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THE FUNDAMENTAL LEMMA AND THE HITCHIN FIBRATION [after Ngô Bao Châu]

by **Thomas C. HALES**

The study of orbital integrals on p-adic groups has turned out to be singularly difficult.

(R. P. Langlands, 1992)

This report describes some remarkable identities of integrals that have been established by Ngô Bao Châu. My task will be to describe why these identities—collectively called the fundamental lemma (FL)—took nearly thirty years to prove, and why they have particular importance in the theory of automorphic representations.

1. BASIC CONCEPTS

1.1. Origins of the fundamental lemma (FL)

To orient ourselves, we give special examples of behavior that the theory is designed to explain.

EXAMPLE 1. — *We recall the definition of the holomorphic discrete series representations of $SL_2(\mathbb{R})$. For each natural number $n \geq 2$, let $V_{n,+}$ be the vector space of all holomorphic functions f on the upper half plane $\mathfrak{h}$ such that*

$$\int_{\mathfrak{h}} |f|^2 y^{n-2} dx\, dy < \infty.$$

$SL_2(\mathbb{R})$ acts on $V_{n,+}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) = (-bz+d)^{-n} f\Big(\frac{az-c}{-bz+d}\Big).$$

Similarly, for each $n \geq 2$, there is an anti-holomorphic discrete series representation $V_{n,-}$. These infinite dimensional representations have characters that exist as locally

integrable functions $\Theta_{n,\pm}$. The characters are equal: $\Theta_{n,+}(g) = \Theta_{n,-}(g)$, except when g is conjugate to a rotation

$$\gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

When g is conjugate to γ, a remarkable character identity holds:

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}. \tag{2}$$

It is striking that numerator of the difference of two characters of infinite dimensional representations collapses to the character of a two dimensional representation $\gamma \mapsto \gamma^{n-1}$ of the group H of rotations. Shelstad gives general characters identities of this sort **[49]**.

We find another early glimpse of the theory in a letter to Singer from Langlands in 1974 **[33]**. Singer had expressed interest in a particular alternating sum of dimensions of spaces of cusp forms of $G = SL_2$ over a totally real number field F. Langlands's reply to Singer describes then unpublished joint work with Labesse **[32]**. Without going into details, we remark that in the calculation of this alternating sum, there is again a collapse in complexity from the three dimensional group SL_2 to a sum indexed by one-dimensional groups H (of norm 1 elements of totally imaginary quadratic extensions of F).

These two examples fit into a general framework that have now led to major results in the theory of automorphic representations and number theory, as described in Section 7. Langlands holds that methods should be developed that are adequate for the theory of automorphic representations in its full natural generality. This means going from SL_2 (or even a torus) to all reductive groups, from one local field to all local fields, from local fields to global fields and back again, from the geometric side of the trace formula to the spectral side and back again. Moreover, interconnections between different reductive groups and Galois groups should be included, as predicted by his general principle of functoriality.

Thus, from these early calculations of Labesse and Langlands, the general idea developed that one should account for alternating sums (or κ-sums as we shall call them because they occasionally involve roots of unity other than ± 1) that appear in the harmonic analysis on a reductive group G in terms of the harmonic analysis on groups H of smaller dimension. The FL is a concrete expression of this idea.

1.2. Orbital integrals

This section provides brief motivation about why researchers care about integrals over conjugacy classes in a reductive group. Further motivation is provided in Section 7.

It is a basic fact about the representation theory of a finite group that the set of irreducible characters is a basis of the vector space of class functions on the group. A second basis of that vector space is given by the set of characteristic functions of the conjugacy classes in the group. We will loosely speak of any linear relation among the set of characteristic functions of conjugacy classes and the set of irreducible characters as a *trace formula.*

More generally, we consider a reductive group G over a local field. Each admissible representation π of G defines a *distribution character:*

$$f \mapsto \operatorname{trace} \int_G f(g)\pi(g)\, dg, \quad f \in C_c^\infty(G),$$

with dg a Haar measure on G. A trace formula in this context should be a linear relation among characteristic functions of conjugacy classes and distribution characters. To put all terms of a trace formula on equal footing, the characteristic function of a conjugacy class must also be treated as a distribution, called an *orbital integral*:

$$f \mapsto \mathbf{O}(\gamma, f) = \int_{I_\gamma \backslash G} f(g^{-1}\gamma g)\, dg, \quad f \in C_c^\infty(G),$$

where I_γ is the centralizer of $\gamma \in G$.

The FL is a collection of identities among orbital integrals that may be used in a trace formula to obtain identities among representations π.

1.3. Stable conjugacy

At the root of these κ-sum formulas is the distinction between *ordinary conjugacy* and *stable conjugacy.*

EXAMPLE 3. — *A clockwise rotation and counterclockwise rotation*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

in $SL_2(\mathbb{R})$ are conjugate by the complex matrix $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, but they are not conjugate in the group $SL_2(\mathbb{R})$ when $\theta \notin \mathbb{Z}\pi$. Indeed, a matrix calculation shows that every element of $GL_2(\mathbb{R})$ that conjugates the rotation to counter-rotation has odd determinant, thereby falling outside $SL_2(\mathbb{R})$. Alternatively, they are not conjugate in $SL_2(\mathbb{R})$ because the character identity (2) separates them.

Let G be a reductive group defined over a field F with algebraic closure $\bar{F}$.

DEFINITION 4. — *An element $\gamma' \in G(F)$ is said to be stably conjugate to a given regular semisimple element $\gamma \in G(F)$ if γ' is conjugate to γ in the group $G(\bar{F})$.*

There is a Galois cohomology group that can be used to study the conjugacy classes within a given stable conjugacy class. Let I_γ be the centralizer of an element $\gamma \in G(F)$. The centralizer is a Cartan subgroup when γ is a (strongly) regular semisimple element. Write $\gamma' = g^{-1}\gamma g$, for $g \in G(\bar{F})$. For every element σ of the Galois group $\mathrm{Gal}(\bar{F}/F)$, we have $g\,\sigma(g)^{-1} \in I_\gamma(\bar{F})$. These elements define in the Galois cohomology group $H^1(F, I_\gamma)$ a class, which does not depend on the choice of g. It is the trivial class when γ' is conjugate to γ.

EXAMPLE 5. — *The centralizer I_γ of a regular rotation γ is the subgroup of all rotations in $SL_2(\mathbb{R})$. The group $I_\gamma(\mathbb{C})$ is isomorphic to $\mathbb{C}^\times$. Each cocycle is determined by the value $r \in I_\gamma(\mathbb{C}) = \mathbb{C}^\times$ of the cocycle on the generator of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. *A given $r \in \mathbb{C}^\times$ satisfies the cocycle condition when $r \in \mathbb{R}^\times$ and represents the trivial class in cohomology when r is positive. This identifies the cohomology group:*

$$H^1(\mathbb{R}, I_\gamma) = \mathbb{R}^\times/\mathbb{R}_+^\times = \mathbb{Z}/2\mathbb{Z}.$$

This cyclic group of order two classifies the two conjugacy classes within the stable conjugacy class of a rotation.

When F is a local field, $A = H^1(F, I_\gamma)$ is a finite abelian group. Every function $A \to \mathbb{C}$ has a Fourier expansion as a linear combination of characters κ of A. The *theory of endoscopy* is the subject that studies stable conjugacy through the separate characters κ of A. Allowing ourselves to be deliberately vague for a moment, the idea of endoscopy is that the Fourier mode of κ (for given I_γ and G) produces oscillations that cause some of the roots of G to cancel away. The remaining roots are reinforced by the oscillations and become more pronounced. The root system consisting of the pronounced roots defines a group H of smaller dimension than G. With respect to the harmonic analysis on the two groups, the mode of κ on the group G should be related to the dominant mode on H.

1.4. Endoscopy

The smaller group H, formed from the "pronounced" subset of the roots of G, is called an *endoscopic group*. Hints about how to define H precisely come from various sources.

- It should be constructed from the data (G, I_γ, κ), with γ regular semisimple.
- Its roots should be a subset of the roots of G (although H need not be a subgroup of G).

- H should have a Cartan subgroup $I_H \subset H$ isomorphic over F to the Cartan subgroup I_γ of G, compatible with the Weyl groups of the two groups H and G.
- Over a nonarchimedean local field, the spherical Hecke algebra on G should be related to the spherical algebra on H.
- It should generalize the example of Labesse and Langlands.

Every reductive group G has a *dual group* $\hat{G}$ that is defined over $\mathbb{C}$. The character group of a Cartan subgroup in the dual group is the cocharacter group of a Cartan subgroup in G, and the roots of the $\hat{G}$ are the coroots of G. The dual of a semisimple simply connected semisimple group is an adjoint group, and vice versa. For example, we have dualities $\hat{G}L(n) = PGL(n)$ and $\hat{S}p(2n) = SO(2n+1)$. The duality between the root systems of $Sp(2n)$ and $SO(2n+1)$ interchanges short and long roots. The groups G and $\hat{G}$ have isomorphic Weyl groups. We write $\hat{T} \subset \hat{G}$ for a Cartan subgroup of $\hat{G}$. There is a somewhat larger dual group ${}^L G$ that is defined as a semidirect product of $\hat{G}$ with the Galois group of the splitting field of G.

There are indications that the groups H should be defined through the dual $\hat{G}$ (or more precisely, ${}^L G$) of G:

- Langlands's principle of functoriality is a collection of conjectures, relating the representation theory of groups when their dual groups are related. Since the examples about SL_2 in Section 1.1 are representation theoretic, we should look to the dual.
- The Satake transform identifies the spherical Hecke algebra with a dual object.
- The Kottwitz-Tate-Nakayama isomorphism identifies the group of characters on $H^1(F, I_\gamma)$ with a subquotient $\pi_0(\hat{T}^\Gamma)$ of the dual torus $\hat{T}$. (This subquotient is the group of components of the set of fixed points of $\hat{T}$ under an action of the Galois group of the splitting field of I_γ.)

DEFINITION 6 (Endoscopic group). — *Let F be a local field. The endoscopic group H associated with (G, I_γ, κ) is defined as follows. By the Kottwitz-Tate-Nakayama isomorphism just mentioned, κ is represented by an element of the dual torus, $\hat{T}$. By an abuse of notation, we will also write $\kappa \in \hat{T}$ for this element. The identity component of the centralizer of κ is the dual $\hat{H}$ of a quasi-split reductive group H over F. The choice of a particular quasi-split form H among its outer forms is determined by the condition that there should be an isomorphism over F of a Cartan subgroup I_H of H with I_γ in G, compatible with their respective Weyl group actions and outer automorphisms.*

We write ρ for the choice of quasi-split form H among its outer forms and refer to the pair (κ, ρ) as *endoscopic data* for H. More generally, if G is defined over any field,

we can use a pair (κ, ρ), with $\kappa \in \hat{T}$, to define an endoscopic group H over that same field.

One of the challenging aspects of the FL is that it is an assertion of direct relation between groups that are defined by a dual relation. Very limited information (such as Cartan subgroups, root systems, and Weyl groups) can be transmitted from the endoscopic group H to G through the dual group.

2. A BIT OF LIE THEORY

2.1. Characteristic polynomials

Let G be a split reductive group over a field k and let $\mathfrak{g}$ be its Lie algebra, with split Cartan subalgebra $\mathfrak{t}$ and Weyl group W. We assume throughout this report that the characteristic of k is sufficiently large (more than twice the Coxeter number of G, to be precise). The group G acts on $\mathfrak{g}$ by the adjoint action. By Chevalley, the restriction of regular functions from $\mathfrak{g}$ to $\mathfrak{t}$ induces an isomorphism

$$k[\mathfrak{g}]^G = k[\mathfrak{t}]^W.$$

We let $\mathfrak{c} = \operatorname{Spec}(k[\mathfrak{t}]^W)$, and let $\chi : \mathfrak{g} \to \mathfrak{c}$ be the morphism deduced from Chevalley's isomorphism. The following example shows that $\chi : \mathfrak{g} \to \mathfrak{c}$ is a generalization of the characteristic polynomial of a matrix.

EXAMPLE 7. — *If $G = GL(n)$, then $k[\mathfrak{g}]^G$ is a polynomial ring, generated by the coefficients c_i of the characteristic polynomial*

$$p(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0 \tag{8}$$

of a matrix $\gamma \in \mathfrak{g} = \mathfrak{gl}(n)$. The morphism $\chi : \mathfrak{g} \to \mathfrak{c}$ can be identified with the "characteristic map" that sends γ to $(c_{n-1}, \ldots, c_0)$.

2.2. Kostant section

Kostant constructs a section $\epsilon : \mathfrak{c} \to \mathfrak{g}$ of $\chi : \mathfrak{g} \to \mathfrak{c}$ whose image lies in the set $\mathfrak{g}^{reg}$ of regular elements of $\mathfrak{g}$. In simplified terms, this constructs a matrix with a given characteristic polynomial.

EXAMPLE 9. — *When $\mathfrak{g} = \mathfrak{sl}(2)$, the Lie algebra consists of matrices of trace zero, and the characteristic polynomial has the form $t^2 + c$. The determinant c generates $k[\mathfrak{g}]^G$. The Kostant section maps c to*

$$\begin{pmatrix} 0 & -c \\ 1 & 0 \end{pmatrix}. \tag{10}$$

EXAMPLE 11. — *If $\mathfrak{g} = \mathfrak{gl}(n)$, we can construct the companion matrix of a given characteristic polynomial $p \in k[t]$, by taking the endomorphism t of $R = k[t]/(p)$, expressed as a matrix with respect to the standard basis $1, t, t^2, \ldots, t^{n-1}$ of R. The companion matrix is a section $\mathfrak{c} \to \mathfrak{g}$ that is somewhat different from the Kostant section. Nevertheless, the Kostant section can be viewed as a generalization of this that works uniformly for all Lie algebras $\mathfrak{g}$.*

2.3. Centralizers

Each element $\gamma \in \mathfrak{g}$ has a centralizer I_γ in G. If two elements of $\mathfrak{g}^{reg}$ have the same image a in $\mathfrak{c}$, then their centralizers are canonically isomorphic. By descent, there is a regular centralizer J_a, for all a in $\mathfrak{c}$, that is canonically isomorphic to I_γ for every regular element γ such that $\chi(\gamma) = a$.

EXAMPLE 12. — *Suppose $G = SL(2)$. We may identify J_a with the centralizer of (10) to obtain the group of matrices with determinant 1 of the form*

$$\begin{pmatrix} x & -yc \\ y & x \end{pmatrix}.$$

EXAMPLE 13. — *If $\mathfrak{g} = \mathfrak{gl}(n)$, then the centralizer of the companion matrix with characteristic polynomial p can be identified with the centralizer of t in $GL(R)$, where $R = k[t]/(p)$. An element of $\mathfrak{gl}(R)$ centralizes the regular element t if and only if it is a polynomial in t. Thus, the centralizer in $\mathfrak{gl}(R)$ is R and the centralizer in $GL(R)$ is $J_a = R^\times$.*

2.4. Discriminant and resultant

Let Φ be the root system of a split group G. The differentials $d\alpha$ of roots define a polynomial called the *discriminant*:

$$\prod_{\alpha \in \Phi} d\alpha \tag{14}$$

on $\mathfrak{t}$. The polynomial is invariant under the action of the Weyl group W and equals a function on $\mathfrak{c}$. The divisor $\mathfrak{D}_G$ of this polynomial $\mathfrak{c}$ is called the *discriminant divisor.*

EXAMPLE 15. — *Let $G = GL(n)$. The Lie algebra $\mathfrak{t}$ can be identified with the diagonal matrix algebra with coordinates $t_1, \ldots, t_n$ along the diagonal. The discriminant is*

$$\prod_{i \neq j} (t_i - t_j).$$

This is invariant under the action of the symmetric group on n letters and can be expressed as a polynomial in the coefficients c_i of the characteristic polynomial. In

particular, the discriminant of the characteristic polynomial $t^2 + bt + c$ is the usual discriminant $b^2 - 4c$.

If H is a split endoscopic group of G, there is a morphism

$$\nu : \mathfrak{c}_H \to \mathfrak{c} \tag{16}$$

that comes from an isomorphism of Cartan subalgebras $\mathfrak{t}_H \to \mathfrak{t}$ and an inclusion of Weyl groups $W_H \subset W$: $\mathfrak{c}_H = \mathfrak{t}/W_H \to \mathfrak{t}/W = \mathfrak{c}$. There exists a *resultant divisor* $\mathfrak{R}$ such that

$$\nu^* \mathfrak{D}_G = \mathfrak{D}_H + 2\,\mathfrak{R}. \tag{17}$$

EXAMPLE 18. — *Let $H = GL(2) \times GL(2)$, embedded as a block diagonal subgroup of $GL(4)$. Identify roots of H with roots of G under this embedding. The morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$, viewed in terms of characteristic polynomials, maps the pair (p_1, p_2) of quadratic polynomials to the quartic $p_1 p_2$. Let t_i^1, t_i^2 be the roots of p_i, for $i = 1, 2$. The resultant is*

$$\prod_{j \neq k} (t_1^j - t_2^k).$$

The resultant is symmetric in the roots of p_1 and in the roots of p_2 and thus can be expressed as a polynomial in the coefficients of p_1 and p_2. It vanishes exactly when p_1 and p_2 have a common root.

3. THE STATEMENT OF THE FL

Let G be a reductive group scheme over the ring of integers $\mathcal{O}_v$ of a nonarchimedean local field F_v in positive characteristic. Let q be the cardinality of the residue field k. The map $\chi : \mathfrak{g} \to \mathfrak{c}$ is compatible with stable conjugacy in the sense that two regular semisimple elements in $\mathfrak{g}(F_v)$ are stably conjugate exactly when they have the same image in $\mathfrak{c}(F_v)$. The results of Section 1.3 (transported to the Lie algebra) show that each element γ stably conjugate to $\epsilon(a)$ carries a cohomological invariant in $H^1(F_v, J_a)$, which is trivial for elements conjugate to $\epsilon(a)$.

For each regular semisimple element $a \in \mathfrak{c}(F_v)$ and character

$$\kappa : H^1(F_v, J_a) \to \mathbb{C}^\times,$$

we write $\langle \kappa, \gamma \rangle$ for the pairing of κ with the cohomological invariant of γ. A κ-orbital integral is defined to be

$$\mathbf{O}_\kappa(a) = \sum_{\{\gamma : \chi\gamma = a\}/\sim} \int_{I_\gamma \backslash G(F_v)} \langle \kappa, \gamma \rangle 1_{\mathfrak{g}(\mathcal{O}_v)}(\mathrm{Ad}\, g^{-1}(\gamma)) dg, \tag{19}$$

where I_γ centralizes γ, and the sum runs over representatives of the conjugacy classes in $\mathfrak{g}(F_v)$ with image a. Here $1_{\mathfrak{g}(\mathcal{O}_v)}$ is the characteristic function of $\mathfrak{g}(\mathcal{O}_v)$. A Haar measure on G has been fixed that gives $G(\mathcal{O}_v)$ volume 1.

The character κ determines a reductive group scheme H over $\mathcal{O}_v$, according to the construction of (1.4). In general, we add a subscript H to indicate quantities constructed for H, analogous to those already constructed for G. In particular, let $\mathfrak{c}_H$ be the Chevalley quotient of the Lie algebra of H. There is a morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$. When κ is trivial, we write $\mathbf{SO}$ for $\mathbf{O}_\kappa$.

Here is the main theorem of Ngô [**46**].

THEOREM 20 (Fundamental lemma (FL)). — *Assume that the characteristic of F_v is greater than twice the Coxeter number of G. For all regular semisimple elements $a \in \mathfrak{c}_H(\mathcal{O}_v)$ whose image $\nu(a)$ in $\mathfrak{c}$ is also regular semisimple, the κ orbital integral of $\nu(a)$ in G is equal to the stable orbital integral of a in H, up to a power of q:*

$$\mathbf{O}_\kappa(\nu(a)) = q^{r_v(a)}\mathbf{SO}_H(a), \quad \textit{where} \quad r_v(a) = \deg_v(a^*\mathfrak{R}). \tag{21}$$

A sketch of Ngô's proof of the FL appears in Section 6.5.

The FL has been obtained from the character identity (2) for SL_2:

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}},$$

by multiple levels of generalization. A general reductive group G replaces SL_2 and a nonarchimedean local field replaces $\mathbb{R}$. Orbital integrals are used rather than characters, roots of unity $\langle \kappa, \gamma \rangle$ rather than signs ± 1, an endoscopic group H rather than the rotation group, and a transfer factor $q^{r_v(a)}$ rather than a denominator $e^{i\theta} - e^{-i\theta}$.

Over the years from the time that Langlands first conjectured the FL until the time that Ngô gave its proof, the FL has been transformed into simpler form [**36**]. The statement of the FL appears here in its simple form. Section 8 makes a series of comments about the original form of the FL and its reduction to this simple form. Except for that section, our discussion is based on this simple form of the FL. In particular, we assume that the field F_v has positive characteristic and that the conjugacy classes live in the Lie algebra rather than the group.

Analogous identities (transfer of Schwartz functions) on real reductive groups have been established by Shelstad [**49**]. Her work gives a precise form to the idea that the oscillations of a character κ cause certain roots to cancel away and others to become more pronounced: normalized κ-orbital integrals extend smoothly across the singular hyperplanes of some purely imaginary roots α, but jump across others. At a philosophical distance, Ngô's use of perverse sheaves can be viewed as p-adic substitute for differential operators, introduced by Harish-Chandra to study invariant distributions near a singular element in the group and adopted by Shelstad as a primary tool.

4. AFFINE SPRINGER FIBERS

4.1. Spectral curves

Calculations in special cases show why the FL is essentially geometric in nature, rather than purely analytic or combinatorial. We recall a favorite old calculation of mine of the orbital integrals for $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$, the rank two odd orthogonal and symplectic Lie algebras [**21**]. Let F_v be a nonarchimedean local field of residual characteristic greater than 2. Let k be the residue field with q elements. Choose $a \in \mathfrak{c}(F_v)$ and let $0, \pm t_1, \pm t_2$ be the eigenvalues of the Kostant section $\gamma = \epsilon(a)$ in $\mathfrak{so}(5) \subset \mathfrak{gl}(5)$. Assume that there is an odd natural number r such

$$|\alpha(\gamma)| = q^{-r/2},$$

for every root α of $\mathfrak{so}(5)$. We use the eigenvalues to construct an elliptic curve E_a over k, given by $y^2 = (1-x^2\tau_1)(1-x^2\tau_2)$, where τ_i is the image of t_i^2/ϖ^r in the residue field, for a uniformizer ϖ. By direct calculation we find that the stable orbital integral $\mathbf{SO}(a, f)$ of a test function f equals the number of points on the elliptic curve:

$$A(q) + B(q)\,\mathrm{card}(E_a(k)), \tag{22}$$

up to some rational functions A and B, depending on f.

Similarly, in the group $\mathfrak{sp}(4) \subset \mathfrak{gl}(4)$, there is an element a' with related eigenvalues $\pm t_1, \pm t_2$. According to the general framework of (twisted) endoscopy, there should be a corresponding function f' on $\mathfrak{sp}(4)$ such that the stable orbital integral $\mathbf{SO}(a', f')$ in $\mathfrak{sp}(4)$ is equal to (22). A calculation of the orbital integral of f' gives a similar formula, with a different elliptic curve $E'_{a'}$, but otherwise identical to (22). The elliptic curves E_a and $E'_{a'}$ have different j-invariants (which vary with a and a'). The proof of the desired identities of orbital integrals in this case is obtained by producing an isogeny between E_a and $E'_{a'}$. (The identities of orbital integrals are quite nontrivial, even though the Lie algebras $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$ are abstractly isomorphic.)

In a similar way, hyperelliptic curves appear in calculations of certain orbital integrals in groups of higher rank. When orbital integrals are computed by brute force, these curves appear as freaks of nature. As it turns out, they are not freaks at all, merely perverse. One of the major challenges of the proof of the FL and one of the major triumphs of Ngô has been to find the natural geometrical setting that combines orbital integrals and spectral curves.

4.2. Orbital integrals as affine Springer fibers

An orbital integral can be computed by solving a coset counting problem. The value of the integrand (19) is unchanged if g is replaced with any element of the coset $g\,G(\mathcal{O}_v)$. The integral is thus expressed as a discrete sum over cosets of $G(\mathcal{O}_v)$

in G modulo the group action by I_γ. Each coset $gG(\mathcal{O}_v)$ contributes a root of unity $\langle \kappa, \gamma \rangle$ or 0 to the value of the integral depending on whether $\operatorname{Ad} g^{-1}\gamma \in \mathfrak{g}(\mathcal{O}_v)$ (again modulo symmetries I_γ). This interpretation as a coset counting problem makes the FL appear to be a matter of simple combinatorics. However, purely combinatorial attempts to prove the FL have failed (for good reason).

Let $\mathcal{M}_v(a, \bar{k})$ be the set of cosets that fulfill the support condition (19) of the integral over $\bar{k}$:

$$\mathcal{M}_v(a, \bar{k}) = \{g \in G(\bar{F}_v)/G(\bar{\mathcal{O}}_v) \mid \operatorname{Ad} g^{-1}\gamma_0 \in \mathfrak{g}(\bar{\mathcal{O}}_v)\}, \quad \gamma_0 = \epsilon(a).$$

Kazhdan and Lusztig showed that the coset space $G(\bar{F}_v)/G(\bar{\mathcal{O}}_v)$ is the set of $\bar{k}$-points of an inductive limit of schemes called the *affine Grassmannian*. Moreover, $\mathcal{M}_v(a, \bar{k})$ itself is the set of points of an ind-scheme $\mathcal{M}_v(a)$, called the *affine Springer fiber* [**28**].

Each irreducible component of $\mathcal{M}_v(a)$ has the same dimension. This dimension, $\delta_v(a)$, is given by a formula of Bezrukavnikov [**8**]. From that formula, it follows that the dimension of the affine Springer fiber of $\nu(a)$ in G exceeds the dimension of the affine Springer fiber of a in H by precisely $r_v(a)$. The factor $q^{r_v(a)}$ that appears in the FL is forced to be what it is because of this simple dimensional analysis.

Goresky, Kottwitz, and MacPherson made an extensive investigation of affine Springer fibers and conjectured that their cohomology groups are pure. Assuming this conjecture, they prove the FL for elements whose centralizer is an unramified Cartan subgroup [**19**]. They prove the purity result in particular cases by constructing pavings of the affine Springer fibers [**20**].

Laumon has made a systematic investigation of the affine Springer fibers for unitary groups. Ngô joined the effort, and together they succeeded in giving a complete proof of the FL for unitary groups [**41**].

Ngô encountered two major obstacles in trying to generalize this earlier work to an arbitrary reductive group. These approaches calculate the equivariant cohomology by passing to a fixed point set in $\mathcal{M}_v(a)$ under a torus action. (In the case of unitary groups, over a quadratic extension each endoscopic group becomes isomorphic to a Levi subgroup of $GL(n)$. The torus action comes from the center of this Levi.) However, in general, a nontrivial torus action on the affine Springer fiber simply does not exist.

The second serious obstacle comes from the purity conjecture itself. In accordance with Deligne's work, Ngô believed that the task of proving purity results should become easier when the affine Springer fibers are combined into families rather than treated in isolation. With this in mind, he started to investigate families varying over a base curve X. This moves us from local geometry of a p-adic field F_v to the global geometry of the function field of X. He found that the Hitchin fibration is the global

analogue of affine Springer fibers. The Hitchin fibers will be described in the next section. Deligne's purity theorem applies in this setting [**18**].

5. HITCHIN FIBRATION

The Hitchin fibration was introduced in 1987 in the context of completely integrable systems [**27**]. Roughly, the Hitchin fibration is the stack obtained when the characteristic map $\mathfrak{g} \to \mathfrak{c}$ varies over a curve X. Ngô carries out all geometry in the language of stacks without compromise, as developed in [**40**]. For this reason, *groupoids* (a category in which every morphism is invertible) appear with increasing frequency throughout this report.

Fix a smooth projective curve X of genus g over a finite field k. We now shift perspective and notation, allowing the constructions in Lie theory from previous sections to vary over the base curve X. In particular, we now let G be a quasi-split reductive group over X that is locally trivial in the etale topology on X. Let $\mathfrak{g}$ be its Lie algebra G and $\mathfrak{c}$ the space of characteristic polynomials, both now schemes over X.

Let D be a line bundle on X. For technical reasons (stemming from the 2 in the structure constants of $\mathfrak{sl}_2$), we assume that D is the square of another line bundle. At one point in Ngô's proof of the FL, it is necessary to allow the degree of D to become arbitrarily large (6.5). We place a subscript D to indicate the tensor product with D: $\mathfrak{g}_D = \mathfrak{g} \otimes_{\mathcal{O}_X} D$, etc.

We let $\mathcal{A}$ be the space of global sections on X with values in $\mathfrak{c}_D = \mathfrak{c} \otimes_{\mathcal{O}_X} D$. The group G acts on $\mathfrak{g}$ by the adjoint action. Twisting $\mathfrak{g}$ by any G-torsor E gives a vector bundle $\mathrm{Ad}(E)$ over X.

DEFINITION 23. — *The Hitchin fibration $\mathcal{M}$ is the stack given as follows. For any k-scheme S, $\mathcal{M}(S) = [\mathfrak{g}_D/G](X \times S)$ is the groupoid whose objects are pairs (E, ϕ), where E is a G-torsor over $X \times S$ and ϕ is a section of* $\mathrm{Ad}(E)_D$.

There exists a morphism $f : \mathcal{M} \to \mathcal{A}$, obtained as a "stacky" enhancement of the characteristic map $\chi : \mathfrak{g} \to \mathfrak{c}$ over X. In greater detail, $\chi : \mathfrak{g} \to \mathfrak{c}$ gives successively

$$[\mathfrak{g}_D/G] \to \mathfrak{c}_D, \qquad [\mathfrak{g}_D/G](X \times S) \to \mathfrak{c}_D(X \times S), \qquad \mathcal{M}(S) \to \mathcal{A}(S), \qquad f : \mathcal{M} \to \mathcal{A}.$$

In words, the characteristic polynomial of ϕ is a section of $X \times S$ with values in $\mathfrak{c}_D$; that is, an element of $\mathcal{A}(S)$. We write $\mathcal{M}_a$ for the fiber of $\mathcal{M}$ over $a \in \mathcal{A}$. This is the *Hitchin fiber.*

The centralizers J_a, as we vary $a \in \mathfrak{c}$, define a smooth group scheme J over $\mathfrak{c}$. Now select on $\mathcal{A}$ an S-point: $a : S \to \mathcal{A}$. There is a groupoid $\mathcal{P}_a(S)$ whose objects are J_a-torsors on $X \times S$. Moreover, $\mathcal{P}_a(S)$ acts on $\mathcal{M}_a(S)$ by twisting a pair (E, ϕ) by a

J_a-torsor. As the S-point a varies, we obtain a *Picard stack* $\mathcal{P}$ acting fiberwise on the Hitchin fibration $\mathcal{M}$.

EXAMPLE 24. — *We give an extended example with $G = GL(V)$, the general linear group of a vector space V. In its simplest form, a pair (E, ϕ) is what we obtain when we allow an element γ of the Lie algebra* $\mathrm{end}(V)$ *to vary continuously along the curve X. As we vary along the curve, the vector space V sweeps out a vector bundle E on X, and the element $\gamma \in \mathrm{end}(V)$ sweeps out a section ϕ of the bundle* $\mathrm{end}(E)_D$.

For each pair (E, ϕ), we evaluate the characteristic map $v \mapsto \chi(\phi_v)$ of the endomorphism ϕ at each point $v \in X$. This function belongs to the set $\mathcal{A}$ of a global section of the bundle $\mathfrak{c}_D$ over X. This is the morphism $f : \mathcal{M} \to \mathcal{A}$.

Abelian varieties occur naturally in the Hitchin fibration. For each section $a = (c_{n-1}, \ldots, c_0) \in \mathcal{A}$, the characteristic polynomial

$$t^n + c_{n-1}(v)t^{n-1} + \cdots + c_0(v) = 0, \quad v \in X, \tag{25}$$

defines an n-fold cover Y_a of X (called the spectral curve). By construction, each point of the spectral curve is a root of the characteristic polynomial at some $v \in X$. We consider the simple setting when Y_a is smooth and the discriminant of the characteristic polynomial is sufficiently generic. A pair (E, ϕ) over the section $a \in \mathcal{A}$ determines a line (a one-dimensional eigenspace of ϕ with eigenvalue that root) at each point of the spectral curve, and hence a line bundle on Y_a. This establishes a map from points of the Hitchin fiber over a to $\mathrm{Pic}(Y_a)$, *the group of line bundles on the spectral curve Y_a. Conversely, just as linear maps can be constructed from eigenvalues and eigenspaces, Hitchin pairs can be constructed from line bundles on the spectral curve Y_a. The identity component* $\mathrm{Pic}^0(Y_a)$ *is an abelian variety.*

5.1. Proof strategies

At this point in the development, it would be most appropriate to insert a book-length discussion of the geometry of the Hitchin fibration, with a full development and many examples. As Langlands speculates in his review of Ngô's proof, "an exposition genuinely accessible not alone to someone of my generation, but to mathematicians of all ages eager to contribute to the arithmetic theory of automorphic representations, would be, perhaps, ... close to 700 pages" [**37**].

To cut 700 pages short, what are the essential ideas?

First, as mentioned above, the Hitchin fibration is the correct global analogue of the (local) affine Springer fiber. This analogy can be made precise; an orbital integral over a local field is computed by counting points on an affine Springer fiber, but an orbital integral over the ring of adeles is computed by counting points on a fiber of the Hitchin fibration. Moreover, the description of the affine Springer fiber as a functor (an S-point is a G-torsor on $X_v \hat{\times} S$ plus a bit more, where X_v is a formal disk)

imitates the description of the Hitchin fibration as a stack (an S-point is G-torsor on $X \times S$ plus a similar bit more). The relationship between the Hitchin fiber $\mathcal{M}_a$ and the affine Springer fiber $\mathcal{M}_v(a)$ can be expressed as a factorization of categories (43): $\mathcal{M}_a$ modulo symmetries as a product of $\mathcal{M}_v(a)$ modulo their symmetries as v runs over closed points of X. Through this relationship, the Hitchin fibration can be used to study orbital integrals and the FL.

Second, the Hitchin fibration should be understood insofar as possible through its Picard symmetries $\mathcal{P}$. The obvious reason for this is that it is generally a good idea to study symmetry groups. The deeper reason for this has to do with endoscopy. The objects of the Picard stack are torsors of the centralizer J_a. Although the relationship between G and H is mediated through dual groups, the relationship between centralizers is direct: over $\mathfrak{c}_H$, there is a canonical homomorphism from the regular centralizer J of G to the regular centralizer J_H of H:

$$\nu^* J \to J_H. \tag{26}$$

Thus, their respective Picard stacks are also directly related and information passes fluently between them. We should try to prove the FL largely at the level of Picard stacks.

Third, by working directly with the Hitchin fibration, the difficult purity conjecture of Kottwitz, Goresky, and MacPherson can be bypassed. Finally, continuity arguments may be used, as explained in (5.4).

5.2. Perverse cohomology sheaves

We give a brief summary without proofs of some of the main results proved by Ngô about the perverse cohomology sheaves of the Hitchin fibration.

There is an etale open subset $\tilde{\mathcal{A}}$ of $\mathcal{A} \otimes_k \bar{k}$ that has the technical advantage of killing unwanted monodromy. The tilde will be used consistently to mark quantities over $\tilde{\mathcal{A}}$. For example, if we write $f^{\mathrm{ani}} : \mathcal{M}^{\mathrm{ani}} \to \mathcal{A}^{\mathrm{ani}}$ for the Hitchin fibration, restricted to the open set of anisotropic elements of $\mathcal{A}$, then $\tilde{f}^{\mathrm{ani}} : \tilde{\mathcal{M}}^{\mathrm{ani}} \to \tilde{\mathcal{A}}^{\mathrm{ani}}$ is the corresponding Hitchin fibration over the anisotropic part of $\tilde{\mathcal{A}}$.

The conditions of Deligne's purity theorem are satisfied [**18**], so that $\tilde{f}_*^{\mathrm{ani}} \bar{\mathbb{Q}}_\ell$ is isomorphic to a direct sum of perverse cohomology sheaves:

$$^p H^n(\tilde{f}_*^{\mathrm{ani}} \bar{\mathbb{Q}}_\ell)[-n].$$

The action of $\tilde{\mathcal{P}}^{\mathrm{ani}}$ on $\tilde{\mathcal{M}}^{\mathrm{ani}}$ gives an action on the perverse cohomology sheaves, which factors through the sheaf of components $\pi_0 = \pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})$. The sheaf π_0 is an explicit quotient of the constant sheaf X_* of cocharacters, and hence X_* acts on the

perverse cohomology sheaves through π_0. As a result, the perverse cohomology sheaves break into a direct sum of κ-isotypic pieces

$$L_\kappa = {}^pH^n(\tilde{f}_*^{\mathrm{ani}}\bar{\mathbb{Q}}_\ell)_\kappa, \tag{27}$$

as κ runs over elements in the dual torus $\hat{T}$. (By duality, the cocharacter group X_* is the group of characters of the dual torus, which gives the pairing between $\hat{T}$ and X_*.)

We use the same curve X and same line bundle D both for G and for its endoscopic groups H. The morphism $\nu : \mathfrak{c}_H \to \mathfrak{c}$ from (16) extends to give $\nu : \mathfrak{c}_{H,D} \to \mathfrak{c}_D$ and then by taking sections of these bundles, we obtain a morphism between their spaces of global sections:

$$\nu : \mathcal{A}_H \to \mathcal{A}. \tag{28}$$

We hope that no confusion arises by using the same symbol ν for all of these morphisms.

For each $\kappa \in \hat{T}$, there is a closed subspace $\tilde{\mathcal{A}}_\kappa$ of $\tilde{\mathcal{A}}$ consisting of elements a whose "geometric monodromy" lies in the centralizer of κ in the dual group ${}^LG$. Each subspace $\tilde{\mathcal{A}}_\kappa$ is in fact the disjoint union of the images of closed immersions

$$\nu : \tilde{\mathcal{A}}_H \to \tilde{\mathcal{A}} \tag{29}$$

coming from endoscopic groups H with endoscopic data $(\kappa, \dots)$. The support of L_κ lies in $\tilde{\mathcal{A}}_\kappa^{\mathrm{ani}}$. The geometric content of the FL is to be found in the comparison of $\nu^* L_\kappa$ with

$$L_{H,st} = {}^pH^{n+2r}(\tilde{f}_{H,*}^{\mathrm{ani}}\bar{\mathbb{Q}}_\ell)_{st}(-r), \text{ where } r = \dim(\mathcal{A}) - \dim(\mathcal{A}_H). \tag{30}$$

The subscript st indicates the isotypic piece with trivial character $\kappa = 1$.

The anisotropic locus $\tilde{\mathcal{A}}^{\mathrm{ani}}$ admits a stratification by a numerical invariant $\delta : \tilde{\mathcal{A}} \to \mathbb{N}$:

$$\tilde{\mathcal{A}}^{\mathrm{ani}} = \coprod_{\delta \in \mathbb{N}} \tilde{\mathcal{A}}_\delta^{\mathrm{ani}}.$$

There is an open set $\tilde{\mathcal{A}}^{\mathrm{good}}$ of $\tilde{\mathcal{A}}^{\mathrm{ani}}$, given as a union of some strata $\tilde{\mathcal{A}}_\delta^{\mathrm{ani}}$ that satisfy:

$$\operatorname{codim}(\tilde{\mathcal{A}}_\delta^{\mathrm{ani}}) \geq \delta. \tag{31}$$

5.3. Support theorem

The proof of the following theorem about the support of the perverse cohomology sheaves of the Hitchin fibration constitutes the deepest part of the proof of the FL.

THEOREM 32 (Support theorem). — *Let Z be the support of a geometrically simple factor of L_κ. If Z meets $\nu(\tilde{\mathcal{A}}_H^{good})$ for some endoscopic group H with data $(\kappa, \dots)$, then $Z = \nu(\tilde{\mathcal{A}}_H^{ani})$. In fact, there is a unique such H.*

A major chapter of the book-length proof of the FL is devoted to the proof of the support theorem. The strategy of the proof is to show that every support Z also appears as the support of some factor in the ordinary cohomology of highest degree of the Hitchin fibration. To move cohomology classes from one degree to another, Ngô uses Poincaré duality and Pontryagin product operations on cohomology coming from the action of the connected component of the identity $\tilde{\mathcal{P}}^{0,\mathrm{ani}}$ on $\tilde{\mathcal{M}}^{\mathrm{ani}}$. This action factors through the action of an abelian variety, a quotient of the Picard stack $\tilde{\mathcal{P}}^{0,\mathrm{ani}}$. To show that the support Z can be pushed all the way to the top degree cohomology, it is enough to show that the dimension of this abelian variety is sufficiently large and that the cohomology of the abelian variety acts freely on the cohomology of the Hitchin fiber. The required estimate on the dimension of the abelian variety comes from the inequality (31). Freeness relies on a polarization of the abelian variety.

Once the support Z is known to appear as a support in the top degree, he shows that the action of $\tilde{\mathcal{P}}^{\mathrm{ani}}$ on the Hitchin fibration leads to an explicit description of the top degree ordinary cohomology as the sheaf associated with the presheaf

$$U \mapsto \bar{\mathbb{Q}}_\ell^{\pi_0(\tilde{\mathcal{P}}^{\mathrm{ani}})(U)}.$$

The supports of π_0 can be described explicitly in terms of data in the dual group, in the style of the duality theorems of Kottwitz, Tate, and Nakayama. By checking that the conclusion of the support theorem holds for the particular sheaf π_0, the general support theorem follows.

We apply the support theorem with H as the primary reductive group and κ as the trivial character. In this context, the only endoscopic group of H with stable data is H itself. Moreover, ν is the identity map on $\tilde{\mathcal{A}}_H$. The support theorem takes the following form in this case.

COROLLARY 33. — *Let Z be the support of a geometrically simple factor of $L_{H,st}$. If Z meets $\tilde{\mathcal{A}}_H^{good}$, then $Z = \tilde{\mathcal{A}}_H^{ani}$.*

5.4. Continuity and the decomposition theorem

The strategy that lies at the heart of the proof of the FL is a continuity argument: arbitrarily complicated identities of orbital integrals can be obtained as limits of relatively simple identities.

The complexity of an orbital integral is measured by the dimension of its affine Springer fiber. Growing linearly with $\deg_v(a^*\mathfrak{D})$, this dimension is unbounded as a function of a. Fortunately, globally, we can view an element a for which this degree at v is large as a limit of elements a' with small degrees: $\deg_w(a'^*\mathfrak{D}) \leq 1$ for all $w \in X$. This follows the principle that a polynomial with repeated roots is a limit of polynomials with simple roots. When the degrees are at most 1, the affine Springer fibers have manageable complexity.

The Beilinson-Bernstein-Deligne-Gabber decomposition theorem for perverse sheaves provides the infrastructure for the continuity arguments [7]. Let S be a scheme of finite type over $\bar{k}$. The support Z of a simple perverse sheaf on S is a closed irreducible subscheme of S. There is a smooth open subscheme U of Z and a local system $\mathcal{L}$ on U such that the simple perverse sheaf can be reconstructed as the middle extension of the local system on U:

$$i_* j_{!*} \mathcal{L}[\dim Z], \quad i : Z \to S, \quad j : U \to Z.$$

We express this as a continuity principle: if two simple perverse sheaves with the same support Z are equal to the same local system on a dense open U, then they are in fact equal on all of S.

More generally, for any irreducible scheme Z of finite type over k, in order to show that two pure complexes on Z are equal in the Grothendieck group, it is enough to check two conditions:

1. Every geometrically simple perverse sheaf in either complex has support all of Z.
2. Equality holds in the Grothendieck group on some dense open subset U of Z.

The purpose of the support theorem (32) and its corollary is to give the first condition for the two pure complexes $\nu^* L_\kappa$ and $L_{H,st}$. The idea is that second condition should be a consequence of identities of orbital integrals of manageable complexity, which can be proved by direct calculation. The resulting identity of pure complexes on all of Z should then imply identities of orbital integrals of arbitrarily complexity. This is Ngô's strategy to prove the FL.

6. MASS FORMULAS

6.1. Groupoid cardinality (or mass)

Let $\mathcal{C}$ be a groupoid that has finitely many objects up to isomorphism and in which every object has a finite automorphism group. Define the *mass* (or *groupoid cardinality*) of $\mathcal{C}$ to be the rational number

$$\mu(\mathcal{C}) = \sum_{x \in \mathrm{obj}(\mathcal{C})/\mathrm{iso}} \frac{1}{\mathrm{card}(\mathrm{Aut}(x))}.$$

EXAMPLE 34. — *Let $\mathcal{C}$ be the category whose objects are the elements of a given finite group G and arrows are given by $x \mapsto g^{-1}xg$, for $g \in G$. Then the set of objects up to isomorphism is in bijection with the set of conjugacy classes, the automorphism group of x is the centralizer of x, and the mass is*

$$\mu(\mathcal{C}) = \sum_{x/iso} \frac{1}{\mathrm{card}(\mathrm{Aut}(x))} = \sum_{x/iso} \frac{\mathrm{card}(\mathrm{orbit}(x))}{\mathrm{card}\, G} = 1.$$

EXAMPLE 35. — *Let P be a group that acts simply transitively on a set M. Let $\mathcal{C}$ be the category whose set of objects is M, and let the set of morphisms be given by the group action of P on M. There is one object up to isomorphism and its automorphism group is trivial. The mass of $\mathcal{C}$ is* 1.

EXAMPLE 36. — *The following less trivial example appears in Ngô. Let P be the group $\mathbb{G}_m \times \mathbb{Z}$ defined over a finite field k of cardinality q. Let $M = (\mathbb{P}^1 \times \mathbb{Z})/\sim$, where the equivalence relation $(\sim)$ identifies the point (∞, j) with $(0, j+1)$ for all j. Thus, M is an infinite string of projective lines, with the point at infinity of each line joined to the zero point of the next line. The group P acts on M by $(p_0, i)\cdot(m_0, j) = (p_0 m_0, i+j)$, where $p_0 m_0$ is given by the standard action of $\mathbb{G}_m$ on $\mathbb{P}^1$, fixing 0 and ∞. Let σ be the Frobenius automorphism of $\bar{k}/k$, and define a twisted automorphism of $P(\bar{k})$ and $M(\bar{k})$ by $\sigma(x_0, i) = (\sigma x_0^{-1}, -i)$. Define a category $\mathcal{C}$ with objects given by pairs*

$$(m,p) \in M(\bar{k}) \times P(\bar{k}) \text{ such that } \sigma(m) = pm. \tag{37}$$

Define arrows by $h \in P(\bar{k})$, where

$$h(m,p) = (m', p'), \text{ provided } hm = m' \text{ and } hp = p'\sigma(h). \tag{38}$$

Then it can be checked by a direct calculation that there are two isomorphism classes of objects in this category, represented by the objects

$$((0,1),(1,1)) \text{ and } ((1,0),(1,0)) \in M(\bar{k}) \times P(\bar{k}) = (\mathbb{P}^1(\bar{k}) \times \mathbb{Z}) \times (\mathbb{G}_m(\bar{k}) \times \mathbb{Z}).$$

The group $P(\bar{k})^\sigma$ of order $q+1$ acts as automorphisms of the first object, and the group of automorphisms of the second object is trivial. The mass of this category is therefore

$$\mu(\mathcal{C}) = \frac{1}{q+1} + 1.$$

More generally, suppose there exists a function $\mathrm{Obj}(\mathcal{C}) \to A$ from the objects of a groupoid into a finite abelian group A and that the image in A of each object depends only on its isomorphism class. Then for every character κ of A, we can define a *κ-mass*:

$$\mu_\kappa(\mathcal{C}) = \sum_{x \in \mathrm{obj}(\mathcal{C})/\mathrm{iso}} \frac{\langle \kappa, x \rangle}{\mathrm{card}(\mathrm{Aut}(x))}.$$

EXAMPLE 39. — *In Example 36, if (m,p) is an object and $p = (p_0, j) \in \mathbb{G}_m \times \mathbb{Z}$, then the image of j in $A = \mathbb{Z}/2\mathbb{Z}$ depends only on the isomorphism class of the object (m,p). If κ is the nontrivial character of A, then the κ-mass of this groupoid is*

$$\mu_\kappa(\mathcal{C}) = -\frac{1}{q+1} + 1.$$

6.2. Mass formula for orbital integrals

Let $\mathcal{M}_v(a)$ be the affine Springer fiber for the element a and let J_a be its centralizer. We write $\mathcal{P}_v(J_a)$ for the group of symmetries of the affine Springer fiber. Let $\mathcal{C}$ be the groupoid of k-points of the quotient stack $[\mathcal{M}_v(a)/\mathcal{P}_v(J_a)]$ with objects (m,p) and morphisms and h defined by the earlier formulas (37) and (38) (substituting $\mathcal{M}_v(a)$ for the space M and $\mathcal{P}_v(J_a)$ for the symmetries P).

For each character of $H^1(k,\mathcal{P}_v(J_a))$ we can naturally define a character κ of $H^1(F_v,J_a)$ as well as a character (also called κ) on a finite abelian group A as above.

The description of orbital integrals in terms of affine Springer fibers takes the following form. It is a variant of the coset arguments of (4.2).

THEOREM 40. — *For each regular semisimple element $a \in \mathfrak{c}(\mathcal{O}_v)$, the κ-mass of the category $\mathcal{C}$ is equal to the κ-orbital integral of a:*

$$\mu_\kappa(\mathcal{C}) = c\ \mathbf{O}_\kappa(a),$$

up to a constant $c = \mathrm{vol}(J_a^0(\mathcal{O}_v), dt_v)$ used to normalize measures.

6.3. Product formula for masses

Recall from (28) that there is a morphism $\nu : \mathcal{A}_H \to \mathcal{A}$. We choose a commutative group scheme J'_a for which there are homomorphisms

$$J'_a \to J_{\nu(a)} \to J_{H,a} \tag{41}$$

extending the homomorphism (26) and that become isomorphisms over a nonempty open set U of X. The group scheme J'_a can be chosen to satisfy other simplifying assumptions that we will not list here. The homomorphisms (41) functorially determine an action of $\mathcal{P}(J'_a)$ on both Hitchin fibrations $\mathcal{M}_{\nu(a)}$ and $\mathcal{M}_{H,a}$. Changing notation slightly, we will assume that henceforth all masses for both G and H are computed with respect to the same Picard stack $\mathcal{P}(J'_a)$ in global calculations and with respect to $\mathcal{P}_v(J'_a)$ in local calculations. This simplifies the comparisons of masses that follow.

For each element $a \in \mathcal{A}_H^{\mathrm{ani}}(k)$, we have a mass $\mu_H(a)$ of the groupoid of k-points of the Hitchin fiber $\mathcal{M}_{H,a}$ modulo symmetry on H. Its image $\nu(a) \in \mathcal{A}^{\mathrm{ani}}$ has a κ-mass of the groupoid of k-points of the Hitchin fiber $\mathcal{M}_a$ modulo symmetry.

For each regular semisimple element $a \in \mathfrak{c}_H(\mathcal{O}_v)$, we have a mass of the affine Springer fiber modulo symmetry on H. We write $\mu_{H,v}(a)$ for this mass. Moreover, if the image $\nu(a)$ under the map $\nu : \mathfrak{c}_H \to \mathfrak{c}$ is also regular semisimple, there is a κ-mass $\mu_{\kappa,v}(\nu(a))$ of the affine Springer fiber modulo symmetry of $\nu(a)$ in G.

$\mathcal{A}_H$ is the set of global sections of $\mathfrak{c}_{H,D}$ over X. For each $v \in X$, we can fix a local trivialization of $\mathfrak{c}_{H,D}$ at v and evaluate a section $a \in \mathcal{A}_H$ at v to get an element $a_v \in \mathfrak{c}_H$. We write $\mathcal{M}_{H,v}(a) = \mathcal{M}_{H,v}(a_v)$ for its affine Springer fiber, and $\mu_{H,v}(a)$ for

the local mass $\mu_{H,v}(a_v)$. Similarly, we write $\mu_{\kappa,v}(\nu(a))$ for $\mu_{\kappa,v}(\nu(a_v))$. With all of these conventions in place, we can state the product formula:

THEOREM 42. — *Let $a \in \mathcal{A}_H^{ani}(k)$. The mass of a Hitchin fiber modulo symmetry satisfies a product formula over all closed points of X in terms of the masses of the individual affine Springer fibers modulo symmetries:*

$$\mu_\kappa(\nu(a)) = \prod_{v \in X} \mu_{\kappa,v}(\nu(a)), \quad \mu_H(a) = \prod_{v \in X} \mu_{H,v}(a).$$

The local factors are 1 for almost all v so that the products are in fact finite.

This theorem is a geometric version of the factorization of κ-orbital integrals over the adele group into a product of local κ-orbital integrals in **[36]**. It confirms the claim that the Hitchin fibration is the correct global analogue of the affine Springer fiber.

Proof sketch. — The proof choses an open set of X over which J'_a is isomorphic to J_a. For a given a, on a possibly smaller open set U of X, the action of $\mathcal{P}(J_a)$ on $\mathcal{M}_a$ induces an isomorphism of $\mathcal{P}(J_a)$ with $\mathcal{M}_a$. It follows that the local masses equal 1 for all $v \in U$. The product in the lemma can be taken as extending over the finite set of points $X \setminus U$. The lemma is a consequence of a wonderful product formula for stacks, relating the Hitchin fibration to affine Springer fibers:

$$[\mathcal{M}_a/\mathcal{P}(J'_a)] = \prod_{v \in X \setminus U} [\mathcal{M}_v(\nu(a))/\mathcal{P}_v(J'_a)]. \tag{43}$$

A similar formula holds on H. □

6.4. Global mass formula

The following is the key global ingredient of the proof of the FL. In fact, it can be viewed as a precise global analogue of the FL.

THEOREM 44 (Global mass formula). — *Assume* $\deg(D) > 2g$, *where g is the genus of X. Then for all $\tilde{a} \in \tilde{\mathcal{A}}_H^{good}(k)$ with images $a \in \mathcal{A}_H^{ani}(k)$ and $\nu(a)$ in $\mathcal{A}(k)$, the following mass formula holds:*

$$\mu_\kappa(\nu(a)) = q^r \mu_H(a), \text{ where } r = \dim \mathcal{A} - \dim \mathcal{A}_H.$$

Proof sketch. — The proof first defines a particularly nice open set $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{A}}_H^{\mathrm{good}} \subset \tilde{\mathcal{A}}_H^{\mathrm{ani}}$. The idea is to place conditions on $\tilde{\mathcal{U}}$ to make it as nice as possible, without imposing so many conditions that it fails to be open. There exists an open set $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{A}}_H^{\mathrm{good}}$ on which both of the following conditions hold:

- Each $\tilde{a} \in \tilde{\mathcal{U}}(\bar{k})$ cuts the divisor $\mathfrak{D}_{H,D} + \mathfrak{R}_D$ transversally.
- For each n, the restriction to $\tilde{\mathcal{U}}$ of the perverse cohomology sheaves $\nu^* L_\kappa$ and $L_{H,st}$ from (27) and (30) are pure local systems of weight n.

The support theorem (32) and decomposition and continuity strategies (5.4) are used to find the pure local systems.

After choosing $\tilde{\mathcal{U}}$, the proof of the lemma establishes the global mass formula on $\tilde{\mathcal{U}}$, then extends it to all of $\tilde{\mathcal{A}}_H^{\mathrm{good}}$.

By imposing such nice conditions on $\tilde{\mathcal{U}}$, Ngô is able to prove the mass formula on this subset by explicit local calculations. By the transversality condition on $\tilde{a}$, at any given point v, the local degree $(d_{H,v}(\nu(a)), r_v(a))$ must be $(0,0)$, $(1,0)$, or $(0,1)$. From Bezrukavnikov's dimension formula (4.2), the dimension of the endoscopic affine Springer fiber $\mathcal{M}_{H,v}(a)$ is 0. In fact, $\mathcal{P}_v(J'_a)$ acts simply transitively on the affine Springer fiber, and the mass is 1.

It is therefore enough to compute the κ-mass of $\nu(a)$ and compare. The transversality condition determines the possibilities for the dimension $\delta_v(\nu(a))$ in G. The affine Springer fiber in this case is at most one and the κ-masses of the groupoids can be computed directly. In fact, (36) is a typical example of the computations involved.

The result of these local calculations is that for every point $\tilde{a}$ in $\tilde{\mathcal{U}}$, with images $a \in \mathcal{A}_H$ and $\nu(a) \in \mathcal{A}$, a local mass formula holds for all closed points v of X:

$$\mu_{\kappa,v}(\nu(a)) = q^{\deg(v) r_v(a)} \mu_{H,v}(a). \tag{45}$$

The exponents satisfy

$$r = \sum_v \deg(v) r_v(a). \tag{46}$$

These two identities, together with the product formula for the global mass, give the lemma for elements a of $\tilde{\mathcal{U}}$.

The extension from $\tilde{\mathcal{U}}$ to all of $\tilde{\mathcal{A}}_H^{\mathrm{good}}$ is a global argument. Through the Grothendieck-Lefschetz trace formula (adapted to stacks), this identity of global masses over $\tilde{\mathcal{U}}$ can be expressed as an identity of alternating sums of trace of Frobenius on local systems. These calculations can be repeated for all finite extensions k'/k. By Chebotarev density as we vary k', the semisimplifications of the local systems are isomorphic on G and H.

Following the decomposition and continuity strategy (5.4), this isomorphism of local systems on $\tilde{\mathcal{U}}$ extends to an isomorphism between (the semisimplifications of) $\nu^* L_\kappa$ and $L_{H,st}$. This isomorphism, again by Grothendieck-Lefschetz, translates back into a mass formula for the Hitchin fibration modulo symmetries, and hence the result. □

6.5. Local mass formula and the FL

We recall some notation from Section 3. Let G_v be a reductive group scheme over the ring of integers $\mathcal{O}_v$ of a nonarchimedean local field F_v in positive characteristic. Let q be the cardinality of the residue field k. Let (κ, ρ) be endoscopic data defining

an endoscopic group H_v. Let $a \in \mathfrak{c}_H(\mathcal{O}_v)$ and $\nu(a)$ be its image in $\mathfrak{c}(F)$. Assume that $\nu(a)$ is regular semisimple. Let $r_v(a) \in \mathbb{N}$ be the local invariant.

Assume that the characteristic of k is large. By standard descent arguments (8.1), we also assume without loss of generality that the center of H_v does not contain a split torus.

THEOREM 47 (Local mass formula). — *The following local mass formula holds for general anisotropic affine Springer fibers (both masses being computed with respect to the same symmetry group $\mathcal{P}_v(J'_a)$ acting on the fibers):*

$$\mu_{\kappa,v}(\nu(a)) = q^{r_v(a)}\mu_{H,v}(a).$$

COROLLARY 48 (Fundamental lemma (FL)). —

$$\mathbf{O}_\kappa(\nu(a)) = q^{r_v(a)}\mathbf{SO}_H(a).$$

The corollary follows from the theorem by the mass formula (40) for orbital integrals.

Proof sketch. — The proof of the FL is based on the global mass formula on $\tilde{\mathcal{A}}_H^{\mathrm{good}}$. We can use standard strategies to embed the local setting into a global context. We pick a smooth projective curve X over k, such that a completion of the function field at some place v is isomorphic to F_v and $\deg(v) = 1$. We choose a global endoscopic group H of a reductive group G, a divisor D on X, a global element[(1)] a' in the Hitchin base $\mathcal{A}_H$ of H. These global choices are to specialize to the given data G_v and H_v at v. If the degree of D is sufficiently large, then we can assume that a' is the image of $\tilde{a}' \in \mathcal{A}_H^{\mathrm{good}}(k)$. The element a' and its image $\nu(a')$ in $\mathcal{A}$ are chosen to approximate the given local elements a and $\nu(a)$ so closely that their affine Springer fibers together with their respective symmetries are unaffected at v.

The global mass formula (44) for $\tilde{a}'$ asserts:

$$\mu_\kappa(\nu(a')) = q^r\mu_H(a').$$

By the product formula (42) and (46), each global mass is a product of local masses:

$$\prod_w \mu_{\kappa,w}(\nu(a')) = \mu_\kappa(\nu(a')) = q^r\mu_H(a') = \prod_w q^{\deg(w)r_w(a')}\mu_{H,w}(a'). \tag{49}$$

The global data is chosen in such a way that at every closed point $w \neq v$, the transversality conditions hold, so that the calculation (45) of the previous section gives the local mass formula at w:

$$\mu_{\kappa,w}(\nu(a')) = q^{\deg(w)r_w(a')}\mu_{H,w}(a'), \quad w \neq v.$$

(1) More accurately, Ngô shows that a suitable element a' exists over every sufficiently large finite field extension k'/k. He makes the global arguments over the extensions k' and uses a Frobenius eigenvalue argument at the end to go back to k.

These masses are nonzero and can be canceled from the products in (49). What remains is a single uncanceled term on each side:

$$\mu_{\kappa,v}(\nu(a')) = q^{\deg(v)r_v(a')}\mu_v(a'), \text{ with } \deg(v) = 1.$$

Since a' was chosen as a close approximation of a at v, we also have

$$\mu_{\kappa,v}(\nu(a)) = q^{r_v(a)}\mu_v(a).$$

This is the desired local mass formula at v. □

7. USES OF THE FL

"Lemma" is a misleading name for the Fundamental Lemma because it went decades without a proof, and its depth goes far beyond what would ordinarily be called a lemma. Yet the name FL is apt both because it is fundamental and because it is expected to be used widely as an intermediate result in many proofs. This section mentions some major theorems that have been proved recently that contain the FL as an intermediate result. In each case, the FL appears to be an unavoidable ingredient.

The FL appears as a specific collection of identities that are needed to stabilize the Arthur-Selberg trace formula. If G is a reductive group defined over a number field F, the trace formula for G is an identity of the general form

$$\sum_{\gamma \in G(F)/\sim} c_\gamma \mathbf{O}(\gamma, f) + \cdots = \sum_\pi m(\pi) \operatorname{trace} \pi(f) + \cdots$$

for compactly supported smooth functions f on the adele group $G(\mathbb{A}_F)$. On the left-hand side appears a sum of orbital integrals and on the right-hand side the sum runs over discrete automorphic representations π of G. The trace formula contains more complicated terms that have been suppressed.

By *stabilization of the trace formula*, we mean that the terms on the left-hand side of the trace formula that are associated with a given stable conjugacy class have been gathered together, rearranged into κ-orbital integrals, and then replaced with stable orbital integrals on the endoscopic groups. These manipulations are justified by the FL and by a product formula that relates the adelic orbital integrals $\mathbf{O}(\gamma, f)$ to orbital integrals on local fields. Another Bourbaki seminar gives further details about the role of the FL in the stable trace formula [**16**]. All applications of the FL come through the stable trace formula.

Before going into recent uses of the FL, we might also mention various special cases of the FL that have been known for years. These classical cases of the FL already give abundant evidence of the usefulness of the lemma. For example, Langlands proves the FL for cyclic base change for $GL(2)$ in his book [**35**, Lemma 5.10]. From there, it

enters into the proof of the tetrahedral and octahedral cases of the Artin conjecture (the Langlands-Tunnell theorem), which in turn is used by Wiles in the proof of Fermat's Last Theorem. Waldspurger's proof of the FL for $SL(n)$ is taken up by Henniart and Herb in their proof of local automorphic induction for $GL(n)$, which becomes part of the proof of the local Langlands correspondence for $GL(n)$ in Harris and Taylor [**53**],[**26**],[**25**].

Shimura varieties provided much of the early motivation for endoscopy and the FL [**34**]. When expressing the Hasse-Weil zeta function of Shimura varieties as a product of automorphic L-functions, the formula involves the L-functions associated with endoscopic groups H as well as those of G. This can be most clearly seen through a comparison of the stable trace formula with the Grothendieck-Lefschetz trace formula of the Hasse-Weil zeta function. An early application of the FL carries this out for Picard modular varieties [**1**]. From there, the FL becomes relevant to the theory of Galois representations through the representations associated with Shimura varieties.

We turn to more recent uses of the FL. For most applications to date, the FL for unitary groups is used as well as the twisted FL between $GL(n)$ and unitary groups. Applications of the trace formula to Shimura varieties often rely on a base change FL, which arises because of the description of that Kottwitz gives of points on certain Shimura varieties in terms of twisted orbital integrals [**30**].

The original proof by Clozel, Harris, Shepherd-Barron and Taylor of the Sato-Tate conjecture for elliptic curves over $\mathbb{Q}$ was restricted to elliptic curves with non-integral j-invariants [**10**]. With the advent of the general FL, it has become possible to remove the non-integrality assumption and to greatly extend the theorem, in particular to elliptic curves over a totally real number field [**6**].

Shin and Morel use the FL in recent work on the cohomology of Shimura varieties and associated Galois representations [**50**][**43**]. Other advances rely on their work. In particular, Skinner and Urban have proved the Iwasawa-Greenberg main conjecture for many modular forms and in particular for the newforms associated with many elliptic curves over $\mathbb{Q}$ [**52**],[**51**]. Their work ultimately relies on the work of Shin and Morel and on the FL to prove the existence of certain Galois representations.

Last year, Bhargava and Shankar proved that when elliptic curves E over $\mathbb{Q}$ are ordered by height, a positive fraction of them satisfy the Birch and Swinnerton-Dyer conjecture [**9**]. Specifically, a positive fraction of them have rank 0 and analytic rank 0. First they construct a set (of positive density) of elliptic curves with rank 0. Second, they construct a subset (again of positive density) of the rank 0 set, consisting of elliptic curves with analytic rank 0. This second step relies on conditions in Skinner and Urban for the analytic rank to be zero, and hence indirectly on the FL.

Moeglin classifies the discrete series representations of unitary groups over a nonarchimedean local field [**42**]. Again, this relies on the FL for unitary groups and related

variants. Finally, we mention that Arthur's forthcoming book uses the twisted FL between $GL(n)$ and the classical groups [**5**]. His work uses the trace formula to give a classification of the discrete automorphic representations of classical groups in terms of cuspidal automorphic representations of $GL(n)$. It also gives a classification locally, for p-adic fields.

I will leave a further discussion of the uses of the FL to those whose research in this area is fresher than my own.

8. REDUCTIONS

Langlands first expressed the FL in these words: "Mais même après avoir vérifié que les facteurs de transfert existent, il reste à vérifier ce que j'appelle le lemme fondamental, qui affirme que pour des G, H et ϕ_H non-ramifiés, on a $f \mapsto c\,\phi_H^*(f)$... pour toute fonction $f \in \mathcal{H}_G$". [**36**, p. 49].

In this notation, ϕ_H^* is the homomorphism given by the Satake transform, from the spherical Hecke algebra $\mathcal{H}_G$ with respect to a hyperspecial maximal compact subgroup of an unramified reductive group G to the spherical Hecke algebra on H. The arrow $f \mapsto c\,\phi_H^*(f)$ is his assertion that for every strongly G-regular element γ in H, the transfer (specified by transfer factors) of each κ-orbital integral of a spherical function f on G (over a stable conjugacy class in G matching γ) is equal to the stable orbital integral of $\phi_H^*(f)$ on the stable conjugacy class of γ in H.

This final section describes some theorems related to the FL that simplify it from the form in which it was initially conjectured, to the final form in which it was proved by Ngô. Waldspurger's work has been particularly significant in transforming the conjecture into a friendlier form. In the initial conjecture, the existence of transfer factors was part of the conjecture. Langlands and Shelstad later defined the transfer factors explicitly [**38**]. We also mention some extensions of the FL.

8.1. Descent to the Lie algebra

A lemma of Harish-Chandra's asserts the transfer of an orbital integral on G near a singular semisimple element $z\gamma_0$, with z central, to an orbital integral on the centralizer I_{γ_0}. This is called the *descent* of orbital integrals. Langlands and Shelstad made hard calculations in Galois cohomology to prove that their transfer factors are compatible with Harish-Chandra's descent of orbital integrals [**39**]. The point of their calculations was to reduce identities of orbital integrals involving transfer factors to a neighborhood of $\gamma_0 = 1$, arguing by induction on the dimension of the centralizer. In a neighborhood of $\gamma_0 = 1$, identities can be pushed to the Lie algebra, using the exponential map.

The original FL has been supplemented by a twisted FL, conjectured by Kottwitz and Shelstad, where the data is twisted by a nontrivial outer automorphism θ of the group G [**31**]. In the untwisted case, the centralizer of an element fails to give a group of smaller dimension precisely when the element is central. By contrast, a twisted centralizer (with respect to a nontrivial outer automorphism) always has dimension less than G. As a consequence, descent *always untwists* the twisted FL into identities of ordinary orbital integrals. If the (standard) FL is then applied, each κ-orbital integral can be replaced with a stable orbital integral. By combining both descent and stabilization, the twisted FL of Kottwitz and Shelstad takes the form of identities of stable orbital integrals on the Lie algebra (from which the automorphism and the character κ have entirely vanished). The corresponding long calculations in Galois cohomology that establish descent properties of the transfer factors for the twisted FL have been carried out by Waldspurger [**56**]. Ngô proves the general twisted FL in its untwisted stable form on the Lie algebra.

8.2. Hecke algebras

A global argument based on the trace formula shows that the FL holds for the full Hecke algebra for an arbitrary nonarchimedean local field of characteristic zero, provided it holds for the unit element of the Hecke algebra for local fields of sufficiently large residual characteristic (and for groups of smaller dimension) [**22**]. The idea of the proof is to choose suitable global functions for which the comparison of stable trace formulas yields an obstruction to the FL. This obstruction, which comes from the spectral side of the trace formula, takes the form of a set of linear functionals

$$L : \mathcal{H}_G \to \mathbb{C}, \quad L(f) = \sum_{\pi} a(\pi) \operatorname{trace} \pi(f)$$

on the local spherical Hecke algebra $\mathcal{H}_G$ of the reductive group G, each given by a finite sum over irreducible admissible representations with an Iwahori fixed vector. By purely local arguments, it can be shown that no nonzero linear map L exists of the form prescribed by the global theory. Because these obstructions L are zero, the FL can be shown to hold on the full spherical Hecke algebra.

8.3. Smooth transfer

Langlands's book on the stabilization of the trace formula contains two separate conjectures: the transfer of smooth functions and the FL [**36**]. An important result of Waldspurger links the two conjectures, by proving that the FL implies the transfer of smooth functions. His key local lemma shows how to obtain simultaneous control over the orbital integrals of test functions f on the Lie algebra $\mathfrak{g}$ and the orbital integrals of their Fourier transform $\hat{f}$ [**54**, Prop. 8.2]. In view of the uncertainty principle, it is a remarkable feat to control both f and $\hat{f}$ as he does. His proof is a global argument,

based on a stable Poisson summation trace formula on the Lie algebra over the ring of adeles. The key local lemma allows Waldspurger to pick global test functions for which the comparison of trace formulas asserts a local identity: the Fourier transform of a semisimple κ-orbit on G equals the Fourier transform of the corresponding stable orbit on H. By a purely local argument, this stabilization identity of Fourier transforms implies smooth transfer.

8.4. Weighted orbital integrals

Langlands's book is a *début*: he stabilizes the terms in the trace formula that come from regular elliptic conjugacy classes, but this is insufficient for general applications of the trace formula. Kottwitz extended the analysis to singular elliptic conjugacy classes [**29**]. Arthur has completed the full stabilization without restrictions on the conjugacy classes. The non-elliptic conjugacy classes lead to significant complications. Arthur truncates the trace formula to obtain the convergence of the non-elliptic terms. Because of truncation, the non-elliptic terms bear "weights," non-invariant factors that appear in the integrand of orbital integrals. Arthur conjectured a weighted FL needed for stabilization of the non-elliptic terms [**3**]. Chaudouard and Laumon have used the Hitchin fibration to prove Arthur's weighted FL [**13**][**14**].

8.5. Transfer to characteristic zero

The FL for nonarchimedean local fields in characteristic zero can be deduced from the FL in positive characteristic [**55**][**15**]. Cluckers and Loeser have developed a general abstract theory of integration as a combination of primitive operations such as taking the volume of a ball of given radius, enumerating points on a variety over the residue field, summing an infinite q-series, and making a change of variables. Since each of the primitive operations manifestly depends only on the residue field rather than the field itself, their theory allows many identities of integrals to be transfered from one field to another with the same residue field. The FL lemma and its weighted and twisted variants are identities that fall within the scope of this theory. Waldspurger's approach is also an abstraction of p-adic integration, but it requires more detailed properties of the specific integrals appearing in the FL.

8.6. Etc.

These separate variations on the FL can be considered in concert: a weighted twisted FL, the twisted FL on the full Hecke algebra, transfer of the weighted FL to characteristic zero, and so forth. Most combinations have now been proved.

9. LITERATURE

I recommend Ben-Zvi's video lecture for a presentation of the big ideas of Ngô's proof. Drinfeld's lecture notes contain many worked examples and exercises that are helpful in learning the geometric concepts. I also recommend Nadler's survey [**44**], Casselman for an in-depth treatment of SL_2 with history going back to Hecke [**11**], my summer school lecture for the detailed statement of the FL [**23**], Arthur's Fields medal laudation [**4**], and [**12**]. Several articles in the book project deal with the FL [**2**], particularly [**17**].

Ngô's book is superb, both as mathematics and as exposition [**46**]. It is helpful to read it with his earlier paper [**45**]. He has several supplementary accounts, especially the expository account [**47**], his article in the book project [**48**], and ICM lectures.

While there have been numerous applications that quote the FL as a finished product, Yun, Chaudouard, and Laumon are noteworthy in following Ngô in their direct use of the Hitchin fibration to prove new results in the field [**57**].

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