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On regular closed curves in the plane [1])

by

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We consider in this note closed curves with continuously turning tangent, with any singularities. To each such curve may be assigned a „rotation number" γ, the total angle through which the tangent turns while traversing the curve. (For a simple closed curve, $\gamma = \pm 2\pi$.) Our object is two-fold; to show that two curves with the same rotation number may be deformed into each other, [4]) and to give a method of determining the rotation number by counting the algebraic number of times that the curve cuts itself (if the curve has only simple singularities, — see Lemma 2).

This paper may be considered as a continuation of a paper of H. Hopf [2]); we assume a knowledge of the first part of his paper.

1. *Regular closed curves.*

Ordinarily, a curve in the plane is defined as a point set with certain properties; but when we allow singularities, this mode of definition cannot be used. (See footnote [3]).) Our first purpose is therefore to define a regular closed curve.

Let E be the Euclidean plane. Let E' be the vector plane (which we might let coincide with E), with origin O. Let I be the closed interval $(0, 1)$. Any differentiable function $f(t)$ with values in E has, as its derivative, a function $f'(t) = \frac{df(t)}{dt}$ with values in E'. By a *parametrized regular closed curve*, or parametrized curve for short, we shall mean a differentiable function $f(t)$ defined in I and with values in E, such that

$$(1) \qquad f(1) = f(0), \quad f'(1) = f'(0), \quad f'(t) \neq 0 \text{ for } t \text{ in } I.$$

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[1]) Presented to the American Mathematical Society, Sept., 1936.

[2]) Heinz Hopf, Über die Drehung der Tangenten und Sehnen ebener Kurven [Compositio Math. 2 (1935), 50—62].

The first two conditions are the conditions for the curve to be closed; the last condition makes t a „regular parameter". To any such f there corresponds a unique differentiable function $\bar{f}$ defined in $(-\infty, \infty)$, such that

$$\bar{f}(t) = f(t) \text{ in } I,\ \bar{f}(t+1) = \bar{f}(t),\ \bar{f}'(t) \neq 0, \tag{2}$$

and conversely.

It is natural to call two parametrized curves equivalent if one can be obtained from the other by a change of parameter (preserving orientation). The exact definition is: f and g are *equivalent* ($f \sim g$) if there exists a function $\eta(t)$ in $(-\infty, \infty)$ whose first derivative is continuous and positive, and is such that

$$\eta(t+1) = \eta(t) + 1,\ \bar{g}(t) = \bar{f}(\eta(t)). \tag{3}$$

Obviously $f \sim f$, $f \sim g$ implies $g \sim f$, and $f \sim g$ and $g \sim h$ imply $f \sim h$. Hence the parametrized curves fall into classes; we call each of these a *regular closed curve*, or curve for short. With any curve C is associated many (equivalent) *parametrizations* f. Let $\bar{C}$ be the corresponding set of points in the plane E (all points $f(t)$). C is by no means determined by $\bar{C}$[3]).

Given any C, a parametrization g may be chosen so that $|g'(t)|$ is constant, that is, so that the parameter is a constant times the arc length.

To prove this, set

$$L(t) = \int_0^t |\bar{f}'(s)|\, ds, \quad L = L(1). \tag{4}$$

$L = L(C)$ is the length of C. As $\bar{f}'(t) \neq 0$, $L(t)$ is a differentiable increasing function; hence we may solve $L \cdot s = L(t)$ for t, giving $t = \eta(s)$. The derivative $\eta'(s)$ is continuous and positive. As $\bar{f}$ is periodic,

$$L(t+1) - L(t) = \int_t^{t+1} |\bar{f}'(s)|\, ds = \int_0^1 |\bar{f}'(s)|\, ds = L;$$

hence $\eta(s+1) = \eta(s) + 1$. Therefore

$$\bar{g}(t) = \bar{f}(\eta(t))$$

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[3]) Let $\bar{C}$ be the unit circle in E; then for each integer $n \neq 0$ there is a corresponding curve C_n with $\bar{C}_n = \bar{C}$, determined by letting $f(t)$ traverse $\bar{C}$ in the positive sense n times while t runs over I. Again, if we take an ellipse and pull the ends of the minor axis together till they are tangent, then there are four corresponding curves, in each of which the corresponding $f(t)$ traverses each point but one of the ellipse only once.

is a parametrization of C. Moreover,

$$\bar{g}'(t)=\bar{f}'(\eta(t))\frac{L}{L'(\eta(t))}, \qquad |\bar{g}'(t)|=L. \tag{5}$$

If h is any parametrization with $|h'(t)|=k$, then $k=L$ and $\bar{h}(t)=\bar{g}(t+a)$ for some constant a.

First, as $h\sim g$, there is an η such that $\bar{h}(t)=\bar{g}(\eta(t))$. As

$$h'(t)=g'(\eta)\eta'(t), \text{ hence } k=L\eta'(t),$$

we have

$$1=\eta(1)-\eta(0)=\int_0^1\eta'(t)dt=\int_0^1\frac{k}{L}dt=\frac{k}{L},$$

and $k=L$. Hence $\eta'(t)=1$, and $\eta(t)=t+a$.

Let f_0 and f_1 be parametrized curves. We say one may be *deformed* into the other if $f_u(t)$ may be defined for $0<u<1$ such that it is continuous in both variables for $0\leqq t\leqq 1$, $0\leqq u\leqq 1$, and each f_u is a parametrized curve.

If f_0 and f_1 are parametrizations of C, then one may be deformed into the other within C, that is, we can make each f_u a parametrization of C.

To prove this, say $f_1(t)=f_0(\eta(t))$. Set

$$\eta_u(t)=u\eta(t)+(1-u)t, \quad f_u(t)=\bar{f}_0(\eta_u(t)), \tag{6}$$

for $0\leqq u\leqq 1$. Then $\eta_0(t)=t$, $\eta_1(t)=\eta(t)$, so that $\bar{f}_0$ and $\bar{f}_1$ bear the proper relation to f_0 and f_1. As

$$\eta_u(t+1)=u[\eta(t)+1]+(1-u)(t+1)=\eta_u(t)+1,$$

$$\frac{d\eta_u(t)}{dt}=u\frac{d\eta(t)}{dt}+(1-u)>0 \text{ for } 0\leqq u\leqq 1,$$

each f_u is a parametrized curve equivalent to f_0.

We say C may be deformed into C' if some parametrization of C may be deformed into one of C'. By the above statement, this is independent of the parametrizations chosen.

2. *The deformation theorem.*

The following lemma is fundamental in this section.

Lemma 1. *Let $f'(t)$ be a continuous vector function in I, such that $f'(t)\neq 0$. If p is a point of E, then*

$$f(t)=p+\int_0^t f'(s)ds \tag{7}$$

is a parametrized curve if and only if

$$f'(1) = f'(0), \quad \int_0^1 f'(s)\,ds = 0. \tag{8}$$

This is obvious. The last relation may be stated as follows: The average value of $f'(s)$ is O.

Given any parametrized curve f, we define its rotation number $\gamma(f)$ as the total angle through which $f'(t)$ turns as t traverses I. The function $f^*(t) = \dfrac{f'(t)}{|f'(t)|}$ is a map of I into the unit circle; $\gamma(f)$ is 2π times the degree of this map. (See Hopf. loc. cit., 1c, and our equation (11).)

If f may be deformed into g, then $\gamma(f) = \gamma(g)$.

For $\gamma(f_u)$ is continuous in u, and is an integral multiple of 2π; hence it is constant. Hence, by **1**, we may define $\gamma(C)$ for a curve C as $\gamma(f)$ for any parametrization f of C.

THEOREM 1 [4]. *The curves C_0 and C_1 may be deformed into each other if and only if $\gamma(C_0) = \gamma(C_1)$.*

One half of the theorem was proved above. Suppose now that $\gamma(C_0) = \gamma(C_1) = \gamma$. Let g_0 and f_1 be parametrizations of C_0 and C_1 such that

$$|g_0'| \equiv L(C_0) = L_0, \quad |f_1'| \equiv L(C_1) = L_1.$$

Set

$$g_u(t) = g_0(0) + \left[u\frac{L_1}{L_0} + (1-u)\right][g_0(t) - g_0(0)];$$

this deforms the parametrized curve g_0 into one g_1. Set $f_0 = g_1$; then $|f_0'| \equiv |g_1'| \equiv L_1$. We must deform f_0 into f_1.

The proof runs as follows. We consider the maps f_0' and f_1' of I into the circle K of radius L_1. They are both of degree $\frac{\gamma}{2\pi}$; hence one map may be deformed into the other, say by the maps h_u. We alter each h_u by a translation to obtain a map f_u' whose average lies at O; these functions then define the required deformation, at least if $\gamma \neq 0$.

We begin by defining the vector function

$$\theta(t) = (L_1 \cos t,\ L_1 \sin t); \tag{9}$$

this gives an angular coordinate t in K. Suppose first that $\gamma \neq 0$. By rotations in the plane E we may alter f_0 and f_1 so that

[4]) This theorem, together with a straightforward proof, was suggested to me by W. C. GRAUSTEIN.

$f_0'(0) = f_1'(0) = \theta(0)$. As $f_i'(t)$ lies on K, we may give it an angular measure $F_i(t)$:

$$f_i'(t) = \theta(F_i(t)), \text{ with } F_i(0) = 0 \qquad (i=0,1). \tag{10}$$

(See Hopf., loc. cit., 1a.) Then, by definition of γ,

$$F_i(1) = \gamma \qquad (i=0,1). \tag{11}$$

Set

$$\begin{aligned} F_u(t) &= uF_1(t) + (1-u)F_0(t), \\ h_u(t) &= \theta(F_u(t)) \end{aligned} \qquad (0 \leqq t \leqq 1), \tag{12}$$

$$\begin{aligned} f_u'(t) &= h_u(t) - \int_0^1 h_u(s)\,ds, \\ f_u(t) &= f_0(0) + u[f_1(0) - f_0(0)] + \int_0^t f_u'(s)\,ds. \end{aligned} \tag{13}$$

It is clear that $\int_0^1 f_u'(t)\,dt = 0$. As $F_u(0) = 0$, $F_u(1) = \gamma$, and γ is an integral multiple of 2π,

$$f_u'(1) - f_u'(0) = \theta(F_u(1)) - \theta(F_u(0)) = \theta(\gamma) - \theta(0) = 0.$$

Finally, as $\gamma \neq 0$ and hence $h_u(t)$ passes over all of K, its average value lies interior to K; therefore for no t does $h_u(t)$ equal the average, and $f'(t) \neq O$. This proves that each f_u is a regular closed curve. As $f_u(t)$ is continuous in both variables, and it reduces to f_0 and f_1 for $u = 0$ and $u = 1$, it is a deformation of f_0 into f_1, as required.

Suppose now that $\gamma = 0$. If we alter $F_u(t)$ so that it is constant for no u, then again $f'(t) \neq 0$, and the above proof will hold. Choose a t_0 for which $F_1(t_0) \neq 0$, and deform $F_0(t)$ in a small neighborhood of t_0 into $F_1(t)$ in this neighborhood; now deform the new F_0 into F_1 by the process given above. Then (as $F_u(0) = 0$) no F_u is constant.

3. *Crossing points of curves.*

Let $f(t)$ be a parametrized curve. Let p be a point of the plane. If there are exactly two numbers t_1, t_2, such that

$$0 \leqq t_1 < t_2 < 1, \quad f(t_1) = f(t_2) = p,$$

and if $f'(t_1)$ and $f'(t_2)$ are independent vectors, we call p a (*simple*) *crossing point* of the curve. This is evidently independent of the parametrization. If the curve has no singularities other than a finite number of simple crossing points, we say the curve is *normal.*

LEMMA 2. *Any curve may be made normal by an arbitrarily small deformation.*

Given $\varepsilon > 0$, cut I into intervals $I_1, \ldots, I_\nu$ so small that each corresponding arc $A_i = f(I_i)$ is of diameter $< \varepsilon$, and the tangents at different points of A_i differ by at most ε. By a small deformation we may clearly obtain arcs A'_i such that neither end of any A'_i touches other points of the curve. Now for any i and j, it is easy to replace A'_j by an arc A''_j arbitrarily near it and with the same ends so that A''_j cuts A'_i in simple crossing points only[5]). Alter thus A'_2 in relation to A'_1; then A'_3 in relation to A'_1; then A'_3 in relation to A'_2, altering it so slightly that its relation to A'_1 is not impaired, etc.

Let f be a parametrized curve, and let $\bar{C}$ be the corresponding set of points $f(t)$ in the plane. We say f has an *outside starting point* if there is a line of support to $\bar{C}$ [6]) containing $f(0)$.

Let $f(t_1) = f(t_2)$, $t_1 < t_2$, be a crossing point. If the vectors $f'(t_1)$ and $f'(t_2)$ are oriented relative to each other in the *opposite* manner to the (fixed) x- and y-axes, we say the crossing point is *positive*; otherwise, *negative*[7]). If we set $\bar{g}(t) = \bar{f}(t+\tau)$ with $t_1 < \tau \leqq t_2$, then the above crossing point changes its type. Corresponding to any normal parametrized curve are the numbers

$$(14) \qquad \left.\begin{matrix} N^+ \\ N^- \end{matrix}\right\} \text{ of crossing points of } \left\{\begin{matrix} \text{positive} \\ \text{negative} \end{matrix}\right\} \text{type.}$$

These may be found by following the curve from its starting point, and watching the intersections with the part of the curve already traversed.

THEOREM 2. *If f is a normal parametrized curve with an outside starting point, then*

$$(15) \qquad \gamma(f) = 2\pi[\mu + (N^+ - N^-)], \qquad \mu = \pm 1.$$

If the axes are moved so that the x-axis is the line of support at

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5) The proof is simplified by first replacing $f(t)$ by a function $g(t)$ with continuous second derivatives. The lemma is contained in Theorem 2 of H. WHITNEY, Differentiable manifolds [Annals of Math. **37** (1936)]. (We replace I by the unit circle M and use (b) of the theorem.)

6) That is, a straight line touching $\bar{C}$ and having each point of $\bar{C}$ on it or on a single side of it.

7) An example of a positive crossing point is given in Fig. 2.

$f(0)$ and the curve is on the same side of this line as the positive y-axis, then $\mu = +1$ or -1 according as $f'(0)$ is in the positive or negative x-direction.

In particular, if the curve has no singularities, then $\gamma = \pm 2\pi$, which is the „Umlaufsatz".

Let T be the triangle of all pairs of numbers

$$(t_1, t_2),\ 0 \leqq t_1 \leqq t_2 \leqq 1.$$

Let $l(t_1, t_2)$ be the smaller of $t_2 - t_1$ and $(1+t_1) - t_2$. Set

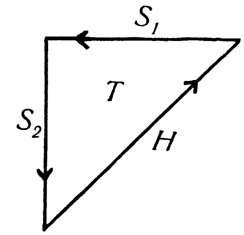

Fig. 1.

$$(16) \qquad \begin{aligned} \psi(t_1, t_2) &= \frac{f(t_2) - f(t_1)}{l(t_1, t_2)} \text{ if } l(t_1, t_2) \neq 0, \\ \psi(t, t) &= f'(t), \qquad \psi(0, 1) = -f'(0). \end{aligned}$$

ψ is continuous in T, and is 0 at (t_1, t_2) if and only if $t_1 < t_2$ (but not $t_1 = 0$, $t_2 = 1$), and $f(t_1) = f(t_2)$, i.e. if and only if $f(t_1)$ is a crossing point [8]).

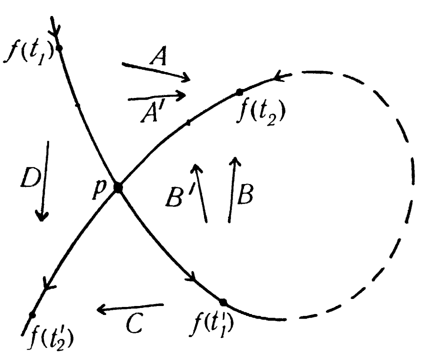

Fig. 2.

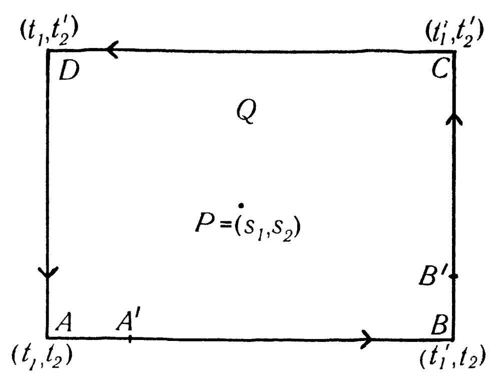

Fig. 3.

Take any crossing point $p = f(s_1) = f(s_2)$; suppose it is positive. As $s_1 < s_2$, $P = (s_1, s_2)$ is not on the hypothenuse of T. As $f(0)$ is an outside point, it is obviously not a crossing point; hence $f(0) \neq f(t)$ for $0 < t < 1$, and P is on neither side of T. As $P \neq (0,1)$, it follows that P is interior to T. Choose numbers t_1, t_1' very close to s_1, and t_2, t_2' very close to s_2, so that

[8]) This function replaces the function $f(s_1, s_2)$ of Hopf (p. 54). It will be seen that $N = N^+ - N^-$ is the algebraic number of times that T covers 0 under ψ (see footnote [10])).

$$t_1 < s_1 < t_1', \quad t_2 < s_2 < t_2',$$

and

$$f(t_1),\ f(t_1'),\ f(t_2),\ f(t_2') \text{ are equidistant from } p.$$

Let Q be the rectangle in T containing P, with coordinates (t_1, t_2), etc. It is easily seen that if we run around Q once in the positive sense, the corresponding ψ runs around O once in the positive sense. For running around each side of Q turns the vector ψ through an angle of approximately $\frac{\pi}{2}$ (see the diagram); hence it turns, in all, approximately 2π; but it turns an integral multiple of 2π, and hence exactly 2π [9]). If the crossing point is negative, the result is obviously -2π.

Let $P_1, \ldots, P_m$ be the points of T corresponding to crossing points, and let $Q_1, \ldots, Q_m$ be corresponding rectangles enclosing them, no two of which have common points. Cut the rest of T into triangles $Q_{m+i}, \ldots, Q_r$. If we run around the boundary of any Q_{m+i}, ψ runs around 0 zero times [10]). To show this, consider the vector $\psi^* = \frac{\psi}{|\psi|}$. This is defined throughout Q_{m+i}, and its values are on the unit circle. Hence an angular coordinate may be defined, giving the position of ψ^* throughout Q_{m+i} (see Hopf, loc. cit., 1b). If we run around the boundary of Q_{m+i}, the angular coordinate comes back to its original value, and hence ψ has turned around zero times.

Let $\alpha_1, \ldots, \alpha_l$ be all sides of triangles or rectangles in T. Let $\alpha_1, \ldots, \alpha_k$ be those lying on the boundary B of T, oriented the same as T; the remaining α_i are oriented arbitrarily. With each α_i we associate a number $\varphi(\alpha_i)$, the angle through which ψ turns when α_i is traversed in the positive direction. Let $\varphi(Q_i)$ be the angle through which ψ turns when the boundary of Q_i is traversed in the positive direction; similarly for $\varphi(T)$. Now

$$(17) \qquad \sum_{i=1}^{r} \varphi(Q_i) = \varphi(T).$$

For each $\varphi(Q_i)$ may be expressed as a sum $\Sigma^{(i)} \pm \varphi(\alpha_j)$, summing over the boundary lines of Q_i; when these sums are added, the two terms corresponding to each α_j interior to T cancel, and we are left with the sum over the α_j on the boundary of T.

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[9]) By choosing the proper degree of approximation, it is easy to make this reasoning rigorous.

[10]) Hence, in all cases, $\varphi(Q_j)$ (see below) is the algebraic number of times that the map ψ of Q_j covers 0.

We have seen above that

$$\sum_{i=1}^{r} \varphi(Q_i) = \sum_{i=1}^{m} \varphi(Q_i) = 2\pi(N^+ - N^-) = 2\pi N. \tag{18}$$

Suppose $\mu = 1$. If S_1, S_2 and H are the positively oriented sides and hypotenuse of T (see Fig. 1), it is easily seen that

$$\varphi(S_1) = \varphi(S_2) = -\pi. \tag{19}$$

(See Hopf, pp. 54—55. The change in sign is caused by the difference in orientation of S_1 and S_2 from that used by Hopf.) Hence, using (17) and (18),

$$2\pi N = \varphi(T) = \varphi(H) + \varphi(S_1) + \varphi(S_2) = \gamma - 2\pi,$$

which gives (15). If $\mu = -1$, the only change is that $\varphi(S_1) = \varphi(S_2) = \pi$, and (15) again follows.

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