COMPOSITIO MATHEMATICA

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A Remark on Fourier Transforms and Functions Analytic in a Half-Plane

by

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Let $\mathfrak{L}_p$ be the class of functions $f(x)$ measurable over $(-\infty, \infty)$ such that the integral

$$(1) \qquad \int_{-\infty}^{\infty} |f(x)|^p dx, \qquad p \text{ fixed}, \ p \geqq 1,$$

is finite. It is well known that $\mathfrak{L}_p$ becomes a complete linear metric vector space if we define its metric by

$$(2) \qquad \|f(x)\|_p = \|f\|_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx \right]^{1/p}.$$

The value $p = \infty$ will also be admitted, with the agreement that $\mathfrak{L}_\infty$ is the class of functions $f(x)$ continuous and bounded over $(-\infty, \infty)$ with the metric

$$(3) \qquad \|f\|_\infty = \sup_{-\infty < x < \infty} |f(x)|.$$

Let $g(x) \subset \mathfrak{L}_p, \ 1 \leqq p < \infty$. Put

$$(4) \qquad G(u ; \ a) = (2\pi)^{-\frac{1}{2}} \int_{-a}^{a} g(x) e^{-iux} dx.$$

If there exists a function $G(u) \subset \mathfrak{L}_q, \ 1 \leqq q \leqq \infty$, such that

$$(5) \qquad \|G(u ; \ a) - G(u)\|_q \to 0 \text{ as } a \to \infty,$$

then we say that $G(u)$ is the Fourier transform of $g(x)$ in $\mathfrak{L}_q$ and write

$$(6) \qquad G(u) = \underset{a \to \infty}{L^q} G(u ; \ a).$$

Such a function is known to exist when $1 \leqq p \leqq 2$ and $q = p' = \dfrac{p}{p-1}$.

Let $f(z)$ be a function of the complex variable $z = x + iy$, analytic in the upper half-plane $y > 0$. If, for almost all x,

$f(z)$ tends to a definite limit, $f(x)$, as $z \to x$ along all non-tangential paths, the function $f(x)$ so defined is said to be the limit-function of $f(z)$. We shall assume that $f(x) \subset \mathfrak{L}_p$, $1 \leqq p < \infty$.

In the present note we are concerned with the class $\mathfrak{A}_p$ of functions $f(z)$, analytic in the half-plane $y > 0$, each possessing a limit-function $f(x) \subset \mathfrak{C}_p$, $1 \leqq p < \infty$, and representable by its Cauchy integral,

$$(7) \qquad f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{\xi - z} \equiv I(z\,;\,f),$$

or, what is equivalent [1]), by its Poisson integral,

$$(8) \qquad f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} \equiv Q(z\,;\,f).$$

The following problem presents itself naturally: find necessary and sufficient conditions which must be satisfied by a given function $f(x) \subset \mathfrak{L}_p$ in order that $f(x)$ be the limit-function of a function $f(z) \subset \mathfrak{A}_p$. In a recent note [2]) we gave a solution of this problem under the assumption that $f(x)$ possesses a Fourier transform (in a certain generalized sense). It turns out that this transform must vanish for negative values of its argument. The purpose of the present note is to investigate the same problem under a different assumption, that $f(x)$ itself is the Fourier transform in $\mathfrak{L}_p$ of a function $g(u) \subset \mathfrak{L}_q$, $1 \leqq q \leqq \infty$.

Theorem. Let $f(x)$ be the Fourier transform in $\mathfrak{L}_p$ of a function $\varphi(u) \subset \mathfrak{C}_q$, $1 \leqq q \leqq \infty$. In order that $f(x)$ be the limit-function of a function $f(z) \subset \mathfrak{A}_p$ it is necessary and sufficient that $\varphi(u)$ vanish for $u > 0$.

Proof. For convenience we replace $\varphi(u)$ by $g(-u)$ and set

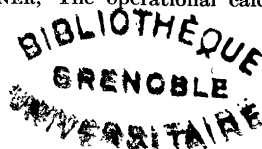$$(9) \qquad f(\xi) = \mathop{L^p F(\xi;\,a)}\limits_{a \to \infty},$$

where

$$(10) \qquad F(\xi\,;\,a) = \int_{-a}^{a} g(u)e^{i\xi u}du,$$

and $g(u) \subset \mathfrak{L}_q$. We first assume that $1 < q < \infty$, and proceed to the computation of the Poisson integral of $f(\xi)$,

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[1]) Cf. HILLE and TAMARKIN, On a theorem of Paley and Wiener [Annals of Math. (2) **34** (1933), 606—614].

[2]) Loc. cit. [1]) For a special case see N. WIENER, The operational calculus [Math. Ann. **95** (1926), 557—584; (580)].

$$(11) \quad Q(z\,;f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} = \lim_{N \to \infty} \frac{1}{\pi} \int_{-N}^{N} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} \,,$$

which obviously converges absolutely. In view of (9) we have

$$(12) \quad \int_{-N}^{N} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} = \int_{-\infty}^{\infty} g(u)du \int_{-N}^{N} \frac{ye^{e\xi u}d\xi}{(\xi - x)^2 + y^2}.$$

On the other hand, by a direct computation,

$$(13) \quad \frac{y}{(\xi - x)^2 + y^2} = \frac{1}{2} \int_{-\infty}^{\infty} e^{i(x-\xi)t}e^{-y|t|}dt.$$

Hence

$$(14) \quad \begin{aligned} \int_{-N}^{N} \frac{ye^{i\xi u}d\xi}{(\xi - x)^2 + y^2} &= \frac{1}{2} \int_{-\infty}^{\infty} e^{ixt-y|t|} \, dt \int_{-N}^{N} e^{i\xi(u-t)}d\xi = \\ &= \int_{-\infty}^{\infty} e^{ixt-y|t|} \, \frac{\sin N(u-t)}{u-t} \, dt. \end{aligned}$$

On substituting into (12) and interchanging the order of integration, which is clearly permissible, we get

$$(15) \quad \begin{aligned} \frac{1}{\pi} \int_{-N}^{N} \frac{yf(\xi)d\xi}{(\xi - x)^2 + y^2} &= \int_{-\infty}^{\infty} e^{ixt-y|t|} \, dt \int_{-\infty}^{\infty} g(u) \frac{1}{\pi} \frac{\sin N(u-t)}{u-t}du = \\ &= \int_{-\infty}^{\infty} e^{ixt-y|t|} \, dt \int_{-\infty}^{\infty} g(u) \, \mathfrak{D}_N\,(u-t)\,du, \end{aligned}$$

where

$$\mathfrak{D}_N(u) = \frac{1}{\pi} \frac{\sin Nu}{u}$$

is the classical Dirichlet kernel. Now put

$$g_N(t) = \int_{-\infty}^{\infty} g(u)\, \mathfrak{D}_N(u-t)du.$$

It is known [1]) that

$$\| g_N(t) - g(t) \|_q \to 0 \text{ as } N \to \infty.$$

Consequently

$$(16) \quad \begin{aligned} Q(z;\,f) &= \int_{-\infty}^{\infty} e^{ixt-y|t|} \, g(t)dt = \\ &= \int_{0}^{\infty} e^{izt} g(t)dt + \int_{-\infty}^{0} e^{i\bar{z}t}g(t)dt \equiv \Omega_1(z) + \Omega_2(\bar{z}), \end{aligned}$$

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[1])   See, for instance, HILLE and TAMARKIN, On the theory of Fourier transforms [Bulletin of the Amer. Math. Soc. 39 (1933)].

where $\bar{z} = x - iy$. Here $\Omega_1(z)$ is analytic in z and $\Omega_2(\bar{z})$ is analytic in $\bar{z}$, while

$$\Omega_1(z) \to 0, \quad \Omega_2(\bar{z}) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Hence the vanishing of $\Omega_2(\bar{z})$ is a necessary and sufficient condition for the analyticity of $Q(z; f)$. Now, if $f(z) \subset \mathfrak{A}_p$, then $f(z)$ is represented by its Poisson integral $Q(z; f)$, and $\Omega_2(\bar{z}) \equiv 0$; conversely, if $\Omega_2(\bar{z}) \equiv 0$, $Q(z; f)$ is analytic and represents a function $f(z) \subset \mathfrak{A}_p$ whose limit-function is precisely $f(x)$. We see therefore that the condition $\Omega_2(\bar{z}) \equiv 0$ is necessary and sufficient in order that $f(x)$ be the limit-function of a function $f(z) \subset \mathfrak{A}_p$. In view of the uniqueness theorem for Fourier integrals, however, the condition $\Omega_2(\bar{z}) \equiv 0$ is equivalent to the condition $g(t) = 0$ for $t < 0$; which is the desired result.

The treatment of the cases $q = 1$, $q = \infty$ is slightly more complicated, but formula (16) and all the subsequent conclusions will still be valid. When $q = 1$ or $q = \infty$, we apply the method of arithmetic means to evaluate the integral (11). Thus

$$\frac{1}{N}\int_0^N dn\, \frac{1}{\pi}\int_{-n}^n \frac{yf(\xi)d\xi}{(\xi-x)^2+y^2} = \lim_{a\to\infty}\int_{-a}^a g(u)du\, \frac{1}{N}\int_0^N dn\, \frac{1}{\pi}\int_{-n}^n \frac{ye^{i\xi u}d\xi}{(\xi-x)^2+y^2} =$$

$$= \lim_{a\to\infty}\int_{-a}^a g(u)du\, \frac{1}{N}\int_0^N dn\int_{-\infty}^\infty e^{ixt-y|t|}\frac{1}{\pi}\frac{\sin n(u-t)}{u-t}dt =$$

$$\text{(17)}$$

$$= \lim_{a\to\infty}\int_{-a}^a g(u)du\int_{-\infty}^\infty e^{ixt-y|t|}\frac{1}{2N\pi}\left[\frac{\sin\frac{N(u-t)}{2}}{\frac{u-t}{2}}\right]^2 dt =$$

$$= \lim_{a\to\infty}\int_{-\infty}^\infty e^{ixt-y|t|}dt\int_{-a}^a g(u)\mathfrak{F}_N(u-t)du = \int_{-\infty}^\infty e^{ixt-y|t|}dt\int_{-\infty}^\infty g(u)\mathfrak{F}_N(u-t)du$$

where

$$\mathfrak{F}_N(u) = \frac{1}{2N\pi}\left[\frac{\sin\frac{Nu}{2}}{\frac{u}{2}}\right]^2$$

is the Fejér kernel. Now put

$$g_N'(t) = \int_{-\infty}^\infty g(u)\mathfrak{F}_N(u-t)du.$$

When $N \to \infty$, the left-hand member of (17) tends to $Q(z; f)$

since the integral (11) converges. On the other hand, when $q = 1$,

$$\left\| g_N'(t) - g(t) \right\|_1 \to 0$$

while, when $q = \infty$, $g_N'(t) \to g(t)$ boundedly, and in fact uniformly over every finite range. Hence allowing $N \to \infty$, we obtain (16) again, and finish the proof as above. This method of course might have been used in the case $1 < q < \infty$ as well.