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A Note on Topological Groups

by

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It is natural to define as „Hausdorff groups", those systems which bear the same relation to Hausdorff spaces as the „L-groups" of Schreier [4] bear to L-spaces.

These systems, which are well-known under various names (including „topological groups") can be defined briefly as follows.

A Hausdorff group is any system G (1) which is a Hausdorff space relative to a certain class of neighborhoods (2) which is an abstract group (3) whose group operations are continuous in its topology — that is, in which

HG1: Given any neighborhood U_{ab} of a group product ab, there exist neighborhoods U_a of a and U_b of b such [1]) that $U_a U_b \subset U_{ab}$.

HG2: Given any neighborhoods U_a of any element $a \varepsilon G$, there exists a neighborhood $U_{a^{-1}}$ of the inverse a^{-1} of a such that $(U_{a^{-1}})^{-1} \subset U_a$.

The main result of the present note is the proof that a Hausdorff group is „metrizible" (i.e., homeomorphic with a metric space) if and only if it satisfies Hausdorff's first countability axiom (the axiom that each point a has a complete [2]) system of neighborhoods which is countable).

Before giving the proof, let us for purposes of orientation recall a few known facts about Hausdorff groups.

[1]) The notations $U_a U_b$ and $(U_a{}^{-1})$ are those of the calculus of complexes. According to this notation, if S and T are any non-vacuous subsets of G, ST denotes the (non-vacuous) set of products st $[s \varepsilon S,\ t \varepsilon T]$, and S^{-1} the set of inverses s^{-1} $[s \varepsilon S]$. $S \cap T$ means the set-theoretic product of S and T.

[2]) A system of neighborhoods of a point is called „complete" if and only if every open set containing the point totally includes a suitable neighborhood of the system.

Any Hausdorff group G is homogeneous — the transformations T_y^x: $T_y^x(g) = xgy$ are a transitive group of homeomorphisms of G with itself. Again, the connected component of G containing the identity is a normal subgroup of G; the other connected components being the group-theoretic cosets of this normal subgroup.

And finally ([2], M. 7 and TG. 14), if G satisfies the second countability axiom of Hausdorff (i.e., the axiom that there exists a countable set of neighborhoods for G, a suitable subset of which forms a complete system for each point), it is known to be metrizible. In fact, if G is Abelian or compact, then the topologizing distance function can be so chosen as to be invariant under the group of transformations T_y^x of the preceding paragraph.

We now come to the proof, which is quite easy.

LEMMA: Let G be any Hausdorff group satisfying the first countability axiom. Then the identity *1* of G has a complete system of neighborhoods $V_1, V_2, V_0, \ldots$ with the properties (2) $V_k = V_k^{-1}$, and (2) $V_k^3 \equiv V_k V_k V_k \subset V_{k-1}$ [whence in particular, $V_1 \supset V_2 \supset V_3 \supset \ldots$].

PROOF: Let $U_1, U_2, U_3, \ldots$ be any countable complete system of neighborhoods of *1*. By HG2, the U_k^{-1} are open. Therefore the $W_k = U_k \cap U_k^{-1}$ form a system of neighborhoods of *1* which is also complete, having the property (1).

Again, one can define $V_1, V_2, V_3, \ldots$ from the rules (α) $V_1 = W_1$, and (β) V_{k+1} is the first W_i such that $W_i^3 \subset V_k \cap W_1 \cap \ldots \cap W_k$. It is obvious that this system exists, is complete, and satisfies both conditions of the Lemma.

THEOREM: A Hausdorff group G is metrizible if and only if it satisfies the first countability axion.

PROOF: That the condition is necessary is obvious. Therefore it is sufficient to prove that if G satisfies the first countability axiom, it is metrizible.

To prove this, add to the neighborhood system of the Lemma, the open set $V_0 = G$. Then define „ecart" through the equation

$$\varrho(x, y) = \operatorname{Inf}_{xy^{-1} \varepsilon V_k} (\tfrac{1}{2})^k.$$

Obviously $\varrho(x, x) = 0$, and $\varrho(x, y) > 0$ if $x \neq y$. Also obviously, the sets $U_e(a)$ of points x satisfying $\varrho(a, x) < e$ $[e > 0]$ are a complete system of neighborhoods for any point a. Moreover since $V_k = V_k^{-1}$, $xy^{-1}\varepsilon V_k$ if and only if $yx^{-1}\varepsilon V_k$, whence $\varrho(x, y) = \varrho(y, x)$. And finally, since $V_h V_i V_j \subset V_k$ if $k > h, i, j$, one sees

(E) If $\varrho(x, y) < e$, $\varrho(y, y') < e$, and $\varrho(y', z) < e$, then $\varrho(x, z) < 2e$.

But Chittenden [1] has shown that it follows from these facts without reference to group properties, that G is metrizible, which completes the proof.

One can also avoid reference to Chittenden's argument by simply defining ,,distance'' through the equation.

$$\varrho^*(x, y) = \operatorname{Inf}_{u_0=x,\ u_n=y} \sum_{k=1}^{n} \varrho(u_{k-1}, u_k).$$

It is obvious that $\varrho^*(x, y)$ is symmetric and satisfies the triangle inequality. The proof is therefore complete if we can show that $\varrho^*(x, y)$ is topologically equivalent to $\varrho(x, y)$. But this follows from the inequalitites

$$\tfrac{1}{2}\varrho(x, y) \leqq \varrho^*(x, y) \leqq \varrho(x, y).$$

The second inequality is obvious; to prove the first, note that given $u_0 = x, u_1, u_2, \ldots, u_n = y$, if one makes the definition $|U| = \varrho(u_0, u_1) + \ldots + \varrho(u_{n-1}, u_n)$, one can always find h such that

$$\sum_{k=1}^{n} \varrho(u_{k-1}, u_k) \leqq \tfrac{1}{2}\,|U| \quad \text{and} \quad \sum_{k=h+1}^{n} \varrho(u_{k-1}, u_k) \leqq \tfrac{1}{2}U.$$

But evidently $\varrho(u_h, u_{h+1}) \leqq |U|$, and by induction on k $\varrho(x, u_h) \leqq \leqq |U|$ and $\varrho(u_{h+1}, y) \leqq |U|$. It follows by (E) that $\varrho(x, y) = 2\,|U|$, whence $|U| \geqq \frac{1}{2}\varrho(x, y)$, completing the proof.

Let us now call a homogeneous space ,,microseparable'', when it contains a separable open set. We then have

Corollary 1: If G is microseparable and connected, then it is separable (satisfies Hausdorff's second countability axiom).

Proof: In metrizible spaces, the properties of being separable and of having everywhere dense sets are equivalent. Hence (by homogeneity), some neighborhood of the identity of G has a countable everywhere dense set. But G is connected, and so the (countable) finite products of the elements of this set are everywhere dense in G.

Corollary 2: If G is locally compact and satisfies the first countability axiom, then it satisfies the second.

Proof: A compact metric space is separable.

Corollary 2 permits one to replace the second countability axiom by the first in the assumption of a theorem of Freudenthal (3) on „end-points" of Hausdorff groups.

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