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Hermitio Zellerus

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MELANGES.

HERMITIO ZELLERUS.

Vir doctissime, summe venerande,

Cessantibus, quas expectaveram, literis Scheringii, trado tibi quæ conscripsi de numeris Bernoullicis, legenda, recensenda et si placet publicanda, eo de quo scribis modo; ita tamen, ut præferrem editionem latinam.

Superioribus addere possum formulas quasdam, in quas incidi continuando reductiones § 14.

Sit

$$K_1 = \frac{1}{x+1} \quad (x = 1, 2, 3 \ldots),$$

$$K_2 = x\Delta \frac{1}{x+1} = x\left(\frac{1}{x+1} - \frac{1}{(x+1)+1}\right) = \frac{x}{(x+1)(x+2)},$$

$$K_3 = x\Delta \frac{x}{(x+1)(x+2)} = \frac{x}{(x+1)(x+2)} - \frac{x}{(x+1+1)(x+1+2)}$$
$$= \frac{x(x-1)}{(x+1)(x+2)(x+3)},$$

$$K_4 = x\Delta \frac{x(x-1)}{(x+1)\ldots(x+3)} = \frac{x^2(x-5)}{(x+1)(x+2)\ldots(x+4)},$$

$$K_5 = x\Delta \frac{x^2(x-5)}{(x+1)\ldots(x+4)} = \frac{(x-1)(x^2-15x-4)}{(x+1)\ldots(x+5)},$$

$$K_6 = x\Delta \frac{(x-1)(x^2-15x-4)}{(x+1)\ldots(x+5)} = \frac{x(x^3-42x^2+119x+42)}{(x+1)\ldots(x+6)},$$

et sic porro, ita ut quisque terminus ex præcedente manet per simplicem legem differentiarum; tum, posito valore $x = 1$, abeunt hi termini in numeros Bernoullii; est scilicet :

$$K_1 = \frac{1}{2} = B_1, \qquad K_2 = \frac{1}{2.3} = B_2, \quad K_3 = 0 = B_3,$$

$$K_4 = -\frac{1}{30} = B_4, \quad K_5 = 0 = B_5, \qquad K_6 = \frac{1}{42} = B_6, \ldots.$$

At, crescentibus potestatibus valoris x, hæ quoque expressiones fiunt prolixiores, ut computum parum juvent, nisi forte calculus differentialis ulteriora præbeat subsidia; qua de re, si tuam sententiam mihi communicare digneris, delectamento mihi erit. Perge favere

Tui observantissimo
CHR. ZELLER, Sem. Real.

Markgrœningen, 2 nov. 1880.

DE NUMERIS BERNOULLII EORUMQUE COMPOSITIONE EX NUMERIS INTEGRIS ET RECIPROCIS PRIMIS;

SCRIPSIT CHR. ZELLERUS Marcagrœningensis.

1. Dicti numeri, quamvis sint finiti et rationales, laborare videntur quadam inconcinnitatis specie; quæ ut diffunderetur, Moivræus, Eulerus et alii prospero successu contenderunt. Eundem ipsum finem nostra spectat commentatio monstratura, quomodo componantur ex differentiis potentiarum et coefficientibus binomii reciprocis, vel omnino ex ejusmodi differentiis et numeris reciprocis.

2. Ordimur a theoremate generali ad summas serierum finitarum pertinente, quod idem alibi (¹) protulimus.

Locum habet in seriebus hujusce formæ

$$M = \alpha a + \beta b + \gamma c + \ldots + \lambda l.$$

Dirimendo factores singulorum terminorum in duplicem seriem

$$\begin{matrix} \alpha & \beta & \gamma & \ldots & \lambda \\ a & b & c & \ldots & l, \end{matrix}$$

et sumendo alterius summas

$$\Sigma_1 = \alpha,\quad \alpha+\beta,\quad \alpha+\beta+\gamma,\quad \ldots,\quad \alpha+\beta+\gamma+\ldots+\lambda,$$

(¹) *Goettinger Gelehrte Anzeigen*, 1879, p. 261.

alterius differentias

$$\Delta_1 = a - b,\quad b - c,\quad c - d,\quad \ldots,\quad k - l,\quad l - 0,$$

habebis, denuo multiplicatis singulis singulatim inter se valoribus, seriem

$$M_1 = \alpha(a - b) + (\alpha + \beta)(b - c) + \ldots,$$

priori valori parem; id quod statim elucet, solutis parenthesibus et instituta multiplicatione.

Ulterius pergendo in fingendis summis et differentiis est omnino

$$M = \Delta_1 . \Sigma_1 = \Delta_2 . \Sigma_2 = \ldots = \Delta_x . \Sigma_x,$$

siquidem producta $\Delta . \Sigma$ id tantum significant, singulos singulatim terminos ejusdem ordinis esse multiplicandos.

Cujus theorematis varius est usus, præsertim in seriebus arithmeticis primi gradus et altioris, cum constet, hic postremo apparere differentias inter se æquales, ita ut denuo sumta differentia plurimi termini evanescant vel ciphræ æquentur. Hinc fit, ut series extensior reduci possit ad multo breviorem.

3. Fiat applicatio in summandas potestates numerorum naturalium.

Problema. — Formulis definire summam potestatum

$$S(n^m) \text{ vel } S(m, n) = n^m + (n - 1)^m + \ldots + 3^m + 2^m + 1^m.$$

Fingens compositam esse hanc seriem ex potentiis et coefficientibus 1, 1 ..., sumas illarum differentias, horum summas et habebis in serie m^{ta} differentiarum, exceptis extremis membris, per omnem terminum $m!$, et in subsequente serie valores evanescentes exceptis m ad finem positis.

Summando autem alteram seriem 1, 1, 1, ..., obtinentur primo numeri naturales 1, 2, 3, ..., tum gradatim numeri figurati secundi, tertii, ..., m^{ti} ordinis. Multiplicando horum m extremos cum m postremis numeris differentialibus, habebis valorem $S(n^m)$, non jam ex n terminis, sed tantum ex tot congerendum, quot unitates habet exponens m.

4. *Exemplum.* — Sit

$$S(n^m) = S(6^3) = 6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3$$

series differentiarum

	216	125	64	27	8	1
Δ_1...	91	61	37	19	7	1
Δ_2...	30	24	18	12	6	1
Δ_3...	6	6	6	6	5	1
Δ_4...	0	0	0	1	4	1

Redactæ sunt differentiæ ad tres terminos, ad quos definiendos tres extremæ potestates suffecissent.

Series summarum

	1	1	1	1	1	1
Σ_1...	1	2	3	4	5	6
Σ_2...	1	3	6	10	15	21
Σ_3...	1	4	10	20	35	56
Σ_4...	1	5	15	35	70	126

conjungendo cum hac quarta serie summatoria quartam differentialem, habebis

$$S(6^3) = \Delta_4 . \Sigma_4 = 1.35 + 4.70 + 1.126 = 441.$$

5. Termini summatorii cum nihil sint, nisi numeri figurati, etiam sine additione ex formulis obtinentur: e. g. numerus n^{tus} seriei Σ_4 par est

$$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}, \quad 126 = \frac{6.7.8.9}{1.2.3.4},$$

sive, quod idem est, = quarto coefficienti binomiali potestatis $n+3$, i. e. $= (n+3)_4$.

Etiam termini differentiales, id quod infra uberius elucescet, per formulas derivari possunt ex antecedentibus, scilicet per formulam binomialem, ita ut, ex. g.,

Termini primi seriei	Δ_2	oriantur ex schemate		$a - 2b + c$,
»	in Δ_3	»		$a - 3b + 3c - d$,
»	in Δ_4	»		$a - 4b + 6c - 4d + e$,

et sic porro; vel positis numeris:

$$30 \text{ in tabula præcedente est } = 6^3 - 2.5^3 + 4^3,$$
$$\text{» } \quad 6 = 6^3 - 3.5^3 + 3.4^3 - 3^3$$
$$= 5^3 - 3.4^3 + 3.3^3 - 1.2^3,$$
....

6. Pro corollario addi potest tabulas summarum et differentiarum etiam per diagonales obliqua directione sibi respondere.

Conjungendo diagonalem tabulæ differentiarum...............	1	7	12	6
cum diagonali alterius tabulæ..	1	5	10	10
habebis.......	$1 + 35 + 120 + 60 = 216 = 6^3$,			

atque

subjungendo eidem diagonali	1	7	12	6
subsequentem tabulæ alterius	6	15	20	16
prodit.......	$6 + 105 + 240 + 90 = 441 = S(6^3)$.			

Termini inferiores hic sunt ex coefficientibus binomialibus ejusdem potestatis, et omnis hic calculus convenit cum usitato modo differentiandi et cum formula

$$S_n = n.a^1 + \frac{n(n-1)}{1.2}\Delta^1 a^1 + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 a^1 + \ldots.$$

7. *Regula generalis.* — Ex dictis patebit, ad inveniendam summam potestatum m numerorum naturalium:

1° Derivandos esse ex potestatibus numerorum $1, 2, \ldots, m$ seriem $(m+1)^{am}$ differentialem, vel istos m terminos ad quos reducitur; eosque

2° Multiplicandos esse cum m extremis terminis summatoriis, sive cum totidem numeris figuratis ejusdem ordinis, qui idem $(m+1)^{os}$ coefficientes binomii repræsentant potestatum $(n+m)$, $(n+m-1)$, ..., usque ad $(n-1)$.

3° m producta resultantia æquiparant summam quæsitam $S(n^m)$, quæ eadem secundum præcedens corollarium par est summæ m productorum ex numeris obliquo ordine per diagonales signatis.

8. Ad calculum facilius instituendum adjumento est tabula differentias extremas potentiarum continens. Habemus pro exponente

$m=2$,	2^2	1^2	$m=3$,	27	8	1	$m=4$,	256	81	16	1, etc.
Δ_1	3	1	Δ_1	19	7	1	Δ_1	175	65	15	1
Δ_2	2	1	Δ_2	12	6	1	Δ_2	110	50	14	1
Δ_3	1	1	Δ_3	6	5	1	Δ_3	60	36	13	1
....			Δ_4	1	4	1	Δ_4	24	23	12	1
....			....				Δ_5	1	11	11	1
....			....				....				

vel si postremæ tantum consignentur differentiæ,

	I.	II.	III.	IV.	V.	VI.	VII.
$m=2\ldots$	1	1					
$m=3\ldots$	1	4	1				
$m=4\ldots$	1	11	11	1			
$m=5\ldots$	1	26	66	26	1		
$m=6\ldots$	1	57	302	302	57	1	
$m=7\ldots$	1	120	1191	2416	1191	120	1

quam tabulam facili negotio continuare poteris, animadvertens legem progressionis vel recursionis, ita comparatam, ut quisque terminus t producatur ex supra apposito a eumque præcedente b per formulam

$$t = ax + by,$$

significante x indicem columnæ verticalis t ab initio, y eundem a tergo, inverso ordine, numeratum, e. g.:

$$120 = 2.57 + 6.1,$$
$$1191 = 3.302 + 5.57 \ldots.$$

Ope hujus tabulæ comparatæ cum tabula numerorum figuratorum omnes summæ potestatum numerorum naturalium facile definiri poterunt.

Est, ex. gr.,

$S(10^6)$, i.e. summa potestatum sextarum 1^6 usque ad 10^6,

$$= 1.(11)_7 + 57.(12)_7 + 302.(13)_7 + 302.(14)_7 + 57\,(15)_7 + 1.(16)_7,$$

designante parenthesi $(16)_7$ coefficientem binomii

$$\frac{16.15.14.13.12.11.10}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}, \text{ etc.}$$

Simili modo ad obtinendam summam $S(n^x)$ opus erit coefficientibus $(n+1)_x$, $(n+2)_x$, ..., $(n+x)_x$.

9. Ejusmodi coefficientes binomiales sive numeri figurati cum oriantur ex multiplicatione numerorum $n+1$, n, $n-1$ et similium, functiones sunt numeri n et ordinari poterunt secundum potestates ejus numeri, quo facto ratio patebit, quæ intercedit inter terminos nostros et numeros Bernoullii.

Scilicet: computanti ope numerorum Bernoullii summam $S(n^6)$ se præbet æquatio

$$n^{6+1} - 2n^6 B_1 + 21 n^5 B_2 - 35 n^4 B_3 + 35 n^3 B_4 - 21 n^2 B_5 + 7 n B_6$$
$$= (6+1) S(n^6) \quad (^1),$$

ubi litteræ B paris indicis cum numeris Bernoullii conveniunt, imparis indicis, præter B_1, evanescunt.

Coefficiens primæ potestatis n^1, i.e. B_6, secundum notissimum illud de coefficientibus theorema par erit coefficienti primæ potestatis numeri n ex superiore expositione et numeris figuratis obtinendo.

Habebamus

$$\begin{aligned} S(n^6) = & \frac{(n+1)n(n-1)\ldots(n-5)}{7!} \times 1 \\ & + \frac{(n+2)(n+1)\ldots(n-4)}{7!} \times 57 \\ & + \frac{(n+3)(n+2)\ldots(n-3)}{7!} \times 302 \\ & + \frac{(n+4)(n+3)\ldots(n-2)}{7!} \times 302 \\ & + \frac{(n+5)(n+4)\ldots(n-1)}{7!} \times 57 \\ & + \frac{(n+6)(n+5)\ldots n}{7!} \times 1. \end{aligned}$$

(1) Cf. *Meier Hirsch*, ed. Bertram, p. 87.

Seligendo coefficientem potestatis n^1 obtinetur

$$-\frac{5!}{7!}\cdot 1+\frac{2!4!}{7!}\cdot 57-\frac{3!3!}{7!}\cdot 302+\frac{4!2!}{7!}\cdot 302-\frac{5!}{7!}\cdot 57+\frac{6!}{7!}\cdot 1,$$

vel inverso ordine

$$\frac{1}{7}\left(1-\frac{5!}{6!}\cdot 57+\frac{4!2!}{6!}\cdot 302-\frac{3!3!}{6!}\cdot 302+\frac{2!4!}{6!}\cdot 57-\frac{5!}{6!}\cdot 1\right),$$

vel

$$\frac{1}{7}\left(1-\frac{1}{6}\cdot 57+\frac{1}{15}\cdot 302-\frac{1}{20}\cdot 302+\frac{1}{15}\cdot 57-\frac{1}{6}\cdot 1\right),$$

qui valor æquiparabit terminum B_6, ita ut sit

$$7B_6=1-\frac{1}{6}\cdot 57+\frac{1}{15}\cdot 302-\frac{1}{20}\cdot 302+\frac{1}{15}\cdot 57-\frac{1}{6}\cdot 1.$$

Habes hic differentias potestatis sextæ multiplicatas in coefficientes binomiales reciprocos ejusdem potestatis; et cum tam luculenta sit lex compositionis, parum habet negotii, pro unoquoque valore B_x eam transcribere.

Est itaque

$$3B_2=1-1.\frac{1}{2}, \qquad B_2=\frac{1}{6},$$

$$4B_3=1-\frac{1}{3}\cdot 4+\frac{1}{3}\cdot 1, \qquad B_3=0,$$

$$5B_4=1-\frac{1}{4}\cdot 11+\frac{1}{6}\cdot 11-\frac{1}{4}\cdot 1, \qquad B_4=-\frac{1}{30},$$

$$6B_5=1-\frac{1}{5}\cdot 26+\frac{1}{10}\cdot 66-\frac{1}{10}\cdot 26+\frac{1}{6}\cdot 1, \qquad B_5=0,$$

$$7B_6=1-\frac{1}{6}\cdot 57+\ldots, \qquad B_6=\frac{1}{42},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots$$

10. Quum, quod supra monstravimus (5), differentiæ oriantur per legem binomialem ex valoribus antecedentibus, ad hos unaquæque differentia reduci potest. Sic, ex. gr., pro differentiis septimis sextæ potestatis valent æquationes

$$
\begin{aligned}
1 &= 1^6,\\
57 &= 2^6 - 7.1^6,\\
302 &= 3^6 - 7.2^6 + 21.1^6,\\
302 &= 4^6 - 7.3^6 + 21.2^6 - 35.1^6,\\
57 &= 5^6 - 7.4^6 + 21.3^6 - 35.2^6 + 35.1^6,\\
1 &= 6^6 - 7.5^6 + 21.4^6 - 35.3^6 + 35.2^6 - 21.1^6.
\end{aligned}
$$

Positis his valoribus in expressione $7B_6$, transibit in

$$
\begin{aligned}
7B_6 = {} & 1^6\\
& - \frac{1}{6}(2^6 - 7.1^6)\\
& + \frac{1}{15}(3^6 - 7.2^6 + 21.1^6)\\
& - \frac{1}{20}(4^6 - 7.3^6 + 21.2^6 - 35.1^6)\\
& + \frac{1}{15}(5^6 - 7.4^6 + 21.3^6 - 35.2^6 + 35.1^6)\\
& - \frac{1}{6}(6^6 - 7.5^6 + 21.4^6 - 35.3^6 + 35.2^6 - 21.1^6),
\end{aligned}
$$

vel, secundum potestates ordinando,

$$
\begin{aligned}
7B_6 = 1^6 & \left(1 + \frac{1}{6}\cdot 7 + \frac{1}{15}\cdot 21 + \frac{1}{20}\cdot 35 + \frac{1}{15}\cdot 35 + \frac{1}{6}\cdot 21\right)\\
& - 2^6\left(\frac{1}{6}\cdot 1 + \frac{1}{15}\cdot\ 7 + \frac{1}{20}\cdot 21 + \frac{1}{15}\cdot 35 + \frac{1}{6}\cdot 25\right)\\
& + 3^6\left(\frac{1}{15}\ 1 + \frac{1}{20}\cdot\ 7 + \frac{1}{15}\cdot 21 + \frac{1}{6}\cdot 35\right)\\
& - 4^6\left(\frac{1}{20}\cdot\ 1 + \frac{1}{15}\cdot\ 7 + \frac{1}{6}\cdot 21\right)\\
& + 5^6\left(\frac{1}{15}\cdot\ 1 + \frac{1}{6}\cdot\ 7\right)\\
& - 6^6\left(\frac{1}{6}\cdot 1\right).
\end{aligned}
$$

Item, repetendo in posteriore parte terminos prioris pro 302, 57, 1 simpliciores,

$$
\begin{aligned}
7B_6 = & \quad 1^6 \\
& - \frac{1}{6}(2^6 - 7.1^6) \\
& + \frac{1}{15}(3^6 - 7.2^6 + 21.1^6) \\
& - \frac{1}{20}(3^6 - 7.2^6 + 21.1^6) \\
& + \frac{1}{15}(2^6 - 7.1^6) \\
& - \frac{1}{6} .1^6.
\end{aligned}
$$

Cæteris, quæ se præberent, transformationibus non immoramur, cum ad computum parum aut nihil faciant.

Id jam ex allatis concludere licet, $7!\,B_6$ vel omnino $(n+1)!\,B_n$ esse numerum integrum, ideoque denominatorem fractionis in B_n, si qua locum habeat, numerum primum, qui valorem $n+1$ transcendat, non continere.

11. Aliquanto aptiores redduntur formulæ adhibita lege recursionis n° 8 tradita. Exemplo sit expressio nostra pro $7B_6$.

Ponendo

$$57 = 2.26 + 5.1,$$
$$302 = 3.66 + 4.26\ldots,$$

prodit

$$
\begin{aligned}
7B_6 = & 1 - \frac{1}{6}\cdot 5 - \frac{2}{6}\cdot 26 + \frac{4}{15}\cdot 26 + \frac{3}{15}\cdot 66 - \frac{3}{20}\cdot 66 - \frac{4}{20}\cdot 26 + \frac{2}{15}\cdot 26 \\
& + \frac{1}{15}\cdot 5 - \frac{1}{6}\cdot 1 \\
= & \frac{1}{6}\cdot 1 - \frac{1}{15}\cdot 26 + \frac{1}{20}\cdot 66 - \frac{1}{15}\cdot 26 + \frac{1}{6}\cdot 1,
\end{aligned}
$$

quæ formula item constat ex coefficientibus binomii reciprocis et differentiis potentiarum, iisque gradus inferioris, et vincit priorem ratione symmetriæ et brevitatis. Præsertim eum habet usum, ut manifestum faciat, cur valores impari indice signati evanescant, scilicet quia apud eos in posteriore parte iidem termini ut in priore reperiantur, sed contrario signo affecti, ita ut se invicem tollant.

Restant formulæ

$$3B_2=\frac{1}{2},\qquad B_2 \text{ sive } \mathfrak{B}_1 \text{ (h. e. num. Bern. I)} = \frac{1}{6},$$

$$5B_4=\frac{1}{4}-\frac{1}{6}\cdot 4+\frac{1}{4},\qquad B_4=\mathfrak{B}_2=-\frac{1}{30},$$

$$7B_6=\frac{1}{6}-\frac{1}{15}\cdot 26+\frac{1}{20}\cdot 66$$

$$-\frac{1}{15}\cdot 26+\frac{1}{6},\qquad \mathfrak{B}_3=+\frac{1}{42},$$

$$9B_8=\frac{1}{8}-\frac{1}{28}\cdot 120$$

$$+\frac{1}{56}\cdot 1191-\frac{1}{70}\cdot 2416$$

$$+\frac{1}{56}\cdot 1191-\frac{1}{28}\cdot 120+\frac{1}{8},\qquad \mathfrak{B}_4=-\frac{1}{30},$$

....

II.

12. Revertamur jam ad n° 6 et eas expressiones summatorias, quæ diagonalibus se applicant.

Propositum sit iterum problema, summam $S(10^6)$ inveniendi. Evolvantur differentiæ potestatum :

Tabula differentiarum.

7^6	6^6	5^6	4^6	3^6	2^6	1^6
117649	46656	15625	4096	729	64	1*
70993	31031	11529	3363	665	63*	1
39962	19502	8162	2702	602*	62	1
20460	11340	6460	2100*	540	61	1
9120	5880	3360*	1560	479	60	1
3240	2520*	1800	1081	419	59	1
720*	720	719	662	360	58	1
0	1	57	302	302	57	1

Tabula summarum vel numerorum figuratorum.

1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10*
1	3	6	10	15	21	28	36	45*	55
1	4	10	20	35	56	84	120*	165	220
1	5	15	35	70	126	210*	330	495	715
1	6	21	56	126	252*	462	792	1287	2002
1	7	28	84	210*	462	924	1716	3003	5005
1	8	36	120*						

Conjungantur, quas asterisco signavimus series lineæ diagonalis

1	63	602	2100	3360	2520	720

cum serie obliqua alterius tabulæ

10	45	120	210	252	210	120

et prodibit summa quæsita.

Ut supra monuimus, coefficientes posteriores pertinent ad binomium potestatis 10, vel omnino pro $S(n^x)$ ad binomium $n^{\text{tæ}}$ potestatis. Sunt hi :

$$\frac{n}{1},\quad \frac{n(n-1)}{1.2},\quad \frac{n(n-1)(n-2)}{1.2.3},\quad \ldots.$$

in quibus occurrit potestas n^1 multiplicata cum

$$\frac{1}{1},\quad -\frac{1}{2},\quad +\frac{1}{3},\quad -\frac{1}{4},\quad +\frac{1}{5},\quad \ldots.$$

Seligendo coefficientem hujus potentiæ n^1 habebis ut supra valorem B_6 parem. Est igitur

$$B_6 = \frac{1}{1}\cdot 1 - \frac{1}{2}\cdot 63 + \frac{1}{3}\cdot 602 - \frac{1}{4}\cdot 2100 + \frac{1}{5}\cdot 3360 - \frac{1}{6}\cdot 2520 + \frac{1}{7}\cdot 720,$$

in qua formula numeris reciprocis naturali ordine progredientibus conjunctæ sunt differentiæ potestatum ex diversis seriebus depromtæ.

Possunt hæ ut supra retroverti in aggregata potentiarum. Est

$$63 = 2^6 - 1,$$
$$602 = 3^6 - 2.2^6 + 1.1^6,$$
$$2100 = 4^6 - 3.3^6 + 3.2^6 - 1.1^6,$$
$$3360 = 5^6 - 4.4^6 + 6.3^6 - 4.2^6 + 1.1^6,$$
$$2520 = 6^6 - 5.5^6 + 10.4^6 - 10.3^6 + 5.2^6 - 1^6,$$
$$720 = 7^6 - 6.6^6 + 15.5^6 - 20.4^6 + 15.3^6 - 6.2^6 \ (= 6!),$$

ut item sit

$$\begin{aligned} B_6 = & \ 1.1^6 \\ & - \frac{1}{2}(2^6 - 1) \\ & + \frac{1}{3}(3^6 - 2.2^6 + 1) \\ & - \frac{1}{4}(4^6 - 3.3^6 + 3.2^6 - 1) \\ & + \frac{1}{5}(5^6 - 4.4^6 + 6.3^6 - 4.2^6 + 1) \\ & - \frac{1}{6}(6^6 - 5.5^6 + 10.4^6 - 10.3^6 + 5.2^6 - 1) \\ & + \frac{1}{7}(7^6 - 6.6^6 + 15.5^6 - 20.4^6 + 15.3^6 - 6.2^6 + 1), \end{aligned}$$

vel digerendo secundum potestates

$$\begin{aligned} B_6 = & \ 1^6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) \\ & - 2^6\left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}\right) \\ & + 3^6\left(\frac{1}{3} + \frac{3}{4} + \frac{6}{5} + \frac{10}{6} + \frac{15}{7}\right) \\ & - 4^6\left(\frac{1}{4} + \frac{4}{5} + \frac{10}{6} + \frac{20}{7}\right) \\ & + 5^6\left(\frac{1}{5} + \frac{5}{6} + \frac{15}{7}\right) \\ & - 6^6\left(\frac{1}{6} + \frac{6}{7}\right) \\ & + 7^6\left(\frac{1}{7}\right). \end{aligned}$$

Ipse conspectus legem compositionis docet, ita ut nihil habeat difficultatis, eam cuilibet valori B_x adaptare.

13. Instituatur tabula differentiarum quæ in his nostris formulis multiplicandæ sunt cum numeris reciprocis. Hanc præbebit faciem :

	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
Pro B_1...	1	1							
B_2...	1	3	2						
B_3...	1	7	12	6					
B_4...	1	15	50	60	24				
B_5...	1	31	180	390	360	120			
B_6...	1	63	602	2100	3360	2520	720		
B_7...	1	127	1932	10206	25200	31920	20160	5040	
B_8...	1	255	6050	46620	166824	317520	332640	181440	40320
.....	..	...	....	.....	......	......	......	......	.....

Valores in columna II formæ sunt $2^x - 1$,

Valores in columna III formæ sunt $3^x - 2.2^x + 1.1^x$, et sic porro.

Verum etiam hic, ut in simili tabula § 8, valet lex recursionis ita comparata, ut pari modo quicumque terminus

$$t = ax + by,$$

significante a terminum supra positum, b antecedentem, dum x et y pares sunt singulis indicibus columnæ verticalis ad a et b pertinentis.

Est, ex. gr.,

$$\begin{aligned} 63 &= 2.31 + 1.1, \\ 602 &= 3.180 + 2.31, \\ 2100 &= 4.390 + 3.180, \\ 3360 &= 5.360 + 4.390, \\ 2520 &= 6.120 + 5.360, \\ 720 &= \quad 0 \quad + 6.120. \end{aligned}$$

Quæ lex non solum adjumentum præbet ad continuandam expedite tabulam, sed etiam subsidium, cujus ope formulæ ipsæ simpliciorem inducunt formam. Introducendo enim in formulam pro B_6

valores jam conscriptos, habes :

$$B_6 = 1 - \frac{1}{2}(1 + 2.31) + \frac{1}{3}(2.31 + 3.180)$$

$$- \frac{1}{4}(3.180 + 4.390) + \frac{1}{5}(4.390 + 5.360) - \frac{1}{6}(5.360 + 6.120) + \frac{1}{7}6.120$$

$$= 1 - \frac{1}{2} - 31\left(\frac{2}{2} - \frac{2}{3}\right) + 180\left(\frac{3}{3} - \frac{3}{4}\right) - 390\left(\frac{4}{4} - \frac{4}{5}\right) + 360\left(\frac{5}{5} - \frac{5}{6}\right) - 120\left(\frac{6}{6} - \frac{6}{7}\right)$$

$$= \frac{1}{2} \quad - \frac{31}{3} \quad + \frac{180}{4} \quad - \frac{390}{5} \quad + \frac{360}{6} \quad - \frac{120}{7},$$

vel revertendo ad aggregata potestatum,

$$\begin{aligned} B_6 = & \ \frac{1}{2} . 1^5 \\ & - \frac{1}{3}(2^5 - 1.1^5) \\ & + \frac{1}{4}(3^5 - 2.2^5 + 1.1^5) \\ & - \frac{1}{5}(4^5 - 3.3^5 + 3.2^5 - 1.1^5) \\ & + \frac{1}{6}(5^5 - 4.4^5 + 6.3^5 - 4.2^5 + 1.1^5) \\ & - \frac{1}{7}(6^5 - 5.5^5 + 10.4^5 - 10.3^5 + 5.2^5 - 1.1^5), \end{aligned}$$

qui numeri ad quemque valorem B_x facile transcribi poterunt.

14. Si in tabula nostra termini columnæ primæ omnino per A, secundæ per B, etc., indigitantur, reputanti, quod supra n° **11** ostendimus, omnes B_x, ubi x numerum imparem significat, ciphræ æquari, patebit seriem $\frac{A}{1} - \frac{B}{2} + \frac{C}{3} - \frac{D}{4} + \cdots$ esse aut = numero Bernoullii aut = o, simulque seriem $\frac{A}{2} - \frac{B}{3} + \frac{C}{4} - \frac{D}{5} + \cdots$ esse aut = o aut = numero Bernoullii. Illud locum habet si A, B, C,... ex serie paris ordinis 2, 4, 6, ..., hoc si ea serie 3, 5, 7,... imparis ordinis sumuntur.

Exemplum I. — Numeri II$^{\text{æ}}$ seriei vel lineæ :

$$1. \quad 3. \quad 2.$$

(*a*) $$\frac{1}{1}-\frac{3}{2}+\frac{2}{3}=\frac{1}{6}=\mathfrak{B}_1,$$

(*b*) $$\frac{1}{2}-\frac{3}{3}+\frac{2}{4}=0.$$

Exemplum II. — Numeri lineæ tertiæ :

$$1. \quad 7. \quad 12. \quad 6.$$

(*a*) $$\frac{1}{1}-\frac{7}{2}+\frac{12}{3}-\frac{6}{4}=0,$$

(*b*) $$\frac{1}{2}-\frac{7}{3}+\frac{12}{4}-\frac{6}{5}=-\frac{1}{30}=\mathfrak{B}_2.$$

Hac significatione utens omnino habebis :

$$\mathfrak{B}_x = B_{2x} = \frac{A_{2x-1}}{2}-\frac{B_{2x-1}}{3}+\frac{C_{2x-1}}{4}-\frac{D_{2x-1}}{5}+\ldots$$
$$=\frac{A_{2x}}{1}-\frac{B_{2x}}{2}+\frac{C_{2x}}{3}-\frac{D_{2x}}{4}+\ldots,$$

indice litteris A, B, ..., addito numerum seriei in tabula nostra indicante.

Item, subtrahendo valores (*a*) et (*b*) :

$$-\mathfrak{B}_x = \frac{A_{2x-1}}{1.2}-\frac{B_{2x-1}}{2.3}+\frac{C_{2x-1}}{3.4}-\frac{D_{2x-1}}{4.5}+\ldots$$
$$\mathfrak{B}_x = \frac{A_{2x}}{1.2}-\frac{B_{2x}}{2.3}+\frac{C_{2x}}{3.4}-\frac{D_{2x}}{4.5}+\ldots$$

III.

15. Restat disquirere, utrum singula expressionis nostræ membra numeri fracti sint an integri, sive, utrum peracta divisione terminorum A, B, C, ... per 2, 3, 4, ..., residuum relinquatur necne.

Quod si eruerimus, simul via patebit ad deducendum ex ipsa formula illustre illud theorema, a Staudtio et Clausenio inventum, cujus ille demonstrationem aliunde repetiit (cf. CRELLE, *Journal für die reine u. angewandte Mathematik,* t. XXI, p. 372, *Beweis eines Lehrsatzes die Bernoullischen Zahlen betreffend*).

Quum singuli valores A, B, C, ... sint formæ

$$n^x - (n-1)(n-1)^x + \frac{(n-1)(n-2)}{1.2}(n-2)^x$$
$$- \frac{(n-1)(n-2)(n-3)}{1.2.3}(n-3)^x + \ldots,$$

quæstio in eo versatur: utrum hæc expressio per n divisa residuum relinquat necne.

Mirabar, excepto casu $n = 4$, id tantum evenire posse, ut residuum aut penitus evanescat aut $= -1$ evadat. Illud sæpissime fit; hoc tum solum accidit, si n est numerus primus, impar, ejus conditionis, ut $n - 1$ exponentem x dividat.

16. Casum hunc alterum ope theorematis Fermatiani facile absolvi posse, exempla doceant.

Exemplum I. — Proposita sit ex formula pro B_{12} vel $\mathfrak{B}_6$ expressio:

$$13^{12} - 12.12^{12} + 66.11^{12} - 220.10^{12} + \ldots.$$

Demonstratio :

$$13^{12} \equiv 0 \,(\text{mod. } 13), \quad 12^{12} \equiv 1 \,(\text{mod. } 13),$$

secundum Fermatii theorema.

Item

$$11^{12},\ 10^{12},\ 9^{12},\ \ldots \equiv 1 \quad (\text{mod. } 13)$$

quare redit expressio ad

$$0 - 12 + 66 - 220 + 495 \ldots.$$

At

$$1 - 12 + 66 - 220 + 495 \ldots = (1-1)^{12} = 0 \equiv 0 \quad (\text{mod. } 13);$$

ergo nostra expressio unitate minor $\equiv -1$ (mod. 13).

Exemplum II.

$$7^{18} - 6.6^{18} + 15.5^{18} - 20.4^{18} + 15.3^{18} - 6.2^{18} + 1^{18}.$$

$$7^{18} \equiv 0 \pmod{7}, \quad 6^{18} \equiv 6^6 \equiv 1 \pmod{7}, \quad \text{item} \quad 6^{18}, \quad 4^{18}, \quad \ldots;$$

ergo expressio reducitur ad

$$0 - 6 + 15 - 20 + 15 - 6 + 1;$$

at

$$1 - 6 + 15 - 20 + 15 - 6 + 1 \equiv 1 - 11^6 \equiv 0 \pmod{7};$$

ideoque aggregatum nostrum $\equiv -1 \pmod{7}$.

Unde sequitur esse omnino

$$a^x - m(a-1)^x + \frac{m(m-1)}{1.2}(a-2)^x - \ldots \equiv -1 \quad (\text{mod. D}),$$

si $D = a$ numerus primus est et x multiplum numeri $D - 1$, ubi m potest esse sive, ut in exemplis, $= a - 1$, sive $< a - 1$.

Etiam e theoremate Wilsonii demonstratio potuisset derivari, quum, ex. gr.,

$$7^{18} - 6.6^{18} + 15.5^{18} \ldots \equiv 7^6 - 6.6^6 + 15.5^6 \ldots \quad \pmod{7},$$

et posterior expressio $= 6! \equiv -1 \pmod{7}$, ex lege Wilsonii.

17. Altera protaseos nostræ pars : esse in cæteris præter dictum casum

$$a^x - m(a-1)^x + \frac{m(m-1)}{1.2}(a-2)^x - \ldots \equiv 0 \quad (\text{mod. } a),$$

magis ardua videtur esse demonstratu.

Post varios irritos conatus recurrebimus ad tabulam differentiarum, quæ pro potestatibus tertiæ dimensionis sic se habet :

	7^3.	6^3.	5^3.	4^3.	3^3.	2^3.	1^3.
	343	216	125	64	27	8	1
Δ_1.......	127	91	61	37	19	7	1
Δ_2.......	36	30	24	18	12	6	1
Δ_3.......	6	6	6	6	6	5	1
Δ_4.......	0	0	0	0	1	4	1

	7^a.	6^a.	5^a.	4^a.	3^a.	2^a.	1^a.
Δ_5.......	0	0	0	-1	-3	3	1
Δ_6.......	0	0	$+1$	$+2$	-6	2	1
Δ_7.......	0	-1	-1	8	-8	1	1
Δ_8.......	$+1$	0	-9	16	-9	0	1
....	....	...	....	...	....	..	..

Sicut hic in tertia linea, ita omnino apud x^{tas} potestates in x^{ta} serie differentiarum plurimi termini, vel omnes præter $x-1$ in fine, inter se pares erunt, ita ut in proxima linea evanescant et præter x extremos nihilo æquentur.

Teneamus porro, quod omnes hujus tabulæ valores secundum legem binomii ex prioribus derivantur. Quum igitur in m^{ta} serie differentiali sint ipsius formæ

$$a^x - m(a-1)^x + \frac{m(m-1)}{1.2}(a-2)^x - \ldots,$$

cernitur ex ipsa tabula, hoc aggregatum nihilo æquari in $(x+1)^a$ serie differentiarum et sequentibus, dummodo item basis a major sit quam exponens x, sive, quod idem est, quoties basis a et index m exponentem potestatum x valore superant.

Atqui in nostris formulis, quamvis sint ejusdem speciei, lemma hoc satis commodum non adhiberi posse videtur, quum hic exponens x ad summum valori a par et nunquam simul utroque numero a et m minor existat. Sunt re vera nostra aggregata valoris positivi et nihilo majora. Verum ostendere valemus, ea singula secundum modulum a vel D congruentia esse aggregatis similibus, ad quæ, quum exponente minori quam a et m affecta sint, dictum lemma pertinet, ita ut exinde deduci possit, etiam illa, cum his congruentia, esse $\equiv 0 \pmod{a}$.

Quod rursus evidentius est eo casu, quando divisor D vel a numerus est primus.

Exemplum :

$$11^{12} - 10.10^{12} + \frac{10.9}{1.2}.9^{12} - \ldots \equiv \quad (\text{mod. } 11),$$

namque

$$11^{12} \equiv 0 \quad (\text{mod. } 11), \quad 10^{12} \equiv 10^{10}.10^2 \quad (\text{mod. } 11);$$

sed

$$10^{10} \equiv 1 \pmod{11};$$

ergo

$$10^{12} \equiv 1 \,.\, 10^2 \ldots \pmod{11},$$

et sic omnino

$$\begin{aligned} 11^{12} - 10 \,.\, 10^{12} \ldots &\equiv 11^2 - 10 \,.\, 10^2 \ldots \pmod{11} \\ &\equiv 0 \pmod{11}, \end{aligned}$$

ex lemmate allato, quum exponens 2 minor sit quam 11 et 10.

Omnino exponenti, quando ut in formulis nostris major est numero a, detrahi possunt $a - 1$ unitates, non immutata congruentia ratione moduli a, siquidem a est numerus primus.

18. Verum etiam eo casu, quando a est numerus compositus, similis valet conclusio.

Sit divisor $D = a = p^{\varpi} . r^{\rho} . s^{\sigma}$; tum, ut constat, numerus numerorum inter 1 et a, qui cum ipso a communem factorem non habent, par est,

$$\varphi(a) = (p-1)p^{\varpi-1} . (r-1)r^{\rho-1} . (s-1)s^{\sigma-1} \quad (^1)$$

et habemus, præter quasdam forsitan exceptiones, nostram tamen argumentationem non attingentes,

$$A^x \equiv A^{\pm\varphi(a)+x} \pmod{a},$$

h. e. : ratio congruentiæ quoad mod. a non immutatur, quamvis exponenti x numerus $\varphi(a)$ addatur vel dematur, semel aut pluries.

$\varphi(a)$ necessario minor est quam a et $a - 1$. Quare, quando ut in casu nostro exponens major adest quam a, detrahendo $\varphi(a)$ vel multiplum ejus numeri, ad valorem $- a$ potest reduci, non immutata ratione congruentiæ respectu mod. a. Quod si factum est, ex lemmate nostro sequitur, aggregatum jam nihilo æquari, ita ut etiam aggregatum prius nihilo debeat esse congruens respectu mod. a.

(1) Cf. Dirichlet, ed. Dedekind, ed. 2, §§ 13, 19.

Addamus exemplum :

$$15^{18} - 14 . 14^{18} + \frac{14 . 13}{1 . 2} . 13^{18} - \ldots,$$

$$\varphi(15) = (3-1)(5-1) = 8,$$

ergo

$$a^{18} \equiv a^{18-2.8} \equiv a^2 \quad (\text{mod. } 15),$$

ideoque

$$15^{18} - 14 . 14^{18} + \ldots \equiv 15^2 - 14 . 14^2 + \ldots \quad (\text{mod. } 15)$$
$$\equiv 0 \,(\text{mod. } 15),$$

ex lemmate superiore.

Absoluto jam eo quoque casu, ubi basis vel divisor est numerus compositus, contendi licebit, aggregatum nostrum omnino esse $\equiv 0$ respectu moduli divisoris D, excepto eo, quem præmisimus casu; unde evidens est, si in forma

$$B_{2n} = \mathfrak{B}_n = \frac{A}{1} - \frac{B}{2} + \frac{C}{3} - \frac{D}{4} \cdots$$

divisio perficiatur, fractiones non orturas esse, nisi quando divisores numeri primi sunt ejus indolis, ut unitate minuti exponentem dividant. Hi fractionem efficient formæ $-\left(-\frac{1}{p}\right)$ vel $\frac{1}{p}$, quare expressio ista pro B_n nullos continebit numeros, nisi integros et reciprocos primos et hancce præbebit speciem

$$M - N + O - P + \ldots \pm \frac{1}{2} + \frac{1}{p} + \frac{1}{q} + \frac{1}{r},$$

ubi litteræ M, N, ... numeros integros significant. Ita ad eundem, ut C. Staudtius finem et propositionem pervenimus.

Admoti eramus his quæstionibus per celeberrimum Dominum C. Hermite, qui, qua est comitate, animadversiones meas de integra numerorum Bernoullii parte, comprehensas in litteris ad Borchardtum datis die 8 sept. 1875 (*Crelle's Journal*, t. LXXVI, p. 93 sq.) ad nos transmittere dignatus est; quibus quum primam hujus rei notitiam debeamus, finem faciamus gratias ipsi persolvendo.