ldots,n).$$

The locus therein we write simply as

$$a_1x_1'+a_2x_2'+\ldots+a_nx_n'+b=0.$$

Since $a_1\neq 0$ if we project this locus onto the plane $x_1'=0$, we determine a topological transformation of the locus onto a certain set in the x_1'-plane. The image set is given by

$$\begin{cases} 0\leqq \dfrac{a_2x_2'+\ldots+a_nx_n'+b}{-a_1}\leqq 1 \\ 0\leqq x_i'\leqq 1 \quad (i=2,\ldots,n). \end{cases}$$

The point P itself projects onto the point Q: $(0,x_2'^0,\ldots,x_k'^0,\ldots,0)$. The point Q must be interior to the strip R:

$$0\leqq \frac{a_2x_2'+\ldots+a_nx_n'+b}{-a_1}\leqq 1.$$

For, since P is on the locus,

$$0<x_1'^0=\frac{a_2x_2'^0+\ldots+a_nx_n'^0+b}{-a_1}<1.$$

By the same reasoning it follows that Q is interior to the region R^α:

$$0 \leqq \frac{a_2^\alpha x_2' + \dots + a_n^\alpha x_n' + b^\alpha}{-a_1^\alpha} \leqq 1$$

corresponding to the projection on $x_1' = 0$ of the locus in each cube C^α having G as a face. This projection is clearly topological on the set of the sum of the different loci. There is now clearly a neighborhood $U(Q)$ in the plane $x_1' = 0$ and lying in all R^α. We can also take $U(Q)$ so small as to be in

$$\begin{cases} 0 \leqq x_s' \leqq 1 & (s = 2, \dots, k) \\ -1 \leqq x_t' \leqq 1 & (t = k+1, \dots, n). \end{cases}$$

Also the part of $U(Q)$ lying in the face $x_1' = 0$ of a cube C^α is clearly a part of the projection onto $x_1 = 0$ of the locus in that cube. $U(Q)$, being a sum of such parts, is thus itself part of the image under the projection. But the projection is a homeomorphism. Hence the point P also possesses a neighborhood $U(P)$, which must be homeomorphic to $U(Q)$, that is, $(n-1)$-dimensional and Euclidean. This proves our assertion.

IV. It remains to consider what happens when (16) passes through a vertex of C. Here it will be simplest to take C as $0 \leqq x_i' \leqq 1$ and (16) as

$$(16^*) \qquad a_1 x_1' + a_2 x_2' + \dots + a_n x_n' = 0$$

passing through the vertex at the origin. There are now two possibilities: that (16*) meet C only in the vertex; or that (16*) passes into the interior of C. The first case occurs if $a_i > 0$ $(i = 1, \dots, n)$ or if $a_i < 0$, $(i = 1, \dots, n)$. Otherwise the second occurs.

Consider the locus in an adjacent quadrant, for example in $-1 \leqq x_1' \leqq 0$, $0 \leqq x_j' \leqq 1$ $(i = 2, \dots, n)$. This is given by

$$a_1^{(-1)} x_1' + a_2 x_2' + \dots + a_n x_n' = 0$$

where $a_1^{(-1)}$ and a_1 have opposite signs. Here the first case occurs if $-a_1^{(-1)}$, $a_2, \dots, a_n$ are all > 0 or are all < 0. Since a_1 and $a_1^{(-1)}$ have opposite signs, this condition is equivalent to the above one; i.e., if the locus exists in one quadrant, it exists in an adjacent one. Proceeding thus through all quadrants, we conclude that either there is a locus in the interior of each of the 2^n quadrants or else the vertex at the origin is an isolated point of the locus.

What are the possible incidences of the cells in the case when there is one in each quadrant? It is easy to see that they are precisely those of

$$\frac{a_1}{|a_1|}|x_1'| + \frac{a_2}{|a_2|}|x_2'| + \ldots + \frac{a_n}{|a_n|}|x_n'| = 0 .$$

That is, it is only the sign of the a_i which matters. For example, consider the locus (16*) in the quadrant

$$0 \leqq x_i' \leqq 1 \qquad (i = 1, \ldots, n)$$

and that in

$$-1 \leqq x_p' \leqq 0 \ (p = 1, \ldots, k); \quad 0 \leqq x_q' \leqq 1 \ (q = k+1, \ldots, n).$$

By condition (17) we find as before that the incidence is precisely that of (16*) with the plane $x_1' = x_2' = \ldots = x_k' = 0$. This intersection is

$$a_{k+1}x_{k+1}' + \ldots + a_n x_n' = 0 \quad \text{in} \quad 0 \leqq x_q' \leqq 1 \ (q = k+1, \ldots, n).$$

The locus is either an $(n-k-1)$-dimensional convex cell or the origin according as the a_q all have the same sign or not. That is, the *signs* of the a_i in (16*) determine fully the nature of the incidences about the origin.

We remark again that the case when all a_i are of the same sign corresponds to the case of extrema of the same type for all ψ_i at the origin. The proportion of maxima and minima among the ψ_i determines the type of incidences in case not all a_i are of one sign.

We summarize our results as follows:

THEOREM IV. *The variety determined by the equation* (14) *can be represented as a cell-complex V' satisfying the conditions*:

(*) *V' is composed of $(n-1)$-dimensional convex cells lying in the different cubes $(q_1^{j_1}, \ldots, q_n^{j_n})$ and a certain set of isolated points lying at the vertices of the cubes.*

(**) *The only topological singularities are the isolated points and branch-points, at vertices, at which 2^n convex cells meet. The nature of the singularity is the same as that of*

$$(18) \qquad \lambda_1|x_1| + \lambda_2|x_2| + \ldots + \lambda_n|x_n| = 0, \quad \lambda_i = \pm 1$$

at the origin, for different values of λ_i.

(***) *The locus is composed of $(n-1)$-dimensional sheets pieced together from the convex cells; those sheets which do not pass through branch points are $(n-1)$-dimensional manifolds.*

(****) *The singularities of the equation* (14) *occur at the points* $(x_1, \ldots, x_n)$ *satisfying it, and such that* $\varphi_1(x_1), \ldots, \varphi_n(x_n)$ *are extrema. The singularity is an isolated point of the locus when the extrema are all of the same type* (*all maxima or all minima*). *The various other types are given by equation* (18).

7. It is possible to attempt to apply our method of **3** to the following very general case:

$$F(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n)) = 0 .$$

Here we have replaced the operation of summation on the φ_i by a general function F, continuous and defined everywhere.

If we now linearize the φ_i as before, we reduce the equation to the form

$$F(l_1^{j_1} x_1' + m_1^{j_1}, l_2^{j_2} x_2' + m_2^{j_2}, \ldots, l_n^{j_n} x_n' + m_n^{j_n}) = 0$$

in the interval $(j_1, \ldots, j_n)$.

It is obvious that this is in general a simplification of the problem of determining the topological structure. If the function F is not too complicated, the actual structure may even be found.

For example, consider

$$(e^x \sin x)(y^2 + 5y + 4)(\cos^2 z) - 1 = 0 .$$

Here a linearization leads us to consider in each interval a surface

$$ax'y'z' + b = 0 .$$

Because of its simplicity this surface can be plotted for each interval and all identifications can be found. With the original equation almost nothing could be done.

§ 3. Pairwise separation of variables.

8. We now consider, without details of proof, the possibility of generalizing our method of Theorem I to the following case:

$$(19) \qquad \varphi_1(x_1, x_2) + \varphi_2(x_3, x_4) + \ldots + \varphi_n(x_{2n-1}, x_{2n}) = 0 .$$

Here we replace our functions $\varphi_i(x_i)$ by functions $\varphi_i(x_{2i-1}, x_{2i})$ of two variables, and want somehow to generalize our original linearization to these more general functions.

We want therefore to formulate a generalized ,,subdivision of the axis'' (as in condition (β)) for each function $\varphi_i(x_{2i-1}, x_{2i})$ so that in each ,,interval of the axis'' φ_i is ,,strictly monotone or constant''. We then ,,join the end-points of the curve φ_i in the interval'' by ,,straight lines'' so that φ_i becomes ,,piecewise linear''.

Actually we are led to choose as ,,interval of the axis'' a *curved triangle* ABC of the following type: $\varphi_i(x_{2i-1}, x_{2i})$ is constant on the arc $\widehat{AB}$; φ_i is strictly monotone on the arcs $\widehat{AC}$ and $\widehat{BC}$; the triangle ABC can be mapped topologically onto a linear triangle $A'B'C'$ in the (y_{2i-1}, y_{2i})-plane so that the level curves of the new function

$$\psi(y_{2i-1}, y_{2i}) \equiv \varphi_i\big(x_{2i-1}(y_{2i-1}, y_{2i}), x_{2i}(y_{2i-1}, y_{2i})\big)$$

in the triangle $A'B'C'$ are the line $A'B'$ and the lines parallel to it.

By a ,,subdivision of the axis'' we now mean a *curved triangulation* of the (x_{2i-1}, x_{2i})-plane, whereby the triangles are all of the type of ABC of the preceding paragraph.

Our fundamental hypothesis, corresponding to (β) of Theorem I, is then that for each function $\varphi_i(x_{2i-1}, x_{2i})$ the (x_{2i-1}, x_{2i})-plane admits the described curved triangulation.

Under this assumption it is then easy to show that φ_i can be reduced to a piecewise *planar* function. Also all the information needed for constructing the new function is the set of values of φ_i at the vertices of the triangulation.

Once we have linearized the functions φ_i, we have a representation of the locus (19) as a cell-complex in a certain m-dimensional Euclidean space. We thus arrive at the same result as in Theorem I. We are again able to give the full topological structure of the variety.

It may be asked, what class of functions satisfy the hypothesis above? The answer here must be given in terms of the structure, both local, and in the large, of the family of level curves of the function. Without giving details here, let us say for the present that a very large class of functions, especially the ones of common practice, have level curves of the desired structure.

Further, we believe it is possible to generalize our type of subdivision of the (x_{2i-1}, x_{2i})-plane to include what might be called ,,infinite triangles'' of the type of ABC, and thereby, by the same method as before, to extend our results to a much larger class of functions φ_i on whose level curves only local assumptions of regularity are made. This we present merely as an intuition which we hope to develop rigorously at a later date.

9. *Applications.* It seems probable that the Theorem IV of 6 can be generalized to an equation of type (19) under the hypothesis of section 8. The essential difference would be more general structures at the critical points. This also we shall develop later.

10. *Generalization to functions of more than two variables.* We believe our method can be extended, at least in theory, to the general equation

$$(20)\quad \varphi_1(x_1, \ldots, x_{p_1}) + \varphi_2(x_{p_1+1}, \ldots, x_{p_2}) + \ldots + \varphi_n(x_{p_{n-1}+1}, \ldots, x_{p_n}) = 0$$

where hypotheses are made concerning the family of level surfaces of the φ_i. It is well-known, however, that to determine the structure of level surfaces for even a function of two variables is very difficult, for a function of more than two variables incomparably more difficult. Thus in practice the generalization to equation (20) would be used very little. Yet theoretically we find it significant in that it reduces the topological problem to one of the level surfaces of the individual functions. This reduction we believe to be basic in the practical analysis of the topological structure of a variety defined by an equation.

11. *Case of a system of equations.* We consider a variety defined by a set of equations

$$(21)\qquad F_i(x_1, \ldots, x_n) = 0 \quad (i = 1, \ldots, k).$$

A very simple device brings us back to an equation of the type of equation (20). For the locus (21) is equivalent to the locus

$$(22)\qquad \Sigma_{i=1}^{i=k} \left(F_i(x_1, \ldots, x_n)\right)^2 = 0$$

defined by a single equation. We now combine terms as much as possible in (22) to reduce it to as simple a form (20) as we can find. We then proceed as originally with equation (20).

Far from trivial, we believe this device to be an important *practical* method of finding the topological structure of (21). Indeed, it is the only way we know of even starting to analyse a variety defined by a system of equations. (We are of course omitting certain special or trivial cases for which the method of analysis is obvious.)

12. *Conclusion.* In this section we have indicated a program of work which we hope to complete at some later date. The result of 8 we have already been able to establish, but shall complete it and indicate its full significance before publishing. The ideas of 9 and 10 are in a more tentative state, but their presentation here may be of use even before their completion in rigorous form.

(Received September 25th, 1937.)