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W. L. FERRAR

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Summation formulae and their relation to Dirichlet's series II

by

W. L. Ferrar
Oxford

1.1. It is well known that, subject to certain conditions governing the behaviour of $f(x)$,

$$\tfrac{1}{2}f(+0) + \sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x)\,dx + 2\sum_{n=1}^{\infty}\int_0^{\infty} f(x)\cos 2n\pi x\,dx. \qquad (1)$$

This 'summation formula', wherein each $f(n)$ on the left hand side is multiplied by unity, corresponds to the Dirichlet's series

$$\zeta(s) = \Sigma\, n^{-s},$$

wherein each n^{-s} has the coefficient unity. Further,

$$2\cos 2n\pi x = \frac{1}{2\pi i}\int \frac{\pi^{s-\frac{1}{2}}\,\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}(nx)^{s-1}ds$$

$$= \frac{1}{2\pi i}\int \frac{\zeta(s)}{\zeta(1-s)}(nx)^{s-1}ds,$$

the integral being taken along a line, parallel to the imaginary axis, between $\sigma = \frac{1}{2}$ and $\sigma = 1$.

In the present paper we show that, under certain conditions, there is, corresponding to the Dirichlet's series

$$\psi(s) = \Sigma\, a_n n^{-s},$$

a summation formula which, for suitable functions $f(x)$, relates

$$\sum_{n=1}^{\infty} a_n f(n)$$

to a sum of the type

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x)\,\alpha(nx)\,dx,$$

wherein

$$\alpha(y) = \frac{1}{2\pi i} \int \frac{\psi(s)}{\psi(1-s)} y^{s-1} ds ,$$

the integral being taken along a line parallel to the imaginary axis of s.

The investigation is closely allied to that of a previous paper [1]) under the same title, but the two papers may be read independently of each other.

1.2. By the methods of the former paper we were not able to derive a general summation formula which corresponded closely to (1): only formulae indirectly related to (1) were obtained. In the present paper we derive a formula directly analogous to (1) containing it as a special case. This we are able to do because we now confine oul attention to functions $f(x)$ which can be expressed in the form [2])

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi(s) x^{-s}\, ds .$$

1.3. In § 2 we prove (1) in a manner which shows what are the essential steps in the derivation of the general formula, which is given in § 3. As a proof of (1) it is inferior to the known, real variable proofs. On the other hand, it shows how (1) can be modified so as to permit of its application to functions $f(x)$ for which $f(+0)$ is infinite.

1.4. For convenience of printing we use

$$\int_{\beta} \text{ instead of } \int_{\beta-i\infty}^{\beta+i\infty} .$$

2. *A proof of formula* (1).

2.1. We base our proof of (1) on two things, the functional equation of the zeta-function and Mellin transforms. In the latter connection we use [3])

LEMMA 1. *Let* $\varphi(s) = \varphi(\sigma+it)$ *be regular and, moreover, let*

$$\int_{-\infty}^{\infty} |\varphi(\sigma+it)|\, dt \tag{2}$$

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[1]) Compositio Mathematica **1** (1935), 344—360.

[2]) This, in virtue of Mellin's transform, includes a large class of functions.

[3]) S. BOCHNER, Vorlesungen über Fouriersche Integrale [Leipzig, 1932], 156, Satz 47. It is sufficient for Bochner's theorem that (2) be bounded in each closed interval (λ_1, μ_1) of the open interval $[\lambda, \mu]$; we need only the, more crude, condition of boundedness in the open interval $[\lambda, \mu]$.

be bounded in the strip $\lambda < \sigma < \mu$. *Then, when* $\lambda < \sigma < \mu$,

$$f(x) = \frac{1}{2\pi i}\int_{\sigma} \varphi(s)x^{-s}\,ds \tag{3}$$

defines, for positive x, *a unique function* $f(x)$ *such that, within the strip,*

$$\varphi(s) = \int^{\infty} f(x)x^{s-1}dx. \tag{4}$$

Moreover, the integral in (4) *converges absolutely.*

2.2. Let $\varphi(s)$ be a function of $s(=\sigma+it)$ satisfying the following conditions:

(I) $\varphi(s)$ *is analytic, save possibly for poles, in a strip* $-b \leqq \sigma \leqq 1+c$ $[b, c > 0]$;

(II) *the only pole in this strip is a simple pole at* $s = 0$, *where the residue is* φ_0;

(III) *we can determine* M, t_0, and η, *with* $\eta > max\,(b, c)$, *so that*

$$|\varphi(\sigma+it)| < M|t|^{\frac{1}{2}\sigma-\frac{3}{2}-\frac{1}{2}\eta}$$

whenever $|t| > t_0$ *and* $-b \leqq \sigma \leqq 1+c$.

We note that, of these conditions, (I) is essential, (II) can be modified at the cost of further detail and modification of the final formula (§ 2.5), while (III) is simply a condition that enables us to carry out the subsequent transformations; a study of these transformations will show just what is required of $|\varphi(\sigma+it)|$ at each step.

With these conditions

$$\frac{1}{2\pi i}\int_{\gamma} \varphi(s)x^{-s}ds \qquad (0 < \gamma \leqq 1+c)$$

defines, for any positive x, a unique function $f(x)$. Also, as we see by moving the path of integration to the right and to the left respectively,

$$f(x) = O(x^{-1-c}) \qquad \text{when } x \text{ is large,}$$
$$f(x) = \varphi_0 + O(x^{b}) \qquad \text{when } x \text{ is small.}$$

Hence the series $\Sigma f(n)$ converges absolutely and

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{1+c} \varphi(s)n^{-s}ds$$
$$= \frac{1}{2\pi i}\int_{1+c} \varphi(s)\zeta(s)ds,$$

since both Σn^{-s} and $\int \varphi(s)ds$ converge absolutely.

The condition (III) enables us to move the path of integration to the line $\sigma = -b$. On using the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{\frac{1}{2}(s-1)}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\zeta(1-s),$$

we then have

$$\sum_{n=1}^{\infty} f(n) = R_0 + R_1 + \frac{1}{2\pi i}\int_{-b} \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\varphi(s)\zeta(1-s)\,ds, \quad (5)$$

where R_0, R_1 represent residues at $s = 0, 1$ respectively.

But, on using (III) again,

$$\left|\varphi(s)\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\right| = O\left(|t|^{\frac{1}{2}\sigma-\frac{3}{2}-\frac{1}{2}\eta}\cdot|t|^{-\sigma+\frac{1}{2}}\right)$$

$$= O\left(|t|^{-\frac{1}{2}\sigma-1-\frac{1}{2}\eta}\right),$$

and so (by absolute convergence)

$$\sum_{n=1}^{\infty} f(n) = R_0 + R_1 + \sum_{n=1}^{\infty}\int_{-b}\frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\Gamma(\frac{1}{2}s)}\varphi(s)n^{s-1}\,ds.$$

Hence, on moving the path of integration [4]) to $\sigma = \beta$, where $\frac{1}{2} < \beta < 1$,

$$\sum_{n=1}^{\infty} f(n) = R_0 + R_1 + \sum_{n=1}^{\infty}\int_{\beta}\frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\Gamma(\frac{1}{2}s)}\varphi(s)n^{s-1}\,ds. \quad (6)$$

2.3. We now use Mellin's transform. Let δ be any positive number less than β; then, if $\delta \leqq \sigma \leqq 1$,

$$\int_{-\infty}^{\infty}|\varphi(\sigma+it)|\,dt$$

$$= \int_{-t_0}^{t_0}|\varphi(\sigma+it)|\,dt + \int_{t_0}^{\infty} O\left(|t|^{\frac{1}{2}\sigma-\frac{3}{2}-\frac{1}{2}\eta}\right)dt$$

$$\leqq K_1 + \frac{K_2}{1+\eta-\sigma} \leqq K,$$

where K_1, K_2, and K are independent of σ. Further, $\varphi(s)$ is regular in the strip $\delta \leqq \sigma \leqq 1$, and, by Lemma 1,

$$\varphi(s) = \int_0^{\infty} f(x)x^{s-1}\,dx,$$

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[4]) $\beta > 0$ would be enough for (6) and (7), but $\beta > \frac{1}{2}$ is necessary for the change from (7) to (8).

the integral being absolutely convergent. Accordingly,

$$\int_{\beta} \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\,\Gamma(\frac{1}{2}s)}\,\varphi(s)n^{s-1}ds$$

$$= \int_{\beta} \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\,\Gamma(\frac{1}{2}s)}\,n^{s-1}\,ds \int_0^\infty f(x)x^{s-1}dx. \tag{7}$$

In this repeated integral we can change the order of integration[5]) and so, after a simple calculation of residues, write it as

$$2\int_0^\infty f(x)\cos 2n\pi x\,dx. \tag{8}$$

2.4. We have thus proved, on combining (6), (7), and (8), that

$$\sum_{n=1}^{\infty} f(n) = R_0 + R_1 + 2\sum_{n=1}^{\infty}\int_0^\infty f(x)\cos 2n\pi x\,dx.$$

On evaluating the residues, we have

$$R_1 = \varphi(1) = \int_0^\infty f(x)\,dx,$$

$$R_0 = -\frac{1}{2}\varphi_0$$

and, further, since $f(x) = \varphi_0 + O(x^b)$ when x is small,

$$R_0 = -\frac{1}{2}f(+0).$$

Hence our final result is

$$\frac{1}{2}f(+0) + \sum_{n=1}^{\infty} f(n) = \int_0^\infty f(x)dx + 2\sum_{n=1}^{\infty}\int_0^\infty f(x)\cos 2n\pi x\,dx.$$

2.5. *The modification of* (1) *when* $f(+0)$ *is infinite.*

Suppose that the conditions (I) and (III) of § 2.1 hold, but that (II) is replaced by the following condition:

(II a) *there is a strip* $\lambda \leqq \sigma \leqq \mu$, *where* $\frac{1}{2} \leqq \lambda \leqq \mu < 1$, *within which* $\varphi(s)$ *is regular.*

This condition permits of poles of $\varphi(s)$ in $-b \leqq \sigma \leqq 1 + c$

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[5]) The proof of this statement is a matter of some detail. One proof depends on an integration by parts which exploits the oscillatory character of the leading term in the asymptotic expansion

$$\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\beta-\frac{1}{2}it)}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}it)} = e^{\frac{1}{4}\pi i}(\tfrac{1}{2})^{\frac{1}{2}-\beta-it}t^{\frac{1}{2}-\beta-it}e^{it}\{1+O(t^{-1})\}.$$

other than a simple pole at $s=0$, and the following modifications are to be made in our previous work.

(A) In (5), R_1 represents, as before, the residue of $\varphi(s)\zeta(s)$ at $s=1$, but R_0 now represents the sum of the residues of $\varphi(s)\zeta(s)$ at poles other than $s=1$.

(B) In (6) we need $\lambda<\beta<\mu$ in order that we may, when $\sigma=\beta$, have

$$\varphi(s)=\int_0^\infty f(x)x^{s-1}dx,$$

which is necessary to the derivation of (7).

Accordingly, the integral

$$\int_\beta \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\Gamma(\frac{1}{2}s)}\varphi(s)n^{s-1}\,ds$$

in (6) must be replaced by

$$-S_n+\int_\beta \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\Gamma(\frac{1}{2}s)}\varphi(s)n^{s-1}ds,$$

where S_n is the contribution from the poles passed over in the passage from $\sigma=-b$ to $\sigma=\beta$; that is, S_n is the sum of the residues of

$$\frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\Gamma(\frac{1}{2}s)}\varphi(s)n^{s-1}$$

at its poles in the strip $-b<\sigma<\lambda$.

With these conventions, the final result is that, if

$$f(x)=\frac{1}{2\pi i}\int_{1+c}\varphi(s)x^{-s}ds$$

where $\varphi(s)$ satisfies the conditions (I), (II a), and (III), then

$$-R_0+\sum_{n=1}^{\infty}f(n)=R_1+\sum_{n=1}^{\infty}\left\{-S_n+2\int_0^\infty f(x)\cos 2n\pi x\,dx\right\}. \tag{9}$$

2.51. By giving $\varphi(s)$ suitable values we can give $f(x)$ singularities of a given type at $x=0$. For example, $\varphi(s)=\left\{\Gamma\left(\frac{1}{2}s\right)\right\}^2$ will give $f(x)$ a logarithmic singularity at $x=0$, while $\varphi(s)=\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}\alpha\right)$, where $0<\alpha<1$, will give $f(x)$ an algebraic singularity $[f(x)=O(x^{-\alpha})]$ at $x=0$.

2.52. We conclude with a simple example of (9), namely,

$$2\sum_{n=1}^{\infty} K_0(nx) = \pi\left[\frac{1}{x} + 2\sum_{n=1}^{\infty}\left\{-\frac{1}{2n\pi} + \frac{1}{\sqrt{(x^2+4n^2\pi^2)}}\right\}\right] + \gamma + \log\left(\tfrac{1}{2}x\right) - \log 2\pi .$$

This formula, first proved by Watson [6]), is obtained from our results by using the fact that

$$4K_0(x) = \frac{1}{2\pi i}\int_c \left(\tfrac{1}{2}x\right)^{-s}\left\{\Gamma\left(\tfrac{1}{2}s\right)\right\}^2 ds \qquad (c>0).$$

In this example $\varphi(s)$ has a double pole at $s=0$ and $S_n = \frac{1}{2}n^{-1}$.

3. *The general formula.*

3.1. *Hypotheses concerning the Dirichlet's series.* We suppose that the series

$$\sum_{n=1}^{\infty} a_n n^{-s} \qquad (a_n \text{ real})$$

converges absolutely for $\sigma > 1$ and defines a function $\psi(s)$. We make the following hypotheses concerning this function:

(I) *There is a positive number b such that some process of analytic continuation defines $\psi(s)$ over the strip $-b \leqq \sigma \leqq 1$, and the only singularities of $\psi(s)$ in this strip are poles, finite in number, none of which lie in the strip $-b \leqq \sigma \leqq 0$.*

(II) *The function $\psi(s)$ is of finite order in $\sigma \geqq -b$; that is, for some t_0 and some K,*

$$|\psi(s)| = O(|t|^K) \text{ when } |t| > t_0 \text{ and } \sigma \geqq -b.$$

We now define $A(s)$ by means of the equation

$$\psi(1-s)A(s) = \psi(s),$$

and make the following hypotheses concerning $A(s)$:

(III) *There are positive numbers δ and σ_1 such that*

$$A(s) \text{ has no pole in } -b \leqq \sigma \leqq \delta$$

and, for large values of $|t|$,

$$|A(\sigma_1+it)| = O\left(|t|^{-1-\eta}\right),$$

where $\eta > 0$.

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[6]) G. N. Watson [Quart. J. of Math. (Oxford) 2 (1931)], 301 (6).

(IV) *The function* $A(s)$ *is of finite order in* $-b \leqq \sigma \leqq \sigma_1$; *that is, for some* t_0 *and some* B,

$$|A(s)| = O(|t|^B) \text{ when } |t| > t_0 \text{ and } -b \leqq \sigma \leqq \sigma_1 .$$

If these hypotheses are satisfied, then the originating Dirichlet's series has a corresponding summation formula. This summation formula is not, of course, applicable to all functions $f(x)$, but only to those which behave suitably. In § 3.2 we give a set of conditions which are sufficient to ensure that the summation formula shall be applicable.

3.11. It may be remarked that the preceding hypotheses are satisfied by any function whose general behaviour follows that of the zeta-function sufficiently closely. For example, in view of the known properties of the functions $\zeta(s)$ and $\eta(s)$, derived from

$$\sum n^{-s} \text{ and } \sum (-1)^n (2n+1)^{-s}$$

respectively, the hypotheses (I)—(IV) are satisfied by any function of the form $\{\zeta(s)\}^a \{\eta(s)\}^b$, where a, $b > 0$.

3.2. *Conditions that the summation formula be applicable to a function* $f(x)$.

As in § 2.2, we impose conditions on the function $\varphi(s)$ used in the definition of $f(x)$, rather than upon $f(x)$ itself.

Let $\varphi(s)$ be a function of s which satisfies the following conditions:

(I) $\varphi(s)$ *is analytic, save possibly for poles, in a strip* $-b \leqq \sigma \leqq 1 + c$, *where* b, $c > 0$ *and* $1 + c > \sigma_1$;

(II) *the only pole in this strip is a simple pole at* $s = 0$, *where the residue is* φ_0;

(III) *in the strip* $-b \leqq \sigma \leqq 1 + c$

$$|\varphi(\sigma+it)| = O(|t|^{-\lambda}) \qquad (|t| > t_0),$$

where $\lambda > max\ (1, K, B)$;

(IV) $|A(-b+it)\,\varphi(-b+it)|$ *is integrable in* $(-\infty, \infty)$.

Under these conditions [7]),

$$\frac{1}{2\pi i} \int_{\sigma} \varphi(s) x^{-s}\, ds \qquad (0 < \sigma \leqq 1 + c)$$

[7]) The condition (II), like (II) of § 2.2, may be relaxed at the cost of a suitable modification of the resulting formula. Further, (III), like (III) of § 2.2, is an ad hoc condition that enables the subsequent transformations to be carried out.

defines, for any positive x, a unique function $f(x)$ such that

$$f(x) = O(x^{-1-c}) \qquad \text{when } x \text{ is large,}$$
$$f(x) = \varphi_0 + O(x^b) \qquad \text{when } x \text{ is small.}$$

Further, as in § 2.3,

$$\varphi(s) = \int_0^\infty f(x) x^{s-1} dx \qquad (0 < \sigma < 1 + c).$$

3.3. The same sequence of transformations as that used in § 2 now gives

$$\sum_{n=1}^{\infty} a_n f(n) = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{1+c} a_n \varphi(s) n^{-s} ds$$
$$= \frac{1}{2\pi i} \int_{1+c} \varphi(s) \psi(s) ds$$
$$= R_0 + (R) + \frac{1}{2\pi i} \int_{-b} \varphi(s) A(s) \psi(1-s) ds,$$

where R_0, (R) represent residues at $s = 0$ and at poles of $\psi(s)$ other than $s = 0$.

But, on using the absolutely convergent series

$$\psi(1-s) = \sum a_n n^{s-1},$$

$$\frac{1}{2\pi i} \int_{-b} \varphi(s) A(s) \psi(1-s) ds$$
$$= \sum_{n=1}^{\infty} \frac{a_n}{2\pi i} \int_{-b} \varphi(s) A(s) n^{s-1} ds$$
$$= \sum_{n=1}^{\infty} a_1 \left\{ -\frac{A(0)\varphi_0}{n} - \Theta_n + \frac{1}{2\pi i} \int_{\sigma_1} \varphi(s) A(s) n^{s-1} ds \right\},$$

where Θ_n is the sum of residues at the poles of $A(s)$ in (δ, σ_1). [By hypothesis, $A(s)$ is regular in $(-b, \delta)$.]

Further,

$$\int_{\sigma_1} A(s) n^{s-1} \varphi(s) ds$$
$$= \int_{\sigma_1} A(s) n^{s-1} ds \int_0^\infty f(x) x^{s-1} dx$$
$$= \int_0^\infty f(x) dx \int_{\sigma_1} A(s) (nx)^{s-1} ds,$$

the change of order [8]) being justified by absolute convergence.

Hence, on using the fact that

$$R_0 = \varphi_0 \cdot \psi(0) = f(+0) \cdot \psi(0),$$

and defining the function $\beta(y)$ by means of the equation

$$\beta(y) = \frac{1}{2\pi i} \int_{\sigma_1} A(s) y^{s-1} ds,$$

we see that

$$- \psi(0) \cdot f(+0) + \sum_{n=1}^{\infty} a_n f(n)$$

$$= (R) + \sum_{n=1}^{\infty} a_n \left\{ -\frac{A(0)\varphi_0}{n} - \Theta_n + \int_0^{\infty} f(x)\beta(nx)dx \right\}. \quad (10)$$

3.4. This general formula admits of considerable simplification in most of the special cases that have been investigated. Examples of such simplification are given below.

(a) In some cases, e.g. $\psi(s) = \zeta(s)$,

$$A(0) = \frac{\psi(0)}{\psi(1)} = 0.$$

(b) If the abscissa of convergence of $\psi(s)$ is less than unity, e.g.

$$\psi(s) = \eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - \ldots,$$

then

$$\sum a_n n^{-1} = \psi(1),$$

and the term $-\varphi_0 A(0) \sum a_n n^{-1}$ on the one side of equation (10) merely cancels the term $-\psi(0) \cdot f(+0)$ on the other.

(c) If there is a number σ_3, less than σ_1, such that

$$A(\sigma + it) = o(1)$$

when $\sigma_3 \leqq \sigma \leqq \sigma_1$ and $|t|$ is large, then it is possible to absorb all terms of Θ_n that arise from poles of $A(s)$ in (σ_3, σ_1) into the integral term of (10).

What happens is sufficiently shown by considering the simple case $\psi(s) = \zeta(s)$. Here

$$A(s) = \frac{\pi^{s-\frac{1}{2}} \Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}, \qquad |A(\sigma + it)| = O(|t|^{\frac{1}{2}-\sigma}),$$

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[8]) In the corresponding work of § 2.3 we did not introduce σ_1; we there had only conditional convergence.

and we take $\sigma_1 = \frac{3}{2} + \varepsilon$ [cf. § 3.1 (III)], where ε is small. Further, we take $\sigma_3 = \frac{1}{2} + \varepsilon$. The expression

$$- \Theta_n + \int_0^\infty f(x)\, dx \int_{\sigma_1} \frac{A(s)\,(nx)^{s-1}}{2\pi i}\, ds$$

of (10) is, in this particular case,

$$2\varphi(1) + \int_0^\infty f(x)\, dx \int_{\frac{3}{2}+\varepsilon} \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\,\Gamma(\frac{1}{2}s)}\,(nx)^{s-1} ds$$

$$= \int_0^\infty f(x)\, dx \left\{ 2 + \int_{\frac{3}{2}+\varepsilon} \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\,\Gamma(\frac{1}{2}s)}\,(nx)^{s-1} ds \right\}$$

$$= \int_0^\infty f(x)\, dx \int_{\frac{1}{2}+\varepsilon} \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\pi i\,\Gamma(\frac{1}{2}s)}\,(nx)^{s-1} ds$$

$$= 2 \int_0^\infty f(x) \cos 2n\pi x\, dx\,.$$

The arithmetical details when $A(s)$ has double poles, for example when $\psi(s) = \{\zeta(s)\}^2$, are rather more tedious, but present no serious difficulty.

3.5. Finally, formula (10) may be modified so as to permit of its application to certain functions $f(x)$ derived from functions $\varphi(s)$ which do not satisfy condition (II) of § 3.2. The sort of modification that will be necessary is shown by § 2.5, where the special case $\psi(s) = \zeta(s)$ is considered.

4. *The Dirichlet's series considered by Hecke.*

A further extension of (1) arises in the following way. Consider the Dirichlet's series

$$\psi(s) = \Sigma\, a_n n^{-s}$$

which are such that $(s-k)\,\psi(s)$ is an integral function and, further, are such that

$$R(s) = \gamma R(k-s)\,,$$

where

$$R(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\,\psi(s)$$

and λ, k are positive numbers. Such series have recently been studied by Hecke [9]).

[9]) E. Hecke, Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung [Math. Ann. **112** (1936), 664—699].

The work of § 3 can be modified to prove the existence of a summation formula that corresponds to each such Dirichlet's series. The definition of the function $A(s)$ in § 3.1 must be replaced by

$$A(s) = \frac{\psi(s)}{\psi(k-s)} = \gamma \left(\frac{2\pi}{\lambda}\right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)},$$

and the other modifications are of an equally simple character. We shall not develop the details.

Actually, the basis of Hecke's interesting paper, the transference of his problem to the field of automorphic functions, (*vide* § 2 of that paper) is effected by considering the Mellin integral

$$e^{-x} = \frac{1}{2\pi i}\int_c \frac{\Gamma(s)}{x^s}\, ds \qquad (c > 0).$$

The proof of the summation formula corresponding to the Dirichlet's series is effected by considering, as in the earlier sections of the present paper,

$$f(x) = \frac{1}{2\pi i}\int_c \frac{\varphi(s)}{x^s}\, ds,$$

where c is such that $\Sigma\, |a_n| n^{-c}$ is convergent and $\varphi(s)$ is a function which permits the transformations.

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