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MICHAEL KERNER

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Abstract differential geometry

by

Michael Kerner

Warsaw

1. *Introduction.* When building a differential geometry one starts usually with the conception of an „arithmetic" space of n dimensions whose properties are presumed to be known. The geometric manifold is then n-dimensional.

Recently were also published some papers, as these of Michal, Kawaguchi, Conforto and others [1]), which have replaced the arithmetic space by particular functional spaces.

A farther step in this direction leads to the investigation of the differential geometry built upon an abstract space. To this problem is devoted the present paper. We transplant the geometric research to a space of type (B) as basis. This space is also known as Banach space. It is a *linear, normed and complete* space [2]) which embraces the n-dimensional space, the Hilbert space and many others as special cases.

We attempt to build a geometric (in general an affine) manifold on an underlying space of type (B) with the tendency to get a perfect conformity with differential geometry of n dimensions in the sense, that our theory must become identical with the former when the abstract space is taken in particular as n-dimensional. On account of this tendency one can consider our theory as a didactically new representation of the classical differential geometry which avoids the employing of coordinates and the flood of indices. The new representation yields of course also some inconveniences peculiar to it, and it were scarcely worth the trouble to introduce the new apparatus only for the sake of notation simplicity.

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[1]) A detailed bibliography one can find in STRUIK's Theory of Linear Connections [Berlin 1934] and KAWAGUCHI's paper [Journ. Hokkaido Imp. Univ. (1) 3 (1935), 103—106].

[2]) Cf. BANACH, Théorie des opérations linéaires [Warszawa 1932], 53.

In one regard only we decided in favour of a diversion from the classical theory. We defined the tangent spaces of a manifold and the vectors and tensors in such a manner that they do not undergo any transformations. It is done without any deep-seated artifice and allows us to return easily in the case of n dimensions to the classical treatment. We wished to consider the tangent spaces as something absolute, and to submit to transformations only the representation of their connection.

When passing to a more general ground one must renounce some notions and some properties of the examined object. The most important is that there are in an abstract geometric manifold no tensors of contravariant order greater than one. It seems, however, that these play no natural role in geometry, and the absence of them corresponds somewhat to the essential circumstances. The analogy between covariant and contravariant quantities is to be lost of course.

In the general case we call the object of investigation the „manifold" and not the „space", because it is composed itself of spaces (namely the tangent spaces). Moreover, as to terminology, we follow the classical one though it is not well adapted to our purposes. We do it in order to facilitate the understanding of our notation which is not accepted in geometry. We shall use the word „function" in a general sense replacing by it the words „functional" and „operation". The geometric word „representation" will be also often employed in this aim.

Part I.

Linear spaces.

2. *Vectors in the affine linear space.* We consider an abstract space of type (B), that is *a linear, normed and complete space.* We assume that the norm can be replaced by another one with the condition that both norms lead to the same determination of the limit. Hence it follows that both spaces (with two different allowable norms) are isomorphic [3]) as spaces of type (B), the isomorphism being determined by the linear operation which associates every element with itself. The ratio of norms in both spaces is uniformly limited for all elements from above as from below.

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[3]) Cf. Banach, loc. cit., 180.

Such a space $\mathscr{E}$ with not completely defined norm we shall call an *affine linear space.*

The capitals will denote the elements of the space $\mathscr{E}$, and the small letters the real numbers.

We call every point (element) A of the space $\mathscr{E}$ a contravariant vector of this space. The notion of a contravariant vector is not new (it was also called a vector), but the new term is introduced for distinction from a covariant vector. We stress especially that the notion of a contravariant vector is completely independent of any coordinate transformations.

We call every linear (additive and continuous) functional $f(X)$ *in the space* $\mathscr{E}$ *a covariant vector of this space.* Therein X denotes an indefinite variable which can run all over the space $\mathscr{E}$. We do not deal here with the numerical value of the functional, but with the functional itself, as an operation. We shall often denote the covariant vectors also by the symbol $f(.)$.

The covariant vectors are elements of the space $\overline{\mathscr{E}}$ conjugate to $\mathscr{E}$ [4]). This space (that is the space of linear functionals) is also normed, but, like in $\mathscr{E}$, the norm is therein not completely determined.

When given a covariant vector $f(.)$ *and a contravariant vector* A, *we call the numerical value* $f(A)$, *taken by the functional* $f(.)$ *for the element* A, *the inner (scalar) product of* $f(.)$ *and* A.

If the inner product vanishes,

$$f(A) = 0,$$

the covariant vector $f(.)$ *and the contravariant vector* A *are called orthogonal one to another.* This notion is not new [5]).

If two spaces $\mathscr{E}$ and $\mathscr{E}'$ of type (B) are isomorphic, we shall call also the corresponding affine linear spaces *isomorphic.* Because of the allowed change of the norm in such spaces there is no notion of equivalence of spaces [6]), the latter being not distinguished among the isomorphisms. As an example of isomorphic spaces one can consider a space, vectors of which are forces, and the ordinary configuration space.

One can build inner products of covariant and contravariant vectors which belong to isomorphic spaces. One must only replace

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[4]) Cf. Banach, loc. cit., 188.

[5]) Cf. Banach, loc. cit., 59.

[6]) Cf. Banach, loc. cit., 180.

the contravariant vector of one space by the corresponding vector of the other, the isomorphism being uniquely determined of course.

3. *Tensors in the affine linear space.* We use the word „tensor" in a general sense without any symmetry restrictions.

We define *no contravariant tensors of order greater than one.* This is the most important difference we meet with when dealing with the abstract geometry.

We call every n-linear functional $f(X_1, X_2, \ldots, X_n)$ in the space $\mathscr{E}$ a covariant tensor of order n of this space. We write also $f(.\,,\,.\,,\ldots,\,.)$, but this kind of notation is inconvenient, if the numerical value of n is not given. We remember that n-linear functionals are functionals linear relative to each their argument.

A covariant tensor is said to be *symmetric* (resp. *alternating*), if the n-linear functional $f(X_1, X_2, \ldots, X_n)$ has the same property.

The inner (scalar) product of a covariant tensor $f(X_1, X_2, \ldots, X_n)$ of order n and contravariant vectors $A_1, A_2, \ldots,$ is to be defined in like manner as for covariant vectors. If the number of vectors $A_1, A_2, \ldots, A_n$ equals to the order n of the tensor, the inner product $f(A_1, A_2, \ldots, A_n)$ is the numerical value of the multilinear functional, when for all its arguments are substituted succesively the vectors $A_1, A_2, \ldots, A_n$. The order of these vectors is to be respected. Only in the case of symmetric tensors it is indifferent. In general there are on the contrary $n!$ different products.

If on the other hand the number m of vectors $A_1, A_2, \ldots, A_m$ is less than the order n of the tensor, the different inner products are covariant tensors of order $(n-m)$, for instance

$$f(A_1, A_2, \ldots, A_m, X_{m+1}, \ldots, X_n).$$

In general there are $\frac{n!}{(n-m)!}$ product formations, which become identical, if the tensor is symmetric.

We call every n-linear operation $F(X_1, X_2, \ldots, X_n)$ which associates with the elements $X_1, X_2, \ldots, X_n$ of the space $\mathscr{E}$ an element of the same space, a mixed tensor of contravariant order one and covariant order n of this space (also of total order $n+1$). It can be also written $F(.\,,\,.\,,\ldots,\,.)$.

The real numbers, the covariant and contravariant vectors, the covariant and mixed tensors are called all together *quantities.*

A mixed tensor is said to be *symmetric* (resp. *alternating*), if the n-linear operation $F(X_1, X_2, \ldots, X_n)$ has the same property.

These terms refer only to the covariant behaviour of a mixed tensor.

As to the inner product there are $n!$ different products by n contravariant vectors $A_1, A_2, \ldots, A_n$, as for instance $F(A_1, A_2, \ldots, A_n)$. These are themselves contravariant vectors (as elements of the space $\mathscr{E}$). In the case of m vectors there are $\frac{n!}{(n-m)!}$ products, as for instance $F(A_1, A_2, \ldots, A_m, X_{m+1}, \ldots, X_n)$, which are mixed tensors of covariant order $(n-m)$ (we shall for mixed tensors declare only the covariant order). All these products become identical, if the original mixed tensor is symmetric.

In particular, the product of a mixed tensor $F(.)$ of total order two by a contravariant vector A is a contravariant vector $F(A)$.

We have yet to define the outer product. It is identical with the ordinary product, and may be defined only when at least one of two quantities is purely covariant. In other words, *the outer product of two covariant tensors* $f(X_1, X_2, \ldots, X_n)$ *and* $g(X_1, X_2, \ldots, X_m)$ *of order n and m respectively is the covariant tensor*

$$h(X_1, X_2, \ldots, X_{n+m}) = f(X_1, X_2, \ldots, X_n) \cdot g(X_{n+1}, X_{n+2}, \ldots, X_{n+m})$$

of order $(n+m)$; *the outer product of a covariant tensor* $f(X_1, X_2, \ldots, X_n)$ *of order n by a mixed tensor* $F(X_1, X_2, \ldots, X_m)$ *of covariant order m is the mixed tensor*

$$G(X_1, X_2, \ldots, X_{n+m}) = f(X_1, X_2, \ldots, X_n) \cdot F(X_{n+1}, X_{n+2}, \ldots, X_{n+m})$$

of covariant order $(n+m)$.

One defines likewise the outer product of a mixed tensor by a covariant one. The definitions embrace also the case of vectors, because the contravariant vector is a special case of mixed tensor (of covariant order zero), and the covariant vector is a special case of covariant tensor (of order one). The outer product is not permutable.

4. *Metric linear space.* We introduce in an affine linear space a *quadratic metric* by defining a covariant fundamental tensor $g(X, Y)$ of order two. We assume that

1. the tensor $g(X, Y)$ is symmetric,

$$g(X, Y) = g(Y, X) \tag{4.1}$$

2. there is such a positive number l that

$$g(X, X) \geqq l \cdot \|X\|^2 \tag{4.2}$$

for every X (the metric is „*uniformly*" *positive definite*).

The second condition is independent on eventual allowable change of norm.

An affine linear space with a metric which satisfies both enumerated conditions is called a metric linear space.

The following inequality is true

$$(4.3)\qquad |g(X, Y)| \leqq m \cdot \|X\| \cdot \|Y\|,$$

where m is an appropriate positive number[7]). Hence we obtain in particular the inequality.

$$(4.4)\qquad g(X, X) \leqq m \cdot \|X\|^2,$$

analogous to (4.2).

If $\mathscr{R}$ is a metric linear space, every contravariant vector A has an *absolute value* equal to $\sqrt{g(A, A)}$.

The inequalities (4.2) and (4.4) allow us to prove that the square $g(X, X)$ of the absolute value of X is an allowable norm. We omit this easy demonstration.

In a metric linear space $\mathscr{R}$ to each contravariant vector A corresponds a covariant vector

$$(4.5)\qquad a(X) = g(X, A) = g(A, X)$$

which is to be denoted by the same letter as the former (but small, of course). We call $a(X)$ *the covariant vector conjugate to* A. It is an element of the space $\overline{\mathscr{R}}$ conjugate to $\mathscr{R}$, that is, of the space of linear functionals in $\mathscr{R}$. However the correspondence

$$(4.6)\qquad A \to a(X)$$

is in general not reversible for every $a(X)$. It carries the whole space $\mathscr{R}$ into a linear set $\overline{\mathscr{T}}$ of the conjugate space $\overline{\mathscr{R}}$.

The operation (4.6), where $a(X)$ is defined by (4.5), admits a uniquely determined inverse operation in $\overline{\mathscr{T}}$, and both operations are linear.

Indeed, if there were two contravariant vectors A and B, for which

$$g(X, A) = g(X, B),$$

it would follow for every X

$$g(X, A - B) = 0,$$

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[7]) See KERNER, Die Differentiale in der allgemeinen Analysis [Ann. of Math. (2) **34** (1933), 548].

and in particular

$$g(A-B,\ A-B)=0,$$

contradicting to (4.2), when A and B are not equal. The inversion of (4.6) is thus uniquely determined. As to linearity, both operations are obviously additive. There remains the continuity proof. It follows from (4.3) that

$$|a(X)| \leqq m \cdot \|A\| \cdot \|X\|,$$

and hence, according to the definition of the functional's norm,

$$\|a(X)\| \leqq m \cdot \|A\|.$$

The direct operation is thereby continuous. On the other hand, it follows from (4.2) that

$$a(A) \geqq l \cdot \|A\|^2,$$

and hence

$$\|a(X)\| \geqq l \cdot \|A\|.$$

Thus, the inverse operation is continuous, too.

It follows from this continuity *that the linear set* $\overline{\overline{\mathscr{T}}}$ *of the space* $\overline{\overline{\mathscr{R}}}$ *is closed.* We wish to observe that $\overline{\overline{\mathscr{T}}}$ is *total* [8]).

For, if A is such an element that

$$f(A)=0$$

for every functional $f(X)$ of the set $\overline{\overline{\mathscr{T}}}$, we have in particular

$$a(A)=0,$$

or

$$g(A,\ A)=0,$$

whence, according to (4.2), it follows

$$A=0.$$

The most important is the case when the set $\overline{\overline{\mathscr{T}}}$ coincides with the whole space $\overline{\overline{\mathscr{R}}}$. In other words, the correspondence (4.6) is then reversible for every $a(X)$. To each covariant vector $a(X)$ there corresponds a contravariant vector A, for which

$$g(X,\ A)=a(X).$$

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[8]) Cf. BANACH, loc. cit., 42.

In such a case we call the metric *spanning* (because the conjugate covariant vectors relative to this metric span the conjugate space $\overline{\mathscr{R}}$).

If the metric is spanning, the space $\mathscr{R}$ is isomorphic with the conjugate space $\overline{\mathscr{R}}$. Since this property is attributed to spaces only exceptionally, one cannot in general introduce a spanning metric into a space. We shall soon see that not only a spanning metric, but also an ordinary (positive definite) metric cannot be introduced in general.

We turn now to the special case of *separable* space. According to what was said above we can take the square $g(X, X)$ of the absolute value as norm of X. Then it is easy to see that for our space $\mathscr{R}$ are satisfied all the axioms which characterize the real Hilbert space, when regarding the bilinear functional $g(X, Y)$ as the inner product in this space [9]). This conception of Hilbert space embraces also the case of the euclidean space of a finite number of dimensions.

Hence, if the original space of type (B) is separable, it must be isomorphic to Hilbert space. Thus, not into every space can one introduce a quadratic positive definite metric, if no restrictions about spanning are made.

We return to the separable metric linear space $\mathscr{R}$. If normed by $g(X, X)$ it is a Hilbert space. In such a space every linear functional can be represented by $g(X, A)$, where, as said, $g(X, A)$ denotes the inner product of X by A, and A is an appropriate element of the space $\mathscr{R}$. Therefore every covariant vector is conjugated to a contravariant one. We have to note an important theorem:

In a separable space each quadratic positive definite metric is spanning.

5. *Vectors and tensors in the metric linear space.* In a metric linear space we can define *the inner product of two contravariant vectors A and B.* It is equal to the inner product of the covariant vector $a(X)$ conjugate to A by the second contravariant vector B, that is, to $a(B)$. The inner product is permutable, since

$$b(A) = g(A, B) = g(B, A) = a(B). \tag{5.1}$$

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[9]) See NEUMANN, Mathematische Grundlagen der Quantenmechanik [Berlin 1932], 19—24. The axioms are built for complex Hilbert space.

In a separable space it is identical with the inner product in the isomorphic Hilbert space.

If the metric is *spanning, the inner product of two covariant vectors $a(X)$ and $b(X)$* can be defined. It is equal to the inner product of one vector $a(X)$ by the contravariant vector B, to which $b(X)$ is conjugate, that is, to $a(B)$. It is permutable, too, because of the same equalities (5.1).

When the metric is not spanning, the last definition is of value only if at least one of the covariant vectors $a(X)$ and $b(X)$ belongs to the set $\overline{\overline{\mathscr{T}}}$ of the conjugate space $\overline{\overline{\mathscr{R}}}$.

If the inner product of two covariant or contravariant vectors vanishes, these are called orthogonal one to another.

To each mixed tensor $A(Y, Z, \ldots)$ in the space $\mathscr{R}$ corresponds a conjugated covariant tensor $g[X, A(Y, Z, \ldots)]$ of the same total order (the process leading from the former to the latter is analogous to the usual lowering of a contravariant index).

If the metric is spanning, the inverse process to the former can be defined. Let $a(X, Y, Z, \ldots)$ to be a covariant tensor. We wish, for instance, to „bring up the first index" which corresponds to the variable X. We must for this purpose determine a mixed tensor $A(Y, Z, \ldots)$, for which

$$g[X, A(Y, Z, \ldots)] = a(X, Y, Z, \ldots). \tag{5.2}$$

We have to solve for fixed Y, $Z, \ldots$ the equation

$$g(X, U) = a(X, Y, Z, \ldots) \tag{5.3}$$

relative to U. This equation has a unique solution U which depends on Y, $Z, \ldots$ We ought to prove that this solution is a multilinear operation (mixed tensor). The additiveness of this operation is obvious on account of the unicity of solution of (5.3). We yet must still prove that U is a continuous function of Y for given $Z, \ldots$ The proof for the remaining variables is the same. Let the values of $Z, \ldots$ to be fixed. We have then the inequality.

$$|a(X, Y, Z, \ldots)| \leqq k \cdot \|X\| \cdot \|Y\|$$

for an appropriate positive number k. Hence, on account of (5.3),

$$|g(X, U)| \leqq k \cdot \|X\| \cdot \|Y\|.$$

We put here

$$X = U,$$

and conclude from (4.2)

$$l \cdot \|U\|^2 \leqq k \cdot \|U\| \cdot \|Y\|,$$

or

$$\|U\| \leqq \frac{k}{l} \cdot \|Y\|,$$

whence the continuity relative to Y follows. Conclusively, the solution of (5.3) is a mixed tensor $A(Y, Z, \ldots)$ which satisfies (5.2).

When wishing to „bring up the second index", which corresponds to Y, one must determine a mixed tensor $A(X, Z, \ldots)$ from the equation

$$g[Y, A(X, Z, \ldots)] = a(X, Y, Z, \ldots).$$

If the original covariant tensor $a(X, Y, Z, \ldots)$ is symmetric, all mixed tensors obtained in such a manner are identical.

The notion of inner product of tensors can be also generalized. In a metric linear space with a spanning metric one can for instance obtain the inner product of a covariant tensor $a(X, Y)$ by covariant vectors $b(X)$ and $c(X)$, when forming the inner product $a(B, C)$ of $a(X, Y)$ by the contravariant vectors B and C, to which $b(X)$ and $c(X)$ are conjugate. It is not worth the trouble to enumerate all the possible cases on this account.

Part II.

General manifolds.

6. *Bare manifolds.* Let $\mathscr{S}$ be a space of type (B). We call it a *substratum* (underlying space). We suppose that to every point P of the substratum $\mathscr{S}$ or of a region $\mathscr{B}$ of the space $\mathscr{S}$ there corresponds an affine linear space $\mathscr{T}_P$ which is isomorphic to the substratum $\mathscr{S}$, as a space of type (B). This space $\mathscr{T}_P$ we call *the tangent space* at point P. The tangent spaces at different points of $\mathscr{S}$ are therefore isomorphic to each other. The totality of tangent spaces under their connection with the substratum, which will be stated beneath, is called *a bare (geometric) manifold* and denoted by $\mathscr{X}$.

Because of the isomorphism of all tangent spaces $\mathscr{T}_P$ we can represent them all on an arbitrarily chosen tangent space. There are of course many ways of such a representation. We choose fixed linear transformations of all tangent spaces into one of them, and decide to denote by the same letter, say X, the elements of different spaces, associated one to another by these transformations. The tangent spaces being affine linear, we replace the norms therein by the norm in the distinguished tangent space. In such a manner all the tangent spaces $\mathcal{T}_P$ become identical, and we can speak of a unique tangent space $\mathcal{T}$. The way of transforming all the spaces into one of them does not play any geometric role and can be replaced by another one. But for continuity and differentiation considerations one must choose a fixed transformation and conserve it during all the investigations.

Now we formulate two conditions defining bare manifolds: the connection between the tangent spaces and the substratum, and the allowable substratum changes.

1. The condition of connection between the tangent spaces and the substratum. *For every point P of the substratum $\mathcal{S}$ (resp. of a region $\mathcal{B}$) there is given a one to one linear transformation of the space $\mathcal{S}$ into the whole tangent space $\mathcal{T}$ (or $\mathcal{T}_P$). This transformation and its inverse are continuous functions of the point P.*

We denote the direct linear transformation (function) by

$$X = F_P(Q). \tag{6.1}$$

Here Q is an element of the space $\mathcal{S}$, and X one of the space $\mathcal{T}$. The dependence on the point P is marked by an index. The inverse function is also linear [10]) and will be denoted by

$$Q = \Phi_P(X). \tag{6.2}$$

Both functions (6.1) and (6.2) are continuous with respect to P. It is interesting that the continuity of (6.1) does not imply that of (6.2) [11]).

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[10]) See BANACH, loc. cit., 41.

[11]) We wish to show it on an example. Let P, Q, X to be elements of the Hilbert space formed by sequences: $p_1, p_2, \ldots; q_1, q_2, \ldots; x_1, x_2, \ldots$; let us denote by $n = E\left[\frac{1}{\|P\|}\right]$ the greatest integer contained in $\frac{1}{\|P\|}$. Then let us define the function (6.1) as follows:

when $P \neq 0$: $\quad x_i = q_i$ for $i \neq n$; $x_n = \frac{q_n}{n}$;

when $P = 0$: $\quad x_i = q_i$ for every i.

One can prove that the function is continuous in respect to P for $P = 0$, while the inverse function is not (it is easy to see the last fact when putting $x_i = \frac{1}{i}$). The discontinuities on the spheres $\|P\| = n$ (n integer) can be easily smoothed.

If the preliminary transformation of all tangent spaces into one of them were changed, then the function (6.1) would alter. But now this change should be restricted to be continuous with respect to P. Otherwise the altered function $F_P(Q)$ would cease to be continuous.

In order to understand better the first condition one can imagine the elements Q as „infinitesimal displacements'' at the point of the substratum, and the elements X as something similar in the tangent space. We reject the usual identification of the neighbourhood of a point in the substratum and in the tangent space.

2. The condition of allowable substratum changes. *It is allowed to transform the region $\mathscr{B}$ of the space $\mathscr{S}$ one to one into a region $\mathscr{B}'$ of another space $\mathscr{S}'$ of type (B) isomorphic to $\mathscr{S}$, by means of a function*

$$P = P(P') \text{ or } P = P_{P'} .$$

This function and the inverse function

$$P' = P'(P) \text{ or } P' = P'_P$$

must be continuous and must have the continuous first differentials

$$dP_{P'}(Q') \text{ and } dP'_P(Q). \tag{6.3}$$

The space $\mathscr{S}'$ is then a new substratum under the condition that the function $X = F_P(Q)$ will be replaced by

$$X = F_{P(P')}[dP_{P'}(Q')] = F'_{P'}(Q') . \tag{6.4}$$

When changing the substratum $\mathscr{S}$, we consider the point P and the corresponding point P' of $\mathscr{S}'$ as the same point of the manifold. One can say that the function $F_P(Q)$ is transformed by the linear function tangent to $P_{P'}$.

The tangent space stays invariable! The substratum change induces only the mutual correspondence between the substratum and the tangent space. The situation is the same as in the case of parametric representation of a curve. The substratum plays the role of a parameter. The change of substratum corresponds with the change of the parameter. We shall not have to deal at all with transformation of tangent spaces.

The changes of substratum form of course a group of transformations.

The totality of all tangent spaces with their connection with the substratum (condition 1) under admission of allowable substratum changes (condition 2) is called a bare (geometric) manifold of class one.

The manifold $\mathscr{X}$ is *of class two*, if the functions $F_P(Q)$ and $\Phi_P(X)$ have the continuous first differentials

$$dF_P(Q;\, R) \text{ and } d\Phi_P(X;\, R),$$

and the allowable functions $P_{P'}$ and P'_P the continuous second differentials

$$d^2P_{P'}(Q',\, R') \text{ and } d^2P'_P(Q,\, R), \tag{6.5}$$

and so on. The differentials (6.5) are both symmetric [12]).

It follows from the identity

$$P[P'(P)] = P$$

by differentiating

$$dP_{P'(P)}[dP'_P(Q)] = Q. \tag{6.6}$$

The formula (6.4) implies

$$Q' = \Phi'_{P'}(X)$$

and

$$dP_{P'}(Q') = \Phi_{P(P')}(X),$$

whence

$$\Phi_{P(P')}(X) = dP_{P'}[\Phi'_{P'}(X)] \tag{6.7}$$

or, on account of (6.6),

$$\Phi'_{P'}(X) = dP'_{P(P')}[\Phi_{P(P')}(X)]. \tag{6.8}$$

If the manifold $\mathscr{X}$ is of class two, we can differentiate the identity

$$F_P[\Phi_P(X)] = X,$$

getting thus

$$dF_P[\Phi_P(X);\, Q] + F_P[d\Phi_P(X;\, Q)] = 0$$

or, by substituting $Q = \Phi_P(Y)$,

$$dF_P[\Phi_P(X);\; \Phi_P(Y)] = -\, F_P\{d\Phi_P[X;\; \Phi_P(Y)]\}. \tag{6.9}$$

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[12]) See KERNER, loc. cit. in footnote 7, 549.

We denote by $T_P(X, Y)$ the function obtained from (6.9) by alternating, that is

$$\begin{aligned} &2T_P(X,\ Y) = \\ (6.10)\quad &= dF_P[\Phi_P(X);\ \Phi_P(Y)] - dF_P[\Phi_P(Y);\ \Phi_P(X)] = \\ &= -F_P\{d\Phi_P[X;\ \Phi_P(Y)]\} + F_P\{d\Phi_P[Y;\ \Phi_P(X)]\}. \end{aligned}$$

One can prove that the symmetry of (6.9) [or the vanishing of $T_P(X, Y)$] is a necessary and sufficient condition that the tangent space (already transformed, of course) can be chosen in some sense itself as a particular substratum. The correspondence (6.1) becomes than a simple translation. We omit these considerations, because they have no significance for us, the preliminary transformation of tangent spaces into one of them [which influences the function $F_P(Q)$] having been taken accidentally.

7. *Vectors and tensors in the bare manifold.* If for each point P of a region $\mathscr{B}$ of the substratum $\mathscr{S}$ is defined a geometric quantity in the tangent space, and if this quantity is a continuous function of the point P, we speak of *a geometric quantity in the manifold* $\mathscr{X}$. Properly spoken, we have to deal with a field of quantities. Thus we can consider in the manifold $\mathscr{X}$ scalars (that is real numbers depending on the point P), contravariant and covariant vectors, covariant and mixed tensors.

The geometric quantities are quantities in tangent spaces, which do not depend on the substratum choice. Only when the definition of a function of proper type is bound to the substratum, one must prove the independence of this function on the substratum in order to know it to be a quantity.

The quantities shall be denoted by a_P (scalars), A_P (contravariant vectors), $a_P(X)$ (covariant vectors), $a_P(X, Y)$, $A_P(X)$, and so on, the dependence on the point P being brought into evidence by a subscript. But we shall often omit this subscript.

When the symbols with a „prime" are attached to the changed substratum $\mathscr{S}'$, the following conditions must hold for quantities

$$\begin{aligned} &a'_{P'} = a_P, \\ &A'_{P'} = A_P, \\ &a'_{P'}(X) = a_P(X), \quad \text{and so on.} \end{aligned}$$

A process which leads from geometric quantities to other ones is called *invariantive*.

To each quantity corresponds in every substratum a function of P which arises by representing the quantity on the substratum by means of the transformation $F_P(Q)$, resp. $\Phi_P(X)$. Thus, there corresponds to a contravariant vector A_P the element $\Phi_P(A_P)$, to a covariant vector $a_P(X)$ the linear functional $a_P[F_P(Q)]$, to a mixed tensor $A_P(X)$ the linear operation $\Phi_P\{A_P[F_P(Q)]\}$, and so on. We shall denote these functions by dashed letters, as $\overline{A}_P$, $\overline{a}_P(Q)$, $\overline{A}_P(Q)$. They depend on the substratum choice and coincide in a n-dimensional manifold with the quantities in the usual sense.

In the last paragraph we have introduced for a manifold of class two the function $T_P(X, Y)$, the definition of which has been attached to the substratum. Nevertheless, we shall now prove that it is a mixed tensor of covariant order two.

We differentiate the identity (6.4)

$$dF'_{P'}(Q'; R') = dF_{P(P')}[dP_{P'}(Q'); dP_{P'}(R')] + F_{P(P')}[d^2P_{P'}(Q', R')],$$

and alternate, having regard to the symmetry of the second differential $d^2P_{P'}(Q', R')$,

$$dF'_{P'}(Q'; R') - dF'_{P'}(R'; Q') =$$
$$= dF_{P(P')}[dP_{P'}(Q'); dP_{P'}(R')] - dF_{P(P')}[dP_{P'}(R'); dP_{P'}(Q')].$$

Now we put here

$$Q' = \Phi'_{P'}(X), \quad R' = \Phi'_{P'}(Y),$$

and obtain with respect to (6.7), according to the definition (6.10),

$$T'_{P'}(X, Y) = T_{P(P')}(X, Y),$$

whence it follows that $T_P(X, Y)$ is a tensor.

8. *Tangent differentials.* We shall introduce in a manifold of class two a process which will be called *tangent differentiation.*

According to what was mentioned above each quantity can be represented on the substratum by a function which is obtained by transforming this quantity by means of $F_P(Q)$ or $\Phi_P(X)$. We form the differential of this function, and transform it backwards to the tangent space. The function obtained in such a manner we call the tangent differential and denote by the sign $\mathfrak{d}$. In some sense we have a formal relation

$$\overline{\mathfrak{d}} = d.$$

We shall explain it better by discussing several special cases.

Let f_P be a scalar which has the continuous first differential $df_P(Q)$. It is not necessary to transform it to the substratum. Then, we must differentiate it, getting $df_P(Q)$, and transform this to the tangent space, obtaining

$$\eth f_P(X) = df_P[\Phi_P(X)]. \tag{8.1}$$

We wish to prove that this tangent differential of a scalar is a covariant tensor. Indeed, when effecting the change $P_{P'}$ of substratum, we put

$$f'_{P'} = f_{P(P')},$$

whence it follows by differentiating that

$$df'_{P'}(Q') = df_{P(P')}[dP_{P'}(Q')].$$

Putting

$$Q' = \Phi'_{P'}(X),$$

we obtain

$$df'_{P'}[\Phi'_{P'}(X)] = df_{P(P')}\{dP_{P'}[\Phi'_{P'}(X)]\},$$

or, by virtue of (6.7),

$$df'_{P'}[\Phi'_{P'}(X)] = df_{P(P')}[\Phi_{P(P')}(X)],$$

that is

$$\eth f'_{P'}(X) = \eth f_{P(P')}(X).$$

In such a manner, *tangent differentiating of a scalar is an invariantive process*. The covariant vector (8.1) is called the *gradient of the scalar* f_P.

Now let A_P be a contravariant vector which has the continuous first differential $dA_P(Q)$. To get its tangent differential, we transform it into $\Phi_P(A_P)$, belonging to the substratum $\mathscr{S}$, then differentiate

$$\Phi_P[dA_P(Q)] + d\Phi_P(A_P; Q),$$

and at last represent the obtained function backwards by

$$\eth A_P(X) = dA_P[\Phi_P(X)] + F_P\{d\Phi_P[A_P; \Phi_P(X)]\}. \tag{8.2}$$

The first term of the right member does not depend on the substratum; but the second term, involving the operation $d\Phi_P(Q; R)$, does. Thus, the tangent differentiating of a contravariant vector is not invariantive; no more is it so in general.

We alternate (8.2), after having put $X = B_P$, and respecting (6.10),

$$(8.3)\quad \frac{1}{2}[\eth A_P(B_P) - \eth B_P(A_P)] = \\ = \frac{1}{2}\{dA_P[\Phi_P(B_P)] - dB_P[\Phi_P(A_P)]\} - T_P(A_P, B_P).$$

The right member does not depend on the substratum, and we see that the left member is a contravariant tensor.

Now we pass to the covariant vector $a_P(X)$ which has the continuous first differential $da_P(X; Q)$. The first transformation leads to $a_P[F_P(Q)]$, the differentiation to

$$da_P[F_P(Q); R] + a_P[dF_P(Q; R)],$$

and then the inverse transformation yields

$$(8.4)\quad \eth a_P(X; Y) = da_P[X; \Phi_P(Y)] + a_P\{dF_P[\Phi_P(X); \Phi_P(Y)]\}.$$

Also here the last term destroys the invariance.

By alternating (8.4), having respect to (6.10), we get

$$(8.5)\quad \frac{1}{2}[\eth a_P(X; Y) - \eth a_P(Y; X)] = \\ = \frac{1}{2}\{da_P[X; \Phi_P(Y)] - da_P[Y; \Phi_P(X)]\} + a_P[T_P(X, Y)],$$

whence it follows that the left member is a covariant alternating tensor of order two. This tensor is called the *rotation of the covariant vector* $a_P(X)$.

These examples show how to form the tangent differential of quantities. It is in some sense a representation of the differentiation in the substratum on the tangent space. The tangent differentials obey the same formal rules as the ordinary differential.

If $a_P(X)$ is itself a tangent differential of a scalar f_P, it follows from the way of forming the tangent differential that

$$\eth a_P(X; Y) = \eth^2 f_P(X, Y) = d^2 f_P[\Phi_P(X), \Phi_P(Y)].$$

Thus, if the scalar f_P has the continuous second differential $d^2 f_P(Q, R)$, its *second tangent differential* $\eth^2 f_P(X, Y)$ *is symmetric.* It is evident that the symmetry of $\eth a_P(X; Y)$ is also a sufficient condition that $a_P(X)$ is a gradient.

The analogous consideration for a contravariant vector A_P shows that

$$(8.6)\quad \eth^2 A_P(X, Y) = d^2 A_P[\Phi_P(X), \Phi_P(Y)] + \\ + F_P\{d\Phi_P[dA_P(\Phi_P(X)); \Phi_P(Y)]\} + \\ + F_P\{d\Phi_P[dA_P(\Phi_P(Y)); \Phi_P(X)]\} + \\ + F_P\{d^2\Phi_P[A_P; \Phi_P(X), \Phi_P(Y)]\},$$

whence it follows that $\eth^2 A_P(X, Y)$ is symmetric, if A_P has the continuous second differential.

9. *Affine connection.* We turn now to the principal object of the present paper. We shall define in a bare manifold $\mathscr{X}$ of class two *the covariant differential.* This determines in the manifold *the affine connection.* A manifold with an affine connection is called an *affine manifold* and is denoted by $\mathscr{L}$.

We define the covariant differential by the following five assumptions:

1. *The covariant differential of a quantity is a quantity whose covariant order is one greater than that of the given quantity.*

2. *The covariant differential of a scalar is equal to its tangent differential.*

3. *The covariant differential is a linear function of the quantity and its tangent differential, taken together.*

4. *The covariant differential of the outer product of two quantities obeys the same rule as the ordinary differential.*

5. *The covariant differential of the inner product of two quantities obeys the same rule as the ordinary differential*[13].

We shall denote the covariant differential by δ. The covariant differentials of a, A, $a(X)$, $A(X)$ are according to 1. $\delta a(X)$, $\delta A(X)$, $\delta a(X; Y)$, $\delta A(X; Y)$ (we omit from now the subscript P).

By virtue of 2. we have for a scalar f

$$\delta f(X) = \eth f(X). \tag{9.1}$$

We pass now to the covariant differential $\delta A(X)$ of a contravariant vector A. The assumption 3. means that $\delta A(X)$ is a linear function Ψ of the pair $[A, \eth A(X)]$,

$$\delta A(X) = \Psi\{[A, \eth A(X)]\}.$$

When putting here

$$[A, \eth A(X)] = [0, \eth A(X)] + [A, 0],$$

we get

$$\delta A(X) = \Psi\{[0, \eth A(X)]\} + \Psi\{[A, 0]\}.$$

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[13]) One can make assumptions involving much less than ours, but it were too long to derive the constitution of the covariant differential from them.

This equality is not quite explicit, since the function Ψ can depend also on the element X. By changing of notation and putting this dependence in evidence we can write

$$\delta A(X) = \Lambda[\eth A(X), X] + \Gamma(A, X), \tag{9.2}$$

where Λ and Γ are linear functions of their first arguments. Now we must apply the last formula to the contravariant vector equal to the outer product $f \cdot A$, having regard to the assumption 4. and to (9.1),

$$\begin{aligned} f \cdot \delta A(X) + \eth f(X) \cdot A = \\ &= f \cdot \Lambda[\eth A(X), X] + \eth f(X) \cdot \Lambda(A, X) + f \cdot \Gamma(A, X) = \\ &= f \cdot \delta A(X) + \eth f(X) \cdot \Lambda(A, X), \end{aligned}$$

whence

$$\Lambda(A, X) = A$$

for every A. In other words, Λ denotes the function independent of X and equal to the first argument A. Thus (9.2) becomes

$$\delta A(X) = \eth A(X) + \Gamma(A, X). \tag{9.3}$$

From the assumption 1. it follows that $\delta A(X)$ must be a linear function of X, so that the covariant differential of the contravariant vector A is given by (9.3), where $\Gamma(A, X)$ is a bilinear operation of both arguments A and X.

Every linear operation $\Gamma(X, Y)$ [exactly $\Gamma_P(X, Y)$], *which associates with* X, Y, *belonging to the tangent space, an element of the same space, determines an affine connection* and can be called also the affine connection. We shall see that it is not a mixed tensor. That $\Gamma(X, Y)$ can be taken arbitrary, follows from the fact that the covariant differential (9.3) and those determined beneath satisfy the assumptions 1—5. It is supposed of course that, when changing the substratum, $\Gamma(X, Y)$ is transformed in such manner that the covariant differential rests unaltered (is a quantity).

The next step is to determine the covariant differential $\delta a(X; Y)$ of a covariant vector $a(X)$. We choose an arbitrary contravariant vector A and form the inner product $a(A)$. The covariant differential of this product (which is a scalar) is equal to

$$\eth a(A; Y) + a[\eth A(Y)].$$

We apply on the other hand to $a(A)$ the rule 5. for obtaining the same differential

$$\delta a(A;\, Y) + a[\delta A(Y)].$$

When comparing both expressions, we get with regard to (9.3)

$$\partial a(A;\, Y) + a[\partial A(Y)] = \delta a(A;\, Y) + a[\partial A(Y)] + a[\Gamma(A,\, Y)].$$

The vector A being taken arbitrary, it follows from this equality that

$$(9.4) \qquad \delta a(X;\, Y) = \partial a(X;\, Y) - a[\Gamma(X,\, Y)],$$

that is the rule of covariant differentiating a covariant vector.

To obtain the covariant differential $\delta A(X;\, Y)$ of a mixed tensor $A(X)$, we form the inner product $A(B)$ by an arbitrary contravariant vector B and apply to it (9.3)

$$\partial A(B;\, Y) + A[\partial B(Y)] + \Gamma[A(B),\, Y],$$

and on the other hand the rule 5.

$$\begin{aligned} &\delta A(B;\, Y) + A[\delta B(Y)] = \\ &= \delta A(B;\, Y) + A[\partial B(Y)] + A[\Gamma(B,\, Y)]. \end{aligned}$$

The comparison of both expressions yields the covariant differential of a mixed tensor of total order two

$$(9.5) \quad \delta A(X;\, Y) = \partial A(X;\, Y) + \Gamma[A(X),\, Y] - A[\Gamma(X,\, Y)].$$

As last example we determine the covariant differential $\delta a(X,\, Y;\, Z)$ of a covariant tensor $a(X,\, Y)$. We form the inner product $a(A,\, Y)$, which is a covariant vector, and apply to it (9.4)

$$\partial a(A,\, Y;\, Z) + a[\partial A(Z),\, Y] - a[A,\, \Gamma(Y,\, Z)],$$

and on the other hand by the rule 5.

$$\begin{aligned} &\delta a(A,\, Y;\, Z) + a[\delta A(Z),\, Y] = \\ &= \delta a(A,\, Y;\, Z) + a[\partial A(Z),\, Y] + a[\Gamma(A,\, Z),\, Y]. \end{aligned}$$

The comparison of both expressions gives the covariant differential of the covariant tensor of order two

$$(9.6) \quad \delta a(X, Y; Z) = \partial a(X, Y; Z) - a[\Gamma(X, Z),\, Y] - a[X, \Gamma(Y, Z)].$$

The structure of the formulae (9.1), (9.3), (9.4), (9.5), (9.6) shows how to proceed in the general case of a tensor. It is for instance for a mixed tensor $A(X,\, Y)$ of covariant order two

$$\begin{aligned} (9.7) \quad \delta A(X,\, Y;\, Z) = \partial A(X,\, Y;\, Z) + \Gamma[A(X,\, Y), Z] - \\ - A[\Gamma(X,\, Z), Y] - A[X,\, \Gamma(Y,\, Z)]. \end{aligned}$$

10. *Torsion.* Let us consider the formula (9.3), that is

$$\delta A(X) = \eth A(X) + \Gamma(A, X).$$

According to the assumption 1. of the last paragraph $\delta A(X)$ is a mixed tensor, that is a function which does not depend on the substratum choice. Since $\eth A(X)$, determined by (8.2), has not the same property, the operation $\Gamma(X, Y)$ is not a tensor; it depends on the substratum choice. The process δ is invariantive, while the process determined by $\Gamma(.\,, X)$ is not.

It follows from (8.2) that

$$\Gamma(X, Y) + F_P\{d\Phi_P[X; \Phi_P(Y)]\} \tag{10.1}$$

does not depend on the substratum. Hence one can derive the rule of transformation of $\Gamma(X, Y)$. We omit this matter.

The invariance of (10.1) implies that of the alternated form [see the definition (6.10) of $T(X, Y)$]

$$\frac{1}{2}[\Gamma(X, Y) - \Gamma(Y, X)] - T(X, Y).$$

Since $T(X, Y)$ is a tensor, we conclude that

$$S(X, Y) = \frac{1}{2}[\Gamma(X, Y) - \Gamma(Y, X)] \tag{10.2}$$

is a tensor, too. This *alternating mixed tensor of covariant order two* we call *the torsion of the affine manifold.*

In case the torsion $S(X, Y)$ vanishes, that is, of

$$\Gamma(X, Y) = \Gamma(Y, X),$$

we call the affine manifold *symmetric* and denote by $\mathscr{A}$. If again

$$S(X, Y) = s(X)\cdot Y - s(Y)\cdot X,$$

where $s(X)$ is a covariant vector, we call the manifold *semi-symmetric.*

According to (9.4) the rotation (8.5) of a covariant vector $a(X)$ is equal in a general affine manifold to

$$\frac{1}{2}[\eth a(X; Y) - \eth a(Y; X)] = \frac{1}{2}[\delta a(X; Y) - \delta a(Y; X)] + a[S(X, Y)];$$

in a semisymmetric manifold to

$$\frac{1}{2}[\delta a(X; Y) - \delta a(Y; X)] + s(X)\cdot a(Y) - s(Y)\cdot a(X);$$

and in a symmetric manifold to

$$\frac{1}{2}[\delta a(X;\,Y) - \delta a(Y;\,X)].$$

11. *Parallel displacement.* Let $\mathscr{L}$ to be an affine manifold of class two. To the covariant differential $\delta A(X)$ there corresponds in every substratum the function

$$d\bar{A}(Q) + \bar{\Gamma}(\bar{A},\,Q), \tag{11.1}$$

where, according to paragraph 8,

$$\bar{A} = \Phi_P(A),$$

$$\bar{\Gamma}(Q,\,R) = \Phi_P\{\Gamma[F_P(Q),\,F_P(R)]\}. \tag{11.2}$$

Let $\mathfrak{C}$ to be a curve of class one, represented by the function $P(t)$, which admits the continuous first derivative $\frac{dP}{dt} = \dot{P}$. If the contravariant vector has the continuous first differential, we set $Q = \dot{P}$ in (11.1) and form the corresponding function in the tangent space

$$F_P[d\bar{A}(\dot{P})] + F_P[\bar{\Gamma}(\bar{A},\,\dot{P})].$$

We call this expression *the covariant derivative of A along the curve* $\mathfrak{C}$. It is equal to

$$\frac{\delta A}{dt} = \delta A[F_P(\dot{P})] + \Gamma[A,\,F_P(\dot{P})]. \tag{11.3}$$

One can call $F_P(\dot{P})$ the tangent derivative, denote it by $\frac{\partial P}{dt}$ and prove that it is a contravariant vector.

The same reasonning yields the covariant derivative of a covariant vector

$$\frac{\delta a(x)}{dt} = \partial a[X;F_P(\dot{P})] - a\{\Gamma[X,\,F_P(\dot{P})]\}. \tag{11.4}$$

All properties of the covariant differential, such as the rule of differentiating outer and inner products, can be transported to the covariant derivative.

If, in particular, *the covariant derivative of a quantity vanishes, this quantity is said to be submitted to a parallel displacement along the curve.* According to (11.3) the equation determining the parallel displacement of a contravariant vector is

$$\partial A[F_P(\dot{P})] = -\Gamma[A,\,F_P(\dot{P})],$$

or, if we pass to the substratum,

$$d\bar{A}(\dot{P}) = -\bar{\Gamma}(\bar{A}, \dot{P}).$$

This equation can be written, when putting

$$\bar{A}(t) = \bar{A}_{P(t)}, \quad A(t) = A_{P(t)},$$

in the form

$$\frac{d\bar{A}(t)}{dt} = -\bar{\Gamma}[\bar{A}(t), \dot{P}]. \tag{11.5}$$

That is a linear differential equation relative to the function $\bar{A}(t)$ [14]). It is equivalent to the equation

$$\frac{dA(t)}{dt} = -\Gamma[A(t), F_P(\dot{P})] - F_P\{d\Phi_P[A(t); \dot{P}]\} \tag{11.6}$$

relative to the vector $A(t)$ itself.

If the value A of $A(t)$ for a given t, say $t = 0$, is determined, there is just one solution of (11.6), equal to A for $t = 0$. For a fixed t this solution depends linearly on the initial value A. Since the right member of (11.5) or (11.6) is linear relative to $\dot{P}$, this equation is invariant under some change of the variable t. The solution $A(t)$ at a determined point P of the manifold depends therefore only on the curve $\mathfrak{C}$, but not on its parametric representation, and rests also unaltered, when changing the sense of the curve. Thus, if P_1 and P_2 are two points of the curve $\mathfrak{C}$, to each vector at one point is associated a vector at the other, and vice-versa. This is a linear one to one correspondence of two tangent spaces at P_1 and at P_2. When transported parallelly the tangent space undergoes a linear transformation which is a continuous function of the parameter t.

The relation (11.4) leads to the following differential equation, determining the parallel displacement of a covariant vector $a(X)$[or $a(X; t)$, when bringing in evidence its dependence on t]

$$\frac{d\bar{a}(Q;\, t)}{dt} = \bar{a}[\bar{\Gamma}(Q, \dot{P}); t]$$

which is of more complicated type. We can avoid the discussion of this equation, when reducing the investigation of a covariant vector to that of a contravariant one. This can be done by virtue of the fact that the inner product of $a(X)$ by an arbitrary contravariant vector A is invariant under parallel displacement. In other

[14]) As to the theory of differential equation in abstract spaces, cf. KERNER, Gewöhnliche Differentialgleichungen der allgemeinen Analysis [Prace mat. fiz. **40** (1932), 47—67].

words, the covariant vectors undergo, when displaced parallelly, a transformation which is conjugate to that of contravariant vectors.

The same reasonning allows to determine parallel displacement of other quantities.

12. *Pseudometric.* For the sake of some specializations it is convenient to introduce, if possible, a covariant symmetric tensor of order two, $g(X, Y)$ or $g_P(X, Y)$, which satisfies all assumptions of paragraph 4. Since this tensor does not play in all cases the role of metric, we shall call it *pseudometric.* We suppose in particular it to be spanning.

We denote by $q(X, Y, Z)$ the covariant differential of the pseudometric

$$\delta g(X, Y; Z) = q(X, Y, Z) \tag{12.1}$$

or

$$\partial g(X, Y; Z) - g[\Gamma(X, Z), Y] - g[X, \Gamma(Y, Z)] = q(X, Y, Z).$$

We effect therein two permutations of X, Y, Z.

$$\partial g(Y, Z; X) - g[\Gamma(Y, X), Z] - g[Y, \Gamma(Z, X)] = q(Y, Z, X),$$
$$\partial g(X, Z; Y) - g[\Gamma(X, Y), Z] - g[X, \Gamma(Z, Y)] = q(X, Z, Y).$$

We add the second and the third equation and subtract from the result the first one, having regard to the definition (10.2) of the torsion and to the symmetry of $g(X, Y)$,

$$\begin{aligned} &\tfrac{1}{2}[\partial g(Y, Z; X) + \partial g(X, Z; Y) - \partial g(X, Y; Z)] + \\ &+ \{g[X, S(Y, Z)] + g[Y, S(X, Z)] + g[Z, S(X, Y)]\} + \\ &- \tfrac{1}{2}[q(Y, Z, X) + q(X, Z, Y) - q(X, Y, Z)] = \\ &= g[Z, \Gamma(X, Y)]. \end{aligned} \tag{12.2}$$

If we denote the left member, which depends on g, q, S, in short by $\sigma(X, Y, Z)$, we get

$$g[Z, \Gamma(X, Y)] = \sigma(X, Y, Z). \tag{12.3}$$

Since the pseudometric $g(X, Y)$ is spanning, the equation (12.3) determines in a unique way the function $\Gamma(X, Y)$ for each X, Y. Its linearity is obvious. *The affine connection is therefore uniquely defined by means of the pseudometric* $g(X, Y)$, *its covariant differential* $q(X, Y, Z)$ *and the torsion* $S(X, Y)$.

The first term (12.2) may be called *the three-index symbol of*

Christoffel of first kind, and will be denoted in short by $c(X, Y, Z)$. The unique solution relative to $C(X, Y)$ of the equation

$$g[Z, C(X, Y)] = c(X, Y, Z)$$

may be called *the three-index symbol of Christoffel of second kind.*

If the pseudometric $g(X, Y)$ is not spanning, the resolution of (12.2) is possible only for certain determinations of the tensor $q(X, Y, Z)$. Namely for these ones, for which the left member of (12.2), as functional of Z, belongs for each X, Y to the part $\overline{\mathscr{T}}$ of the space conjugate to the tangent space, according to the results of paragraph 4.

13. *Different specializations of the affine manifold.* If, in particular,

$$\delta g(X, Y; Z) = q(X, Y, Z) = g(X, Y) \cdot q(Z), \tag{13.1}$$

where $q(Z)$ is a covariant vector, we get *the conformal connection.* It follows for such a manifold from (13.1) when putting

$$\gamma(X, Y) = \log g(X, Y),$$

that

$$\delta\gamma(X, Y; Z) = q(Z).$$

Therefore, when transporting parallelly two contravariant vectors along a curve, the ratio of their absolute values (defined by means of the pseudometric) rests invariant. Also their angle, defined by

$$\cos \alpha = \frac{g(X, Y)}{\sqrt{g(X, X)} \cdot \sqrt{g(Y, Y)}},$$

is unchanged.

A symmetric and conformal connection is called a *Weyl connection.* The equation (12.2) takes in this case the form

$$c(X, Y, Z) - \tfrac{1}{2} g[X \cdot q(Y) + Y \cdot q(X), Z] + \tfrac{1}{2} g(X, Y) \cdot q(Z) = g[Z, \Gamma(X, Y)]. \tag{13.2}$$

When denoting by $Q(X, Y)$ the solution of the equation

$$g[Z, Q(X, Y)] = \tfrac{1}{2} g(X, Y) \cdot q(Z),$$

it follows from (13.2)

$$\Gamma(X, Y) = C(X, Y) + Q(X, Y) - \tfrac{1}{2}[X \cdot q(Y) + Y \cdot q(X)].$$

If in the place of (13.1) we assume that

$$\delta g(X, Y; Z) = q(X, Y, Z) = 0, \tag{13.3}$$

the connection is called *metric*. From (13.3) it follows that, when transporting parallelly a contravariant vector along a curve, its absolute value is constant. In a metric manifold the pseudometric $g(X, Y)$ is called *the metric of the manifold.* Each tangent space of such a manifold is metric linear.

A symmetric and metric connection is called a *riemannian connection.* We have in this case

$$\Gamma(X, Y) = C(X, Y).$$

The function $\Gamma(X, Y)$ is identical with the three-index symbol of the second kind.

For a curve $\mathfrak{C}$ of class one represented by $P(t)$ $[t_1 \leqq t \leqq t_2]$ in a riemannian manifold the length is defined by the integral

$$\int_{t_1}^{t_2} \sqrt{g(\dot{P}, \dot{P})}\,dt, \tag{13.4}$$

or

$$\int_{t_1}^{t_2} \sqrt{g[F_P(\dot{P}), F_P(\dot{P})]}\,dt. \tag{13.5}$$

It is obviously a scalar (does not depend on the substratum choice).

An example of riemannian manifold with a Hilbert's substratum is given by the so-called „Riemann-Hilbert space" [15]).

14. *Geodesics.* Let $\mathfrak{C}$ to be a curve of class two represented by $P(t)$ in an affine manifold $\mathscr{L}$ of class three. To $\dot{P}$ corresponds in the tangent space $\mathscr{T}_P$ the contravariant vector

$$Z(t) = F_P(\dot{P}).$$

This vector determines in the tangent space the direction of the curve $\mathfrak{C}$. If the covariant derivative of $Z(t)$ along the curve $\mathfrak{C}$ has in each point the direction of $Z(t)$ itself, the curve $\mathfrak{C}$ is called a *geodesic of the manifold* $\mathscr{L}$. To find geodesics one must solve the equation

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[15]) See KERNER, Extremum dans l'espace hilbertien [Annali di mat. (4) 10 (1932), 198—202].

(14.1) $$\frac{\delta Z(t)}{dt} = k(t) \cdot Z(t),$$

or, according to (11.3),

(14.2) $$\delta Z[t;\, Z(t)] = k(t) \cdot Z(t),$$

where $k(t)$ is an arbitrary continuous numerical function of t. The fact that $Z(t)$ is defined only along $\mathfrak{C}$ can be easily removed.

We shall transform the equation (14.2), that is

$$\partial Z[t;\, Z(t)] + \Gamma[Z(t),\, Z(t)] = k(t) \cdot Z(t),$$

to the substratum $\mathcal{S}$

$$d\dot{P}(\dot{P}) + \overline{\Gamma}(\dot{P},\, \dot{P}) = k(t) \cdot \dot{P},$$

where $\overline{\Gamma}(Q,\, Q)$ is defined by (11.2). The first term is equal to the second derivative $\ddot{P}$ of $P(t)$, and the equation takes the form

(14.3) $$\ddot{P} + \overline{\Gamma}(\dot{P},\, \dot{P}) = k(t) \cdot \dot{P}.$$

Let P_0 to be a fixed point of $\mathcal{L}$ and A a vector at this point. Then $\Phi_P(A)$ is the corresponding element of the substratum. According to the general theory of differential equations [14]) the equation (14.3) has (we assume for an affine manifold of class three that $\Gamma(X,\, Y)$ has a continuous first differential) just one solution $P(t)$ in the neighbourhood of P_0, for which

$$P(0) = P_0 \text{ and } \dot{P}(0) = \Phi_P(A) \text{ or } Z(0) = A.$$

Thus, in each direction through a point P_0 there is a unique geodesic.

If we put

$$u = \int_0^t e^{\int_0^s k(r)\,dr}\, ds,$$

by using the new parameter u we reduce the equation (14.3) to

(14.4) $$\ddot{P} + \overline{\Gamma}(\dot{P},\, \dot{P}) = 0,$$

and (14.1), (14.2) to

(14.5) $$\frac{\delta Z(u)}{du} = 0, \quad \delta Z[u; Z(u)] = 0.$$

The geodesics therefore do not depend on the arbitrary function $k(t)$.

Because of the structure of their equation they depend only on the symmetric part of $\Gamma(X, Y)$ and do not alter when changing the torsion $S(X, Y)$.

In the case of a riemannian connection (14.5) takes the form

$$\partial Z[u, Z(u)] + C[Z(u), Z(u)] = 0,$$

which can be also written

$$g\{X, \partial Z[u; Z(u)]\} + c[Z(u), Z(u), X] = 0,$$

or, according to the definition of $c(X, Y, Z)$,

$$g\{X, \partial Z[u; Z(u)]\} + \partial g[X, Z(u); Z(u)] - \frac{1}{2}\partial g[Z(u), Z(u); X] = 0.$$

We transform this equation to the substratum

$$\bar{g}(Q, \ddot{P}) + d\bar{g}(Q, \dot{P}; \dot{P}) - \frac{1}{2}d\bar{g}(\dot{P}, \dot{P}, Q) = 0, \tag{14.6}$$

where Q is an arbitrary element. This can be also derived from (14.4) by means of transforming to $\mathcal{S}$ the definition equations of $C(X, Y)$ and $c(X, Y, Z)$.

We shall prove that the geodesics of a riemannian manifold are *extremals of the integral* (13.5) or (13.4), which defines the length. The theory of extremals is till now developed only in the special case of Hilbert space [16]), but it is easy to generalize it to arbitrary spaces of type (B) by means of the integral notion.

If we put

$$f(P, \dot{P}) = \sqrt{\bar{g}_P(\dot{P}, \dot{P})},$$

the integral (13.4) becomes

$$\int_{t_1}^{t_2} f(P, \dot{P})dt,$$

and its extremals are defined by the Euler equation

$$\frac{d}{dt}d_{\dot{P}}f(P, \dot{P}; Q) - d_P f(P, \dot{P}; Q) = 0, \tag{14.7}$$

where Q is arbitrary. We specialize the parameter t by the condition

$$f(P, \dot{P}) = \sqrt{\bar{g}(\dot{P}, \dot{P})} = 1.$$

Then we have

$$d_P f(P, \dot{P}; Q) = \frac{1}{2}d\bar{g}(\dot{P}, P; Q),$$
$$d_{\dot{P}} f(P, \dot{P}; Q) = \quad \bar{g}(\dot{P}, Q).$$

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[16]) See KERNER, loc. cit. in footnote 15, 183—202.

The Euler equation (14.7) takes the form

$$\frac{d}{dt}\bar{g}(\dot{P}, Q) - \frac{1}{2}d\bar{g}(\dot{P}, \dot{P}; Q) = 0,$$

or

$$\bar{g}(\ddot{P}, Q) + d\bar{g}(\dot{P}, Q; \dot{P}) - \frac{1}{2}d\bar{g}\,(\dot{P}, \dot{P}; Q) = 0,$$

equivalent to (14.6).

15. *Curvature tensor.* We suppose the affine manifold $\mathscr{L}$ to be of class three and the contravariant vector A to have a continuous second differential. We form the iteration $\delta^2 A(X, Y)$ and alternate it, getting the tensor

$$\delta^2 A(X, Y) - \delta^2 A(Y, X).$$

From

$$\delta A(X) = \eth A(X) + \Gamma(A, X)$$

it follows, according to (9.5), that

$$\begin{aligned}\delta^2 A(X, Y) = \eth^2 A(X, Y) &+ \eth\Gamma(X, A; Y) + \Gamma[\eth A(Y), X] + \\ &+ \Gamma[\eth A(X), Y] + \Gamma[\Gamma(A, X), Y] + \\ &- \delta A[\Gamma(X, Y)].\end{aligned}$$

The term $\eth^2 A(X, Y)$ is symmetric by virtue of (8.6). By alternating we get

$$\begin{aligned}&\delta^2 A(X, Y) - \delta^2 A(Y, X) = \\ &= \eth\Gamma(A, X; Y) - \eth\Gamma(A, Y; X) + \\ &+ \Gamma[\Gamma(A, X), Y] - \Gamma[\Gamma(A, Y), X] + \\ &- \delta A[S(X, Y)].\end{aligned} \tag{15.1}$$

We introduce the notation

$$\begin{aligned}R(Z, X, Y) = &\eth\Gamma(Z, X; Y) - \eth\Gamma(Z, Y; X) + \\ &+ \Gamma[\Gamma(Z, X), Y] - \Gamma[\Gamma(Z, Y), X].\end{aligned} \tag{15.2}$$

The formula (15.1) becomes

$$\begin{aligned}&\delta^2 A(X, Y) - \delta^2 A(Y, X) = \\ &= R(A, X, Y) - \delta A[S(X, Y)].\end{aligned} \tag{15.3}$$

The function $R(X, Y, Z)$ is therefore a mixed tensor of covariant order two alternating in regard to both last variables Y, Z. We call it the *curvature tensor of the affine manifold* $\mathscr{L}$.

All the identities which are usually derived for the curvature tensor can be proved also in our general case.

16. *Teleparallelism.* If two points P_0 and P_1 are joined by two curves $\mathfrak{C}_1$ and $\mathfrak{C}_2$ the parallel displacement of a vector A at P_0 along both curves leads in general to different vectors B_1 and B_2 at P. If, on the contrary, the vectors at P do not depend on the joining curve $\mathfrak{C}$, at least in a neighbourhood of P_0, we say that in the affine manifold $\mathscr{L}$ there is a *teleparallelism.* We wish to find necessary and sufficient conditions for teleparallelism.

Let A_0 to be an arbitrary contravariant vector at P_0, the manifold $\mathscr{L}$ being of class three. The necessary and sufficient condition for teleparallelism is, that for each A_0 there exists in the neighbourhood of P_0 a vector A, for which the right member of (11.3) vanishes along each curve. This is equivalent to the condition

$$\delta A(X) = 0 \tag{16.1}$$

for each X. Thus, the integrability of (16.1) is the necessary and sufficient condition for teleparallelism.

From (16.1) it follows by tangent differentiating that

$$\eth\delta A(X; Y) = 0$$

or

$$\delta^2 A(X, Y) - \Gamma[\delta A(X), Y] + \delta A[\Gamma(X, Y)] = 0.$$

On account of (16.1)

$$\delta^2 A(X, Y) + \delta A[\Gamma(X, Y)] = 0,$$

and by alternating we get a tensorial relation

$$\delta^2 A(X, Y) - \delta^2 A(Y, X) + \delta A[S(X, Y)] = 0.$$

This necessary condition leads by virtue of (15.3) to

$$R(A, X, Y) = 0,$$

and since the point P_0 and the value A_0 were arbitrary, to

$$R(Z, X, Y) = 0. \tag{16.2}$$

Thus, *the vanishing of the curvature tensor is a necessary condition for teleparallelism.*

Before we prove that it is also *sufficient*, we must formulate the following theorem about the partial differential equations in the space of type (B):

Let $\Psi[\Pi, \Lambda, \Xi]$ be a function which associates with each Π

in neighbourhood of Π_0 in Π-space, each Λ in neighbourhood of Λ_0 in Λ-space and each Ξ of Π-space an element of Λ-space. Let Ψ have continuous second differentials relative to Π and Λ, and let it be linear relative to Ξ.

Then a necessary and sufficient condition that the equation

$$dA(\Pi; \Xi) = \Psi[\Pi, A(\Pi), \Xi] \tag{16.3}$$

relative to the unknown function $\Lambda(\Pi)$, have in some neighbourhood of Π_0 just one solution $\Lambda(\Pi)$, for which

$$\Lambda(\Pi_0) = \Lambda_0, \tag{16.4}$$

consists in

the *symmetry* of

$$d_\Pi\Psi(\Pi, \Lambda, \Xi; \mathbf{H}) + d_\Lambda\Psi[\Pi, \Lambda, \Xi; \Psi(\Pi, \Lambda, \mathbf{H})] \tag{16.5}$$

relative to Ξ, $\mathbf{H}$.

We do not prove this theorem, since it can be done in a like manner as for the theory of differentials [17]). It seems that even the assumption about the existence of the second differential of $\Psi(\Pi, \Lambda, \Xi)$ is superfluous.

Now we turn to the proof that (16.2) is a sufficient condition for teleparallelism. On account of what was assumed about the second differential in the last theorem we must suppose the affine manifold $\mathscr{L}$ to be of class four, but this could probably be avoided.

The equation (16.1), that is

$$\partial A_P(X) + \Gamma_P(A_P, X) = 0,$$

is equivalent to the equation

$$d\bar{A}_P(Q) = -\bar{\Gamma}_P(\bar{A}_p, Q) \tag{16.6}$$

in the substratum. But (16.6) is of the form (16.3), where the point P is to be put in the place of Π. The expression (16.5) becomes

$$-d\bar{\Gamma}_P(\bar{A}_P, Q; R) + \bar{\Gamma}_P[\bar{\Gamma}_P(\bar{A}_P, R), Q],$$

to which corresponds in the tangent space the function

$$-\partial\Gamma(A, X; Y) + \Gamma[\Gamma(A, Y), X]. \tag{16.7}$$

And the symmetry of (16.5) is equivalent to that of (16.7),

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[17]) Cf. Kerner, loc. cit. in footnote 7, 546—557.

which is on its part equivalent to the vanishing of $R(X, Y, Z)$. Thus is proved again the necessity and also the sufficiency of the condition (16.2).

17. *Euclidean manifolds.* We call an affine manifold $\mathscr{L}$ *affine euclidean,* if there exists such a substratum $\mathscr{S}_0$ that, when transporting parallelly a contravariant vector A, the corresponding element $\bar{A} = \Phi_P(A)$ of this substratum is constant. In an affine euclidean manifold we have therefore for the substratum $\mathscr{S}_0$

$$\Gamma(X, Y) = 0,$$

and for every substratum $\mathscr{S}$ (because the torsion is a tensor)

$$S(X, Y) = 0. \tag{17.1}$$

There exists obviously in such a manifold teleparallelism, and in virtue of what was said in the last paragraph we have

$$R(X, Y, Z) = 0. \tag{17.2}$$

We wish to prove that the *necessary* conditions (17.1) and (17.2) for an affine euclidean manifold are also *sufficient.*

Indeed, if (17.2) is satisfied, we have teleparallelism, and the equation (16.6) has just one solution $\bar{A}_P$, for which in a fixed point P_0

$$\bar{A}_{P_0} = R,$$

R being an arbitrary element of the substratum $\mathscr{S}$. We denote $\bar{A}_P$ by $\bar{A}_P(R)$, bringing in evidence its dependency on R. According to what was said in paragraph 11 about parallel displacement $\bar{A}_P(R)$ is a linear function of R and has a uniquely determined inverse relative to R.

After having defined $\bar{A}_P(R)$, we form the equation

$$dP_{P'}(R) = \bar{A}_P(R), \tag{17.3}$$

which is also of type (16.3). The unknown function $P_{P'}$ associates with P' belonging to $\mathscr{S}$ points of the same space. The condition of integrability (16.5) takes here the form

$$d\bar{A}_P[R; \bar{A}_P(Q)] = d\bar{A}_P[Q; \bar{A}_P(R)], \tag{17.4}$$

or, according to (16.6),

$$\bar{\Gamma}_P[\bar{A}_P(R), \bar{A}_P(Q)] = \bar{\Gamma}_P[\bar{A}_P(Q), \bar{A}_P(R)].$$

This equality is satisfied by virtue of (17.1). The equation (17.3) is therefore integrable.

The function $P_{P'}$, which is defined by (17.3) with the initial condition

$$P_{P_0} = P_0, \tag{17.5}$$

can be taken to be a substratum change. It transforms the substratum $\mathscr{S}$ into a new substratum $\mathscr{S}'$ which is formed by the same abstract space, but with another correspondence to tangent spaces. To show that $P_{P'}$ is a substratum change we ought to determine also its inverse P'_P. This latter can be defined by the equation

$$dP'_P(Q) = B_P(Q), \tag{17.6}$$

where $B_P(Q)$ is the inverse function of $\bar{A}_P(R)$, with the initial condition

$$P'_{P_0} = P_0. \tag{17.7}$$

From the identity

$$\bar{A}_P[B_P(Q)] = Q$$

we get by differentiating

$$d\bar{A}_P[B_P(Q);\, R] + \bar{A}_P[dB_P(Q;\, R)] = 0,$$

whence, on account of (17.4), there follows the symmetry of $dB_P(Q;\, R)$, and from it the integrability of (17.6). The uniquely determined functions $P_{P'}$ and P'_P are in virtue of (17.5) and (17.7) inverses one of another.

Thus, we change the substratum from $\mathscr{S}$ to $\mathscr{S}'$ by effecting the transformation $P_{P'}$. To the contravariant vector A_P there corresponds in $\mathscr{S}'$ a new element $\bar{A}'_{P'}$, for which

$$\bar{A}_P = dP_{P'}(\bar{A}'_{P'}),$$

or

$$\bar{A}_P(R) = dP_{P'}[\bar{A}'_{P'}(R)].$$

On comparing this with (17.3), we get

$$dP_{P'}[\bar{A}'_{P'}(R)] = dP_{P'}(R),$$

whence it follows, $dP'_{P'}(R)$ admitting a unique inverse, that

$$\bar{A}'_{P'}(R) = R.$$

We see that the vector A_P, arisen from a vector $F_{P_0}(R)$ by parallel displacement, has in the substratum $\mathscr{S}'$ a representing element $A'_{P'}(R)$ which does not depend on the point P'. To vectors at different points, parallel by teleparallelism, corresponds in $\mathscr{S}'$ the same element. Thus $\mathscr{S}'$ is the preferred substratum for affine euclidean manifold, and $\mathscr{L}$ is such a manifold.

If the affine euclidean manifold is metric (and consequently also riemannian), it is called *metric euclidean.* For the preferred substratum $\mathscr{S}$ in such a manifold the transform of metric $g(X, Y)$ to $\mathscr{S}$, that is $\bar{g}(Q, R)$, does not depend on the point P. The length of straight line segment $P = P_1 + (P_2 - P_1)t$ $[0 \leqq t \leqq 1]$ is then equal to

$$\int_0^1 \sqrt{\bar{g}(\dot{P}, \dot{P})}\,dt = \int_0^1 \sqrt{\bar{g}(P_2 - P_1, P_2 - P_1)}\,dt = \sqrt{\bar{g}(P_2 - P_1, P_2 - P_1)}.$$

We content ourselves with these introductory developements hoping to have shown clearly enough how to transport geometric research on the basis of abstract spaces.

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