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ERGODICITY OF TWO DIMENSIONAL TURBULENCE [after Hairer and Mattingly]

by **Antti KUPIAINEN**

INTRODUCTION

The problem of turbulence has been described as the last great unsolved problem of classical physics. Understanding of the complicated motion of fluids in the presence of obstacles or stirring has been a challenge to mathematicians, physicists and engineers for quite a time now. The equations governing macroscopic fluid motion, the Navier Stokes equations, have been known for close to two centuries. For an incompressible fluid in units where the density equals one they read

$$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f. \tag{1}$$

$u(t,x) \in \mathbb{R}^d$ is the velocity field at time t at $x \in \Lambda$, a domain in $\mathbb{R}^d$ subject to the incompressibility condition

$$\nabla \cdot u = 0 \tag{2}$$

and suitable boundary conditions on $\partial\Lambda$. ν is the viscosity coefficient of the fluid, $p(t,x)$ the pressure and $f(t,x)$ the external force that sustains the flow. Given f and $u(0,\cdot)$ the task is to find u and p. It is fair to say that theoretical understanding of the consequences of these equations is still in its infancy. On the mathematical side, existence of smooth solutions for the three dimensional NS equations is wide open and has been chosen by some as one of the major problems of mathematics (`http://www.claymath.org/millennium/`). On the physical side, experimental violations of the Kolmogorov scaling theory of turbulence [**12**] are still waiting for theoretical understanding.

In two dimensions, i.e., for flows on the plane, there has been some progress during the last ten years. On the physical side, 2d turbulence has been the subject of accurate numerical and experimental studies [5], [25] and mathematically the ergodic theory of the NS flow has been under intensive study.

It is important to realize that for the problem of turbulence one is interested in a very particular kind of force in (1), namely one that has a fixed length scale L built into it. Examples of this are flows past obstacles, with L the characteristic size of the obstacle. In such a setup the flow exhibits universal statistical properties as the viscosity parameter tends to zero (actually the control parameter is a dimensionless quantity, the Reynolds number given by $\frac{Lv}{\nu}$ where v is a velocity scale related to the forcing). E.g. time averages of measurements of suitable functions of u seem to show statistical properties only depending on the Reynolds number. It is therefore of some interest to inquire about the foundations for such statistical studies, i.e., about the ergodic properties of the NS flow in the turbulent setup of a fixed scale high Reynolds number forcing.

A convenient model for isotropic and homogeneous turbulence (i.e., in the limit of large Reynolds number and away from the boundary $\partial\Lambda$) is to consider (1) Equation (1) on the torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and take f random, a Fourier series with a finite number of terms and coefficients independent white noises (see below). Then the deterministic dynamics of (1) is replaced by a Markov process and one may pose questions on its ergodic properties: whether the process has a unique stationary state and whether this is reached and with what rate from arbitrary initial conditions.

This Markov process is a diffusion process of a very degenerate type. While the phase space is infinite dimensional the noise is finite dimensional. There are two general mechanisms that can contribute to the ergodic and mixing properties of stochastic flows. One is dissipation, coming in our case from the Laplacian in (1). Dissipation contributes to ergodicity by exponential contraction of phase space under the flow. A second mechanism comes from the spreading of the noise from its finite dimensional subspace due to the nonlinear term in (1). In finite dimensional diffusion processes this leads to hypoellipticity if the noise spreads to the full phase space: the transition kernels are smooth (for equations with smooth coefficients). Combined with some irreducibility of the process ergodicity follows.

In our infinite dimensional setup the dissipation due to the Laplacian leads to strong damping of large enough (depending on the Reynolds number) Fourier modes. If we keep noise on all the other, low, modes then one can reduce the problem to a low mode dynamics, albeit with some (exponentially decaying) memory due to the large modes. Proofs of ergodicity and mixing of the dynamics were given in this case in the works [6], [10] and [18]. However, it seemed far from trivial to extend the hypoellipticity ideas to the infinite dimensional setup to control also the case of very

(1) To get to the turbulent state one actually has to modify (1) a bit, see Section 8.

degenerate forcing where the number of forced modes does not depend on the Reynolds number. This was accomplished by Hairer and Mattingly **[13]**, **[15]** who gave sharp sufficient conditions for the noise to produce ergodic and mixing dynamics. In what follows I will present the main points of their approach focusing on the difference to finite dimensional hypoelliptic diffusions. The papers **[13]**, **[15]** are very clearly written and they contain plenty of background material, especially **[15]** which builds a more general formalism applicable also to some reaction-diffusion equations. **[15]** also corrects a mistake in **[13]** so it should be consulted for a thorough study. In the final section I discuss more informally what we have learned about 2d turbulence and what issues might be accessible to a rigorous mathematical analysis.

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1. 2D NS EQUATIONS

The fundamental fact that is behind both the mathematical and physical understanding of 3d NS equations is energy conservation: in the absence of forces smooth inviscid flow preserves the L^2 norm of $u(t,\cdot)$. In two dimensions there is a second conserved quantity, the *enstrophy*, which is related to the H^1 norm and which leads to quite different physics and to much better regularity.

Let us first define the *vorticity*

$$\omega = \nabla \times u,$$

which in $d = 2$ is a (pseudo)scalar: $\omega = \partial_1 u_2 - \partial_2 u_1$. The NS equation becomes in terms of ω a transport equation:

$$\dot{\omega} = \nu \Delta \omega - u \cdot \nabla \omega + g, \tag{3}$$

where $g = \partial_1 f_2 - \partial_2 f_1$. We will assume the average force vanishes, i.e., $\int f(t,x)dx = 0$. Then (1) preserves the condition $\int u(t,x)dx = 0$ which we will assume. The incompressibility condition (2) allows to write $u = \mathcal{C}\omega$ where the linear operator $\mathcal{C}$ is given in terms of the Fourier transform by

$$\widehat{\mathcal{C}\omega}(k) = i(k_2, -k_1)k^{-2}\hat{\omega}(k) \tag{4}$$

for $k \in \mathbb{Z}^2 \setminus 0$.

The enstrophy $\mathcal{E}$ is defined to be (half of) the L^2-norm of ω:

$$\mathcal{E} = \tfrac{1}{2}\int \omega(t,x)^2 dx := \tfrac{1}{2}\|\omega(t)\|^2.$$

For a smooth u the condition $\nabla \cdot u = 0$ leads to the absence of contribution from the nonlinear term to the evolution of the enstrophy:

$$\frac{d\mathcal{E}}{dt} = -\nu \int (\nabla\omega)^2 dx + \int \omega g dx, \tag{5}$$

where the first term on the RHS can be interpreted as an enstrophy dissipation rate and the second one as an enstrophy injection rate. Using Poincaré inequality $\|\nabla\omega\| \geq \|\omega\|$ and simple estimates one deduces

$$\|\omega(t)\|^2 \leq e^{-\nu t}\|\omega(0)\|^2 + \nu^{-2} \sup_t \|g(t)\|^2. \tag{6}$$

This a priori estimate for the H^1 norm of u is the main ingredient in the proof of global regularity of the 2d NS flow.

We wish now to discuss a version of (3) where the force g is random. We work in the subspace of real valued $L^2(\mathbb{T}^2)$ functions with $\hat{\omega}(0) = 0$. It will be convenient to use the following basis for this space. Let Z^+ be the "upper half plane" in $\mathbb{Z}^2$ consisting of $k = (k_1, k_2)$ with $k_2 > 0$ or $k_2 = 0$ and $k_1 > 0$. Hence $\mathbb{Z}^2 \setminus 0 = Z^+ \cup (-Z^+)$. Let $e_k = \sin kx$ for $k \in Z^+$ and $e_k = \cos kx$ for $-k \in Z^+$. For each $k \in \mathbb{Z}^2$ pick independent Brownian motions $\beta_k(t)$ with unit speed, denoted collectively by $\beta(t)$ and numbers $\gamma_k \in \mathbb{R}$. Let

$$Q\beta(t) = \sum_{k\in\mathbb{Z}^2} \gamma_k \beta_k(t) e_k. \tag{7}$$

The stochastic version of Equation (3) reads

$$d\omega = (\nu\Delta\omega - u \cdot \nabla\omega)dt + Qd\beta. \tag{8}$$

Regularity of the stochastic flow proceeds in parallel with the deterministic case as long as γ_k have enough decay at infinity. The analog of the enstrophy conservation Equation (5) is obtained by an application of the Ito formula

$$d\mathcal{E} = \tfrac{1}{2} d\|\omega\|^2 = -\nu\|\nabla\omega\|^2 dt + (\omega, Qd\beta) + \epsilon dt \tag{9}$$

where $\epsilon = 2\pi^2 \sum_k \gamma_k^2$ can be interpreted as the enstrophy injection rate. Taking averages we get a probabilistic analog of (5) and (6):

$$\frac{d}{dt}\mathbb{E}\mathcal{E} = -\nu\mathbb{E}\|\nabla\omega\|^2 + \epsilon \tag{10}$$

and

$$\mathbb{E}\|\omega(t)\|^2 \leq e^{-2\nu t}\|\omega(0)\|^2 + \nu^{-1}\epsilon. \tag{11}$$

Actually (9) can be used to control exponential moments of the enstrophy **[6]**, **[13]** Lemma A.1:

$$\mathbb{E}\exp(\eta\|\omega(t)\|^2) \leq 2\exp(\eta e^{-\nu t}\|\omega(0)\|^2) \tag{12}$$

for all $\eta \leq \frac{\nu}{\epsilon}$, i.e., probability for large L^2 norm is exponentially small. (9) also allows to control the time integral of the H^1 norm:

$$\mathbb{E}\exp(\eta\nu\int_0^t \|\nabla\omega(t)\|^2) \leq 2\exp(\eta\epsilon t + \eta\|\omega(0)\|^2) \tag{13}$$

again for all $\eta \leq \frac{\nu}{\epsilon}$. Such a priori estimates allow one to prove the existence and pathwise uniqueness of strong solutions to Equation (8) under quite general conditions on the noise coefficients γ_k, see e.g. **[11]** and **[22]**. Of course the less the γ_k decay at infinity, the harder it is to establish the regularity of the PDE's. As explained in the introduction, for the turbulence problem only a finite number of the γ_k are nonzero. Thus from the point of view of regularity the turbulent case is easy (this is not true in 3d!). However, the less noise there is, the harder it is to establish the ergodicity of the flow.

2. INVARIANT MEASURE

Let us now specialize to the case where $\gamma_k = 0$ for $k \notin K$ where K is a finite set. Thus the noise is finite dimensional: $\beta = \{\beta_k\}_{k\in K}$ can be identified with the Wiener process in $\Omega = C([0,\infty), \mathbb{R}^D)$ where $D = |K|$, equipped with the Wiener measure $W(db)$. The solution of Equation (8) is a one parameter family of continuous maps $\Phi_t : \Omega \times L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ such that $\omega(t) = \Phi_t(\beta, \omega_0)$ solves Equation (8) with initial condition ω_0 and noise realization β. Actually, Φ_t is (Fréchet) differentiable in β and ω_0.

$\omega(t)$ is a Markov process with state space $H = L^2(\mathbb{T}^2)$. It gives rise to transition probabilities $P_t(\omega_0, A)$ which are probability measures on H, giving the probability of entering the set $A \subset H$ at time t given that at time 0 we have $\omega(0) = \omega_0$:

$$P_t(\omega_0, A) = \mathbb{E}1_A(\omega(t)).$$

The transition probabilities generate a semigroup $\mathcal{P}_t$ on bounded measurable functions on H by the same formula:

$$\mathcal{P}_t\phi = \int P_t(\cdot, d\omega)\phi(\omega) \tag{14}$$

and the adjoint semigroup acting on bounded (Borel) measures:

$$\mathcal{P}_t^*\mu = \int \mu(d\omega_0)P_t(\omega_0, \cdot). \tag{15}$$

We are interested in the invariant (or stationary) probability measures μ^* satisfying the equation

$$\mathcal{P}_t^*\mu^* = \mu^*. \tag{16}$$

Existence of an invariant measure is straightforward given the strong probabilistic control of the flow. One considers the family of time averages $\mu_t^{(\omega_0)} = t^{-1} \int_0^t ds P_s(\omega_0, \cdot)$ and shows it is tight. Prohorov's theorem then yields a limit point which is shown to be invariant.

Uniqueness of the invariant measure is much more subtle. It implies *ergodicity*, i.e., in particular the equivalence of time averages and ensemble averages: $\lim_{t\to\infty} \mu_t^{(\omega_0)}(\phi) = \mu(\phi)$ for all $\phi \in L^2(H, \mu)$ and μ-a.s. in ω_0. In practice one would like to have more, i.e., the convergence in some sense of the measures $P_t(\omega_0, \cdot)$ to μ^* as $t \to \infty$. This leads to various *mixing* concepts.

3. DISSIPATION AND SMOOTHING

For finite dimensional diffusion processes it is well known that the uniqueness of the invariant measure follows from recurrence and smoothing properties of the transition probabilities. Let us sketch a special version of this argument having the application to NS in mind.

The semigroup $\mathcal{P}_t$ is called *strong Feller* if the image is continuous for ϕ measurable. This has drastic consequences for the supports of invariant measures. Recall that x belongs to the support of a finite Borel measure μ on a Polish space (our setup) if $\mu(U) > 0$ for all open U containing x. Then *the supports of two distinct ergodic invariant probability measures for a strong Feller semigroup are disjoint.* To see this, suppose $\mu \perp \nu$ and $x \in \operatorname{supp} \mu \cap \operatorname{supp} \nu$. Pick A with $\mu(A) = 1$ and $\nu(A) = 0$. By strong Feller there exists a U containing x such that $\sup_{y,z\in U} |P_t(y, A) - P_t(z, A)| \le \frac{1}{2}$. Moreover by assumption $\alpha := \min\{\mu(U), \nu(U)\} > 0$. Write $\mu = (1-\alpha)\bar{\mu} + \alpha\mu_U$ with $\bar{\mu}$ and μ_U probability measures with $\mu_U(U) = 1$ (i.e., $\mu_U = \mu 1_U/\mu(U)$) and ν similarly. Then, by invariance $|\mu(A) - \nu(A)| = |\mathcal{P}_t^*\mu(A) - \mathcal{P}_t^*\nu(A)|$ and thus

$$
\begin{aligned}
1 = |\mu(A) - \nu(A)| &\le (1-\alpha)|\mathcal{P}_t^*\bar{\mu}(A) - \mathcal{P}_t^*\bar{\nu}(A)| + \alpha|\mathcal{P}_t^*\mu_U(A) - \mathcal{P}_t^*\nu_U(A)| \\
&\le (1-\alpha) + \alpha \int_{U\times U} |\mathcal{P}_t(y, A) - \mathcal{P}_t(z, A)| \mu_U(dy)\nu_U(dz) \le 1 - \tfrac{1}{2}\alpha,
\end{aligned}
$$

a contradiction.

Suppose now that we knew that there exists an x that necessarily belongs to the support of every invariant measure of a strong Feller semigroup. We could then conclude uniqueness. This is a reasonable strategy for the NS equation. Indeed, $\omega = 0$ is such a point. This follows since the NS equation is dissipative. Without forcing the fluid slows down, i.e., the L^2 norm decays exponentially (see Equation (6)). There is a non zero probability for the force to stay small enough so that any neighborhood of 0 can be reached. More precisely, the Ito formula (10) combined with Poincaré inequality $\|\omega\| \leq \|\nabla\omega\|$ yields

$$\int \mu(d\omega)\|\omega\|^2 \leq \epsilon/\nu$$

for every invariant probability measure μ. Hence, there exists $R < \infty$ such that every such measure has at least half its mass in the ball B_R of radius R centered at 0 in H. Thus one needs to show: for all $r > 0$, there exits $T_r < \infty$ such that

$$I_r := \inf_{\omega_0 \in B_R} P_{T_r}(\omega_0, B_r) > 0$$

(see [**9**], Lemma 3.1). Then $\mu(B_r) = \mathscr{P}_t^* \mu(B_r) \geq \frac{1}{2} I_r > 0$ for all $r > 0$.

This strategy does not quite work in our case since the strong Feller property is very hard to show for $\mathscr{P}_t$ and might very well not be true. One of the main accomplishments of Hairer and Mattingly was to replace it with a condition that is more natural for NS and yet allows one to conclude that the supports of invariant measures are disjoint.

4. ASYMPTOTIC STRONG FELLER PROPERTY

A strong Feller semigroup maps bounded functions to continuous ones. Often the easiest way to prove this is to show a bit more [**8**], Lemma 7.1.5:

PROPOSITION 4.1. — *A semigroup on a Hilbert space H is strong Feller if for all $\phi : H \to \mathbb{R}$ with $\|\phi\|_\infty := \sup_{x\in H} |\phi(x)|$ and $\|D\phi\|_\infty$ finite one has*

$$\|D\mathscr{P}_t\phi(x)\| \leq C(\|x\|)\|\phi\|_\infty \,, \tag{17}$$

where $C : \mathbb{R}_+ \to \mathbb{R}$ and D is the Fréchet derivative.

We will now argue that the condition (17) is not very natural for the NS dynamics. As mentioned in the introduction there are (at least) two ways ergodicity can result. One is due to smoothing by the noise, the other is due to dissipation that erases memory of the initial conditions. The former effect leads to a condition like (17), the latter not. Let us next discuss the latter effect in our case.

Let $J_{s,t}$ with $s < t$ be the derivative of the NS flow (8) between times s and t, i.e., for every $\xi \in H$, $J_{s,t}\xi := \xi(t)$ is the solution of the linear equation

$$\partial_t\xi(t) = \nu\Delta\xi(t) + \mathscr{C}\omega(t)\cdot\nabla\xi(t) + \mathscr{C}\xi(t)\cdot\nabla\omega(t) := \mathscr{L}_{\omega(t)}\xi(t) \tag{18}$$

for $t > s$ and $\xi(s) = \xi$. This linear equation is readily controlled in terms of the H^1 norm of ω ([**13**], Lemma 4.10):

$$\|\xi(t)\| \leq \exp\big(C(\delta,\nu)(t-s) + \delta\int_s^t \|\nabla\omega(r)\|^2 dr\big)\|\xi(s)\| \tag{19}$$

for any $\delta > 0$. Combining with the a priori estimate (13) then

$$\mathbb{E}\|J_{s,t}\|^p \leq 2^p \exp\left(C(\epsilon, \nu, \eta, p)(t-s) + \eta\|\omega(s)\|^2\right) \tag{20}$$

for all $\eta > 0$, all $p < \infty$.

Equations (19) and (20) indicate a possible exponential separation of trajectories. However, since the Laplacian is the Fourier multiplier $-k^2$ it is not surprising that the high Fourier modes of ξ are strongly damped for a time that can be taken as large as we wish as N is increased. This is expressed by **[13]**, Lemma 4.17:

LEMMA 4.2. — *For every $p \geq 1$, every $T > 0$, and every two constants $\gamma, \eta > 0$, there exists an orthogonal projector π_ℓ onto a finite number of Fourier modes such that*

$$\mathbb{E}(\|(1-\pi_\ell)J_{0,T}\|^p + \|J_{0,T}(1-\pi_\ell)\|^p) \leq \gamma e^{\eta\|w_0\|^2}.$$

For such contracting dynamics (17) is not a natural condition to try to prove. Indeed, let $\xi_h = (1-\pi_\ell)\xi$ be the projection of ξ to the high modes and consider the toy problem where we apply $(1-\pi_\ell)$ to Equation (18) and drop altogether the ω-dependent terms:

$$\partial_t \xi_h(t) = \nu \Delta \xi_h(t).$$

Then for a function $\phi(\omega) = \psi((1-\pi_\ell)\omega)$ depending only on the high modes we have $D\mathcal{P}_t\phi(\omega_0)\xi = \mathbb{E}D\phi(\omega(t))\xi_h(t)$. Since in this toy case $\|\xi_h(t)\| \leq e^{-At}\|\xi\|$ for $A > 0$ we conclude

$$\|D\mathcal{P}_t\phi(x)\| \leq e^{-At}\|D\phi\|_\infty.$$

This toy model and Lemma 4.2 motivate the following definition by Hairer and Mattingly (**[13]**, Proposition 3.12).

DEFINITION 4.3. — *A semigroup $\mathcal{P}_t$ on a Hilbert space $\mathcal{H}$ is* asymptotically strong Feller *if there exist two positive sequences t_n and δ_n with $\{t_n\}$ nondecreasing and $\{\delta_n\}$ converging to zero such that for all $\phi : \mathcal{H} \to \mathbb{R}$ with $\|\phi\|_\infty$ and $\|D\phi\|_\infty$ finite,*

$$|D\mathcal{P}_{t_n}\phi(x)| \leq C(\|x\|)\left(\|\phi\|_\infty + \delta_n\|D\phi\|_\infty\right) \tag{21}$$

for all n, where $C : \mathbb{R}_+ \to \mathbb{R}$.

(Hairer and Mattingly actually give a "topological" definition of the asymptotically strong Feller condition which is implied by the one above.) The main point is the following result whose proof is similar to the one given above in the strong Feller case (**[13]**, Theorem 3.16):

PROPOSITION 4.4. — *If the semigroup is asymptotically strong Feller at x then x belongs to the support of at most one ergodic invariant measure.*

We saw above that it is not unreasonable to expect that the high mode dynamics give rise to the second term in (21). Thus the question remains: why would the low mode dynamics be strong Feller? The answer to this question lies in the *hypoellipticity* of the low mode dynamics.

5. HYPOELLIPTICITY

Let us first think about the low mode dynamics in the *Galerkin approximation*, i.e., by putting the high modes to zero. More formally, consider the equation

$$d\omega = (\nu\Delta\omega - \pi_\ell(u\cdot\nabla\omega))dt + Qd\beta, \tag{22}$$

where we assume the forcing is on low modes $(1-\pi_\ell)Q\beta = 0$ and set $(1-\pi_\ell)\omega = 0$. Equation (22) defines a diffusion process in a finite dimensional space which we may identify with $\mathbb{R}^N$, $N = \dim \pi_\ell H$. The diffusion process is thus degenerate with the dimension D of the noise (much) smaller than N. The strong Feller property follows for such diffusions provided the generator of the diffusion process is hypoelliptic. Let us discuss this next.

Recall the Fourier basis $\{e_k\}$ for H. Let the range of π_ℓ be the span of $\{e_k\}$ with $|k| \le M$. Write $\omega = \sum_k \omega_k e_k$. Then the equation (22) reads

$$d\omega_k = v_k(\omega)dt + \gamma_k d\beta_k, \tag{23}$$

where v_k is given by

$$v_k(\omega) = -\nu|k|^2\omega_k - \frac{1}{8\pi^2}\sum_{j+\ell=k}(j_1\ell_2 - j_2\ell_1)\Big(\frac{1}{|\ell|^2} - \frac{1}{|j|^2}\Big)w_j w_\ell$$

and $w_k = \frac{1}{2}\omega_{-k} + \frac{1}{2i}\omega_k$ for $k \in Z^+$ and $w_{-k} = \bar{w}_k$. The generator of this diffusion is

$$L = X_0 + \sum_{k\in K} X_k^2 \tag{24}$$

where we recall that $\gamma_k = 0$ for $k \notin K$. The vector fields X_α are given by

$$\begin{aligned} X_0 &= \sum_k v_k \partial_{\omega_k} \\ X_k &= \gamma_k \partial_{\omega_k}. \end{aligned}$$

An operator of the form (24) with smooth vector fields X_α is known to generate a semigroup $\mathcal{P}_t$ with smooth kernel (hence it is strong Feller) provided the Hörmander bracket condition is satisfied (L is then hypoelliptic). The condition is that the span of the vector fields X_j, $j \neq 0$ and $[X_{i_1}, [X_{i_2}, \ldots [X_{i_{k-1}}, X_{i_k}]]\ldots]$ for $k > 1$ and $i_j \in \{0\}\cup K$ at each $\omega \in \mathbb{R}^N$ equals R^N.

To check this condition in the NS case is a purely algebraic exercise and the result is the following [**9, 13**]:

PROPOSITION 5.1. — *The following conditions for the set $K \subset \mathbb{Z}^2 \setminus \{0\}$ are sufficient for the Hörmander bracket condition to be satisfied:*

(a) *K is invariant under the reflection $k \to -k$*

(b) *K contains at least two elements of unequal length*

(c) *K spans $\mathbb{Z}^2$ under linear combinations with integer coefficients.*

An example of a very degenerate forcing that suffices is given by the set $K = \{(1,0), (-1,0), (1,1), (-1,-1)\}$, i.e there is forcing only on two wave vectors and their reflections.

Note that Proposition 5.1 is true for arbitrary (large enough) Galerkin cutoff N. Hence the full infinite dimensional generator formally satisfies the Hörmander condition and one might be tempted to try to use this to return to the attempt to prove the strong Feller property for $\mathscr{P}_t$. However, it is likely that, as we let N increase, the derivatives of the kernel of $\mathscr{P}_t$ with respect to the high modes blow up since the smoothing is very weak for them. It is much more natural to try to use in that regime the dissipation as coded in the asymptotic strong Feller condition.

Let us finally remark that if all the γ_k in (23) are nonzero the generator L is elliptic. If N is large enough (of the order ϵ/ν^3) then one may use the dissipativity of the high mode dynamics to solve for the high modes in terms of the (temporal history) of the low modes and use the ellipticity of the latter to prove ergodicity and mixing of the full dynamics [**6, 10, 18**].

6. MALLIAVIN MATRIX

Why does elliptic diffusion produce smoothness in transition kernels? One way to think about this is to consider trajectories of the flow. Noise will make the trajectories non-unique: a change in the initial condition can be compensated by the noise. In elliptic diffusions noise spans the whole space and the compensation is immediate, in hypoelliptic diffusions the nonlinearity spreads the noise in all directions thanks to the bracket condition. Thus a derivative of the solution in the initial condition should equal its derivative in a particular direction in the (history of) noise space. Since we are integrating over the noise the latter derivative can be integrated by parts and hence an estimate like the strong Feller property can emerge.

To be more explicit recall that we wrote the solution of the stochastic NS equation as $\omega(t) = \Phi_t(\beta, \omega_0)$ with Φ_t smooth in the noise $\beta \in C([0,\infty)), \mathbb{R}^D)$ and the initial condition ω_0. Also we have denoted the derivative in the initial condition by $\langle D_{\omega_0}\omega(t),\xi\rangle = J_{0,t}\xi = \xi(t)$ for $\xi \in H$. Thus

$$\langle D\mathscr{P}_t\phi(\omega_0),\xi\rangle = \mathbb{E}\langle (D\phi)(\omega(t)),\xi(t)\rangle. \tag{25}$$

Consider next the infinitesimal change in the solution corresponding to the change of the noise β in the direction $V \in C([0,\infty)),\mathbb{R}^D)$: $\langle D_\beta\omega(t),V\rangle := \zeta(t)$. $\zeta(t)$ satisfies the same linearized NS equation (18) but with forcing QV:

$$d\zeta(t) = \mathscr{L}_{\omega(t)}\zeta(t)dt + QdV(t) \tag{26}$$

and zero initial condition. The natural space to vary the noise is the Cameron-Martin space, i.e., to take V of the form $V(t) = \int_0^t v(s)\,ds$ with $v \in L^2_{\mathrm{loc}}([0,\infty],\mathbb{R}^D)$. By variation of constants, $\zeta(t)$ is then given by

$$\zeta(t) = \int_0^t J_{s,t}Qv(s)ds := A_t v. \tag{27}$$

Actually, the v one will eventually use is itself a function of the noise (see Equation (40)), but it will be a.s. in L^2_{loc}. The upshot is that $A_t : L^2([0,t],\mathbb{R}^D) \to H$ is an a.s. bounded random operator and so the Fréchet derivative can be written as

$$\langle D_\beta\omega(t),V\rangle = \sum_{k\in\mathscr{K}}\int_0^t \mathscr{D}^k_s\omega(t)v_k(s)ds$$

where the operator $\mathscr{D}^k_s$ is called the Malliavin derivative and heuristically corresponds to an instantaneous kick at time s to the direction k in noise space. Explicitly

$$\mathscr{D}^k_s\omega(t) = J_{s,t}\gamma_k e_k. \tag{28}$$

Suppose now we can find a v such that

$$\xi(t) = \zeta(t)\,,\ \text{i.e.,}\ J_{0,t}\xi = A_t v. \tag{29}$$

Inserting this to Equation (25) we get

$$\langle D\mathscr{P}_t\phi(\omega_0),\xi\rangle = \mathbb{E}\langle (D\phi)(\omega(t)),\zeta(t)\rangle = \mathbb{E}\langle D_\beta\phi(\omega(t)),V\rangle. \tag{30}$$

The derivative D_β in Equation (30) can be integrated by parts in the Gaussian Wiener measure to obtain

$$\mathbb{E}\langle D_\beta\phi(\omega(t)),V\rangle = \mathbb{E}(\phi(\omega(t))D^*_\beta V). \tag{31}$$

In other words, the expression D^*_β is the adjoint of D_β in $L^2(\Omega,W)$. If the process v is adapted to the Brownian filtration its expression is simply $D^*_\beta V = \sum_k \int v_k(s)d\beta_k(s)$, the Ito integral. Otherwise a derivative of v with respect to the noise also appears and $D^*_\beta V$ is called the Skorokhod integral of v. Combining (31) with (30) the desired bound follows:

$$|\langle D\mathscr{P}_t\phi(\omega_0),\xi\rangle| \le \|\phi\|_\infty\mathbb{E}|D^*_\beta V|. \tag{32}$$

It remains to solve Equation (29) for V (i.e., for v). Let A_t^* be the Hilbert space adjoint of A_t, i.e., explicitly

$$(A_t^*\xi)(s) = Q^* J_{s,t}^* \xi \tag{33}$$

for $s \leq t$. Then the Malliavin matrix is defined by

$$M(t) := A_t A_t^* = \int_0^t J_{s,t} Q Q^* J_{s,t}^* ds. \tag{34}$$

Suppose $M(t)$ is invertible. Then, clearly a solution to (29) is given by

$$v = A_t^* M_{0,t}^{-1} J_{0,t} \xi. \tag{35}$$

To sketch the rest of the story in the finite dimensional setup we need a bound for the Skorokhod integral appearing in Equation (32) **[24]**:

$$\mathbb{E}(D_\beta^* V)^2 \leq \mathbb{E} \int_0^t |v(s)|^2 ds + \sum_{kl} \mathbb{E} \int \mathcal{D}_s^k v_l(r) \mathcal{D}_r^l v_k(s)) ds dr. \tag{36}$$

The first term is the usual identity for the L^2 norm of the Ito integral, the second term appears for a non-adapted v, as is the one given by (35). To compute the Malliavin derivative of v in (36) note that all we need is to compute $\mathcal{D}_r J_{s,t}$ since A_t and $M(t)$ are expressed in terms of $J_{s,t}$. This in turn is obtained by differentiating the equation (18): $\eta := \mathcal{D}_r^k \xi$ satisfies

$$\partial_t \eta(t) = \nu \Delta \xi(t) + \mathcal{C}\omega(t) \cdot \nabla \xi(t) + \mathcal{C}\xi_t \cdot \nabla \omega(t) := \mathcal{L}_{\omega(t)} \eta(t) + B(J_{r,t} \gamma_k e_k, \eta(t))$$

where B is the bilinear form in appearing in $\mathcal{L}$. By variation of constants an expression involving only J emerges. Thus in the finite dimensional setup (so e.g. for the Galerkin NS) the main work to be done is to show that $M(t)^{-1}$ has good probabilistic bounds. Indeed it turns out $\|M(t)^{-1}\|$ is in $L^p(\Omega)$ for all $p < \infty$. In the infinite dimensional case with degenerate noise it is unlikely that $M(t)$ is a.s. invertible. QQ^* is proportional to the projection in H to the subspace generated by the noise. In the expression for $M(t)$ the dynamics spreads the range beyond this subspace, however we expect the projection of the result to the high modes to be very small. The key estimate on the Malliavin matrix Hairer and Mattingly prove is that $M(t)$ is unlikely to be small on vectors that have large projection to low modes:

Proposition 6.1. — *For every α, η, p and every orthogonal projection π_ℓ on a finite number of Fourier modes, there exists C such that*

$$\mathbb{P}\left(\inf_{\|\pi_\ell \phi\| \geq \alpha \|\phi\|} \frac{(M\phi, \phi)}{\|\phi\|^2} < \epsilon \right) \leq C \epsilon^p \exp(\eta \|\omega_0\|^2) \tag{37}$$

holds for every $\epsilon \in (0,1)$, and for every $\omega_0 \in H$.

We will not discuss the details of the proof which is the technical core of the paper **[15]** (see also **[21]**). However, for experts we want to make the following comments. One major difficulty Hairer and Mattingly face is that the integrand in the expression for the Malliavin matrix is not adapted, i.e., depends on the future noise. The usual way out of this problem in the finite dimensional theory is to use the semigroup property $J_{0,t} = J_{s,t}J_{0,s}$ to rewrite $M(t) = J_{0,t}\hat{M}(t)J^*_{0,t}$ with

$$\hat{M}(t) := \int_0^t J^{-1}_{0,s}QQ^*{J^*_{0,s}}^{-1}$$

the reduced Malliavin matrix (and the control $v(s) = Q^*{J^*_{0,s}}^{-1}\hat{M}(t)^{-1}\xi)$. In finite dimensions $J_{0,t}$ is invertible and now the integrand is adapted. The proof then uses Norris' lemma **[23]** which states that if a semimartingale is small then both its bounded variation part and local martingale part are small. In the infinite dimensional case with degenerate noise, $J_{0,t}$ is not invertible due to dissipation of the high modes. Hence one needs to work with non-adapted processes. The way out for Hairer and Mattingly is the polynomial nature of the nonlinearity. In the iterative proof, to show that $(\phi, M(t)\phi)$ is small implies that $s \to (J_{s,t}P(u(s))\phi, \phi)$ is small for the various multiple commutators P ; the P will always be a polynomial. One then writes $u(s) = v(s) + Q\beta(s)$ where v is more regular and expands $P(u(s))$ in powers of $Q\beta(s)$, ending up with a polynomial in the Wiener process $\beta(s)$ with coefficients that are nonadapted processes, but with higher regularity. The basic lemma one now needs is that such a Wiener polynomial can be small only if all the coefficients are small (up to events of small probability).

7. LOW MODE CONTROL

The approach to prove smoothness sketched in the previous section is a form of *stochastic control* where the noise is used to force solution to a prescribed region in phase space (for results on stochastic control in our setup, see also **[2]**, **[1]**). We saw that an exact compensation of the change of initial condition by a change in the noise seems impossible, but Proposition 6.1 gives reason to hope that partial compensation is possible for the low modes. Since by Lemma 4.2 the high modes are contracted the idea of Hairer and Mattingly is to do an approximate control such that instead of the full control (29) we have $\xi(t) - \zeta(t) \to 0$ as $t \to \infty$. Thus, as before let $v \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^D)$ be a shift in the noise and $\zeta(t) = A_{0,t}v$ be the corresponding (infinitesimal) shift in the solution. Let

$$\rho(t) = \xi(t) - \zeta(t). \tag{38}$$

Then, instead of the identities (30) and (31) we obtain

$$(39)\qquad \begin{aligned}\langle D\mathcal{P}_t\phi(\omega_0),\xi\rangle &= \mathbb{E}(\phi(\omega(t))D^*_\beta V)+\mathbb{E}\langle D\phi(\omega(t)),\rho(t)\rangle\\ &\le \|\phi\|_\infty\mathbb{E}|D^*_\beta V|+\|D\phi\|_\infty\mathbb{E}\|\rho(t)\|.\end{aligned}$$

The asymptotic strong Feller property will follow provided v can be chosen such that $\mathbb{E}|D^*_\beta V|$ stays bounded as $t\to\infty$ and $\mathbb{E}\|\rho(t)\|$ tends to zero exponentially.

To find v, Hairer and Mattingly use a construction where at successive time intervals two steps are alternated, one where high modes contract, the second where low modes are controlled by the noise. Suppose at some time t we knew $\rho(t)$ is mostly in the high mode subspace, i.e., $\|\pi_\ell\rho(t)\|\ll\|\rho(t)\|$. Then, at least for a short time it pays to set $v=0$ since the linearized dynamics contracts such a ρ strongly. However, we cannot do this for too long since the low mode part of ρ will increase. Then provided we can find a v that will compensate the low mode part on a fixed time interval while leaving the high mode part approximately intact we can iterate the procedure.

The low mode control is a simple modification of the full control explained in the previous section. Let us take the time intervals as $[n,n+1]$ with n an odd integer for the first step and an even integer for the second step. Thus we set $v(t)=0$ for $t\in[n,n+1]$, n odd. Let $A_n:=A_{n,n+1}$, $M_n:=A_nA_n^*$ and $J_n:=J_{n,n+1}$. For n even take

$$(40)\qquad v_n:=v|_{[n,n+1]}=A_n^*(M_n+\beta)^{-1}J_n\rho(n).$$

Note that except for the parameter β this agrees with the full control (35). While for $\beta=0$ the inverse in (40) most likely does not exist, for $\beta>0$ it does. The point now is that for small enough β (40) does a good job for the low mode control while the high modes remain approximately intact. To see this, compute

$$(41)\qquad \begin{aligned}\rho(n+1) &= \xi(n+1)-\zeta(n+1)\\ &= J_n\xi(n)-(J_n\zeta(n)+A_nA_n^*(M_n+\beta)^{-1}J_n\rho(n))\\ &= \beta(M_n+\beta)^{-1}J_n\rho(n).\end{aligned}$$

By Proposition 6.1, eigenvectors of M_n with small eigenvalues have small projections to the low modes. Hence one expects that for small β the operator $\beta(M_n+\beta)^{-1}$ is small on vectors ψ with $\|\pi_\ell\psi\|\ge\alpha\|\psi\|$ whereas it is obviously bounded by one elsewhere. Combining the two steps we get the iteration

$$(42)\qquad \rho(n+2)=J_{n+1}\beta(M_n+\beta)^{-1}J_n\rho(n).$$

Combining Lemma 4.2, the bound (20) and Proposition 6.1, Hairer and Mattingly prove ([**13**], Lemma 4.16)

PROPOSITION 7.1. — *For every two constants $\gamma, \eta > 0$ and every $p \geq 1$, there exists a constant $\beta_0 > 0$ such that for n even*

$$\mathbb{E}\big(\|\rho_{n+2}\|^p \mid \mathcal{F}_n\big) \leq \gamma e^{\eta\|\omega_n\|^2}\|\rho_n\|^p$$

holds almost surely whenever $\beta \leq \beta_0$.

Iterating Proposition 7.1 the exponential decay of $\mathbb{E}\|\rho(t)\|$ then follows ([**13**], Lemma 4.13).

What remains is to bound the term $\mathbb{E}|D^*_\beta QV|$ in (39) uniformly in t, i.e., to bound the two integrals in (36). The crux of the matter here is that both terms can be written as a sum over n of factors proportional to $\rho(n)$ which provides a convergence factor. For the first term this is obvious by (40). For the second one we need to go back to the integration by parts formula Equation (31). By construction $V(t) = \int_0^t v(s)ds = \sum_n V_n$ where V_n is $\mathcal{F}_{n+2}$ measurable. Thus since the integration by parts is local in time $D^*_\beta V = \sum_n D^*_{\beta|_{[n,n+2]}} V_n$ and the second factor becomes

$$\sum_n \mathbb{E}\int_{[n,n+2]^2} \operatorname{tr}(\mathcal{D}_s v(r), \mathcal{D}_r v(s))dsdr. \tag{43}$$

For details of how to finish the argument we refer the reader to Section 4.8 in [**13**].

8. TURBULENCE

We have seen that the NS dynamics has a unique stationary state under very general forcing conditions. Moreover, it can be proven that the dynamics is mixing [**14**] and the stationary state is reached exponentially fast from arbitrary initial conditions and for arbitrary large Reynolds numbers R (for earlier proofs of mixing in the case where an R-dependent number of modes are forced, see [**6**], [**20**]). Does this mean we have reached some understanding on the properties of this state, in particular on the phenomenon of turbulence? The proof outlined in the previous sections uses properties of the system that have counterparts in the phenomenological theory of turbulence. These are the dissipation of the high Fourier modes and the transfer of the noise from the forced modes to the unforced ones due to nonlinearity. The latter point is significant because most results of the NS dynamics are based on the energy and enstrophy conservation laws alone, and those bounds would hold even if the nonlinearity was zero. Therefore, the properties of the latter are not used.

This being said it must be stressed that we have gained very little understanding of the actual nature of the invariant state. Crucial part of the proof is irreducibility which is based on the fact that $\omega = 0$ belongs to the support of every invariant measure. Recall that this holds, because there is a small probability that the random forces are close to zero for any given time interval so the fluid flow slows down due to viscosity. This is clearly not the true reason one sees fast approach to stationarity in physical experiments. The mixing times resulting from visits to the origin will be much larger than the ones observed. To understand the real mechanism for mixing one has to understand much better the transfer of energy and enstrophy from the forcing scale to other scales.

It was Kraichnan's observation **[16]** that we should expect this transfer to be in two dimensions quite different from the three dimensional case. In three dimensions, according to the Richardson-Kolmogorov picture the forcing in low modes injects into the system energy which is transported due to the nonlinearity in NS equation to the higher modes and eventually dissipated by the viscous term by large enough modes. This transport of energy through scales in wave number space (i.e., $|k| := \kappa$) is called the Richardson energy cascade. In fact the theory predicts a constant flux of energy from the injection scale (in our case 1) to the dissipation scale κ_ν (these claims can be formulated in terms of various correlation functions in the putative stationary state, see e.g. the review **[19]**). Kraichnan noted that the existence in 2d of the second conserved quantity of the inviscid flow, the enstrophy, means that one has to pose the question at what scales (if any) energy and enstrophy are dissipated and if there exist separate fluxes for the two. His observation was that the fluxes of energy and enstrophy are to opposite directions, energy flows towards *low* modes and enstrophy towards high ones. Moreover, energy tends to be not dissipated at all whereas enstrophy is dissipated at high modes like energy in the 3d case. The presence of the two cascades, the *direct cascade* of enstrophy and the *inverse cascade* of energy is very well established both numerically **[5]** and experimentally **[25]**. In what follows we will point out a couple of mathematical questions regarding this picture which would be nice to understand.

To state the Kraichnan picture more precisely it is convenient to work on a torus of size N, i.e., $\mathbb{T}_N^2 := (\mathbb{R}/(2\pi N\mathbb{Z})^2$ rather than $N = 1$ we had before. Of course by simple scaling we can get rid of the N at the expense of changing ν and the forcing scale, but since the theory involves large separations of the scales of dissipation, forcing and injection it is natural to take N large (eventually to infinity) we rather not do that. Consider now the NS dynamics on $\mathbb{T}_N^2$ with the random forcing on Fourier modes of size $|k| \sim \kappa_f \gg N^{-1}$ (observe that now $k \in (N^{-1}\mathbb{Z})^2$, i.e., $|k| \geq 1/N$). We shall add to the NS equation (3) an extra term that damps the low Fourier modes more strongly than the viscous term does (note that νk^2 can be as small as ν/N^2). This is the Ekman friction term $-\tau\omega$ for $\tau > 0$. Stationary states for this system exist for the same reasons as before and uniqueness should follow in the presence of the friction as without provided the conditions of Proposition 5.1 hold. The Kraichnan theory makes predictions on this stationary state, call it $\mu_{\nu,\tau,N}$ in the various limits $N \to \infty$, $\tau \to 0$ and $\nu \to 0$.

The starting point is conservation laws of energy and enstrophy following from the enstrophy balance equation (10) and a corresponding one for energy and taking into account the extra friction term in the equation. Since the unique stationary state is translation invariant these become local identities, for enstrophy

$$\nu \mathbb{E}_{\nu,\tau,N}(\nabla\omega(x))^2 + \tau \mathbb{E}_{\nu,\tau,N}(\omega(x))^2 = \epsilon \tag{44}$$

and analogously for energy

$$\nu \mathbb{E}_{\nu,\tau,N}(\nabla u(x))^2 + \tau \mathbb{E}_{\nu,\tau,N}(u(x))^2 = \epsilon' \tag{45}$$

with ϵ' the energy injection rate (per unit volume) which is proportional to $\epsilon\kappa_f^{-2}$. $\mathbb{E}_{\nu,\tau,N}$ denotes expectation in the measure $\mu_{\nu,\tau,N}$.

The first question to pose is what happens to the *viscous* dissipation of energy and enstrophy as $\nu \to 0$. All the evidence points to vanishing of energy dissipation

$$\lim_{\nu\to 0} \nu \mathbb{E}_{\nu,\tau,N}(\nabla u(x))^2 = 0. \tag{46}$$

Enstrophy dissipation is more subtle as we will see below, but again it is believed [**3**] that it vanishes:

$$\lim_{\nu\to 0} \nu \mathbb{E}_{\nu,\tau,N}(\nabla\omega(x))^2 = 0. \tag{47}$$

It would be interesting to prove these statements and also to understand whether a limiting measure $\lim_{\nu\to 0}\mu_{\nu,\tau,N}$ exists and is supported on solutions of the damped randomly forced Euler equation. Indeed, some indications that this could be done come from the work [**7**] where time averages of solutions and statistical solutions are controlled in that limit. They are shown to be given in terms of solutions of the Euler equation and in particular [**7**] prove the relation (47) in that setup.

The main predictions of the Kraichnan theory come from the limit $N \to \infty$ and $\tau \to 0$. The limit $N \to \infty$ means we are considering the NS dynamics in $\mathbb{R}^2$. It is an interesting problem to try to prove that the (weak) limit $\lim_{N\to\infty}\mu_{\nu,\tau,N} = \mu_{\nu,\tau,\infty}$ exists. Note that we do not expect this state to be supported on L^2 but rather on polynomially bounded (and presumably smooth) functions. The reason the large volume limit might exist is the damping of the low modes by the friction term. It produces an effective low wave number cutoff (which turns out to be $\sim \tau^{3/2}\epsilon'^{-\frac{1}{2}}$).

Granting this, what happens if we now take $\tau \to 0$? Is there also a measure $\mu_{\nu,0,\infty}$? The prediction of the Kraichnan theory is that the viscous energy dissipation (46) vanishes as $\nu \to 0$ *uniformly* in τ. Thus in that limit $\mathbb{E}_{\nu,\tau,\infty}(u(x))^2 = (\epsilon' - o(\nu))/\tau$, i.e., the average energy density is not bounded in the putative limiting measure $\mu_{\nu,0,\infty}$. However, it is believed that $\mu_{\nu,0,\infty}$ is supported on smooth ω and in particular

$\lim_{\tau\to 0}\tau\mathbb{E}_{\nu,\tau,\infty}(\omega(x))^2 = 0$. Then (44) implies *dissipative anomaly* for enstrophy: enstrophy dissipation remains nonzero as $\nu \to 0$, i.e., $\lim_{\nu\to 0}\nu\mathbb{E}_{\nu,0,\infty}(\nabla\omega(x))^2 = \epsilon > 0$.

The Kraichnan theory makes more quantitative predictions of the distribution of energy and enstrophy according to wave number. Define the *energy spectrum* for $\kappa \in \mathbb{R}_+$

$$e(\kappa) = 2\pi\hat{g}(\kappa)/\kappa \tag{48}$$

where $\hat{g}(|k|)$ is the Fourier transform of the vorticity 2-point function

$$g(x-y) = \mathbb{E}_{\nu,\tau,\infty}\omega(x)\omega(y).$$

Then energy density is given by

$$\mathbb{E}_{\nu,\tau,\infty}u(x)^2 = \int_0^\infty e(\kappa)d\kappa \tag{49}$$

and enstrophy density by $\int_0^\infty e(\kappa)\kappa^2 d\kappa$. Kraichnan theory predicts

$$e(\kappa) \sim \begin{cases} \varepsilon^{2/3}\kappa^{-3}, & \kappa_f \ll \kappa \ll \kappa_\nu \\ \epsilon'^{2/3}\kappa^{-5/3}, & \kappa_\tau \ll \kappa \ll \kappa_f, \end{cases} \tag{50}$$

where $\kappa_\nu \sim \nu^{-\frac{1}{2}}\epsilon^{\frac{1}{6}}$ is the dissipation scale and $\kappa_\tau \sim \tau^{3/2}\epsilon'^{-\frac{1}{2}}$ the friction scale.

The picture painted by the Kraichnan theory on 2d turbulence is thus quite complex. With well separated scales of viscous dissipation, injection and friction energy flows from the injection scale towards small wave numbers and is eventually dissipated by the friction. In the absence of friction and in infinite volume energy flows to ever smaller wave numbers and energy density is not defined in the stationary state. Enstrophy in turn flows to high wave numbers and is dissipated there by the viscosity. Only in the state $\mu_{\nu,0,\infty}$ as $\nu \to 0$ one expects to have constant fluxes of energy and enstrophy, for some exact calculations (subject to regularity assumptions), see **[3]**. One has to be careful with the order of limits as is seen from the behavior of enstrophy dissipation. Note in particular that the stationary state $\mu_{\nu,0,N}$ which we have been discussing in the previous sections does not exhibit turbulence in the sense of cascades of energy and enstrophy. Here energy will reside in low modes, indeed, in experiments one often sees the formation of a few large vortices in the flow. If ν is taken to zero in this state then both energy and enstrophy will blow up and indeed, no limit measure exists **[17]**. In **[17]** it is proven that only by taking the injection rate ϵ (and thus also ϵ') proportional to ν a nontrivial limiting measure exists. Formally this limit still corresponds to diverging Reynolds number, but one does not expect it to be a turbulent state with near constant fluxes of energy and enstrophy.

What makes the Kraichnan theory intriguing is that e.g. the spectrum (50) seems to be very well verified numerically and experimentally. Moreover, the invariant measure seems to possess strong scale invariance properties, at least in the inverse cascade regime. There are even indications of conformal invariance [4]. Thus it is not excluded that some of its properties could be mathematically accessible.

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