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PERIODIC TWISTED COHOMOLOGY AND T-DUALITY

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PERIODIC TWISTED COHOMOLOGY AND T-DUALITY

Ulrich BUNKE, Thomas SCHICK and Markus SPITZWECK

Abstract. — Using the differentiable structure, twisted 2-periodic de Rham cohomology is well known, and showing up as the target of Chern characters for twisted K-theory. The main motivation of this work is a topological interpretation of two-periodic twisted de Rham cohomology which is generalizable to arbitrary topological spaces and at the same time to arbitrary coefficients.

To this end we develop a sheaf theory in the context of locally compact topological stacks with emphasis on:

— the construction of the sheaf theory operations in unbounded derived categories
— elements of Verdier duality
— and integration.

The main result is the construction of a functorial periodization associated to a $U(1)$-gerbe.

As an application we verify the T-duality isomorphism in periodic twisted cohomology and in periodic twisted orbispace cohomology.

***Résumé* (Cohomologie périodique tordue et T-dualité).** — La cohomologie de de Rham tordue (periodique de période 2) est une construction bien connue, elle est importante en tant que codomaine d'un caractère de Chern pour la K-theorie tordue.

La motivation principale de notre livre est une interprétation topologique de la cohomologie de de Rham tordue, une interprétation avec géneralisations á des espaces et coefficients arbitraires.

Dans ce but, nous développons une théorie des faisceaux sur des piles topologiques localement compactes, et plus particulièrement :

— la construction des opérations de la théorie des faisceaux dans les catégories derivées non-bornées,
— les élements de la dualité de Verdier,
— et l'integration.

Notre résultat principal est la construction d'une périodisation fonctorielle associé a une $U(1)$-gerbe.

Parmi les applications, citons la vérification d'un isomorphisme de T-dualité pour la cohomologie périodique tordue et celle des orbi-espaces.

CONTENTS

1. Introduction 1
1.1. Periodic twisted cohomology 1
1.2. T-duality 4
1.3. Duality for sheaves on locally compact stacks 6

2. Gerbes and periodization 11
2.1. Sheaves on the locally compact site of a stack 11
2.2. Algebraic structures on the cohomology of a gerbe 13
2.3. Identification of the transformation D_G in the smooth case 17
2.4. Two-periodization — up to isomorphism 22
2.5. Calculations 25

3. Functorial periodization 29
3.1. Flabby resolutions 29
3.2. A model for the push-forward 34
3.3. Zig-zag diagrams and limits 39
3.4. The functorial periodization 43
3.5. Properties of the periodization functor 48
3.6. Periodicity 54

4. T-duality 57
4.1. The universal T-duality diagram 57
4.2. T-duality and periodization diagrams 60
4.3. Twisted cohomology and the T-duality transformation 63

5. Orbispaces 69
5.1. Twisted periodic delocalized cohomology of orbispaces 69
5.2. The T-duality transformation in twisted periodic delocalized cohomology 71
5.3. The geometry of T-duality diagrams over orbispaces 73
5.4. The T-duality transformation in twisted periodic delocalized cohomology is an isomorphism 74

6. Verdier duality for locally compact stacks 79
6.1. Elements of the theory of stacks on Top and sheaf theory 79
6.2. Tensor products and the projection formula 91

6.3. Verdier duality for locally compact stacks in detail 100
6.4. The integration map 106
6.5. Operations with unbounded derived categories 109
6.6. Extended sites 127

Bibliography 133

CHAPTER 1

INTRODUCTION

1.1. Periodic twisted cohomology

1.1.1. — The twisted de Rham cohomology $H_{dR}(M,\omega)$ of a manifold M equipped with a closed three form $\omega \in \Omega^3(M)$ is the two-periodic cohomology of the complex

$$\Omega(M,\omega)_{\mathrm{per}}\colon \cdots \to \Omega^{\mathrm{ev}}(M) \stackrel{d_\omega}{\to} \Omega^{\mathrm{odd}}(M) \stackrel{d_\omega}{\to} \Omega^{\mathrm{ev}}(M) \to \cdots , \tag{1.1.1}$$

where $d_\omega := d_{dR}+\omega$ is the sum of the de Rham differential and the operation of taking the wedge product with the form ω. The two-periodic twisted de Rham cohomology is interesting as the target of the Chern character from twisted K-theory [**1**], [**19**], [**3**], or as a cohomology theory which admits a T-duality isomorphism [**4**], [**7**].

1.1.2. — In [**9**] we developed a sheaf theory for smooth stacks. Let $f\colon G \to X$ be a gerbe with band $U(1)$ over a smooth stack X, and consider a closed three-form $\omega \in \Omega^3_X(X)$ which represents the image of the Dixmier-Douady class of the gerbe $G \to X$ in de Rham cohomology. The main result of [**9**] states that there exists an isomorphism

$$Rf_*f^*\underline{\mathbb{R}}_{\mathbf{X}} \stackrel{\sim}{\leftarrow} \Omega_X[[z]]_\omega \tag{1.1.2}$$

in the bounded below derived category $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ of sheaves of abelian groups on X. Here $\underline{\mathbb{R}}_{\mathbf{X}}$ denotes the constant sheaf with value $\mathbb{R}$ on X. Furthermore, $\Omega_X[[z]]_\omega$ is the sheaf of formal power series of smooth forms on X, where $\deg(z) = 2$, and its differential is given by $d_\omega := d_{dR} + \omega\frac{d}{dz}$. The isomorphism is not canonical, but depends on the choice of a connection on the gerbe G with characteristic form ω.

1.1.3. — The complex (1.1.1) can be defined for a smooth stack X equipped with a three-form $\omega \in \Omega^3_X(X)$. It is the complex of global sections of a sheaf of two-periodic complexes $\Omega_{X,\omega,\mathrm{per}}$ on X. The complex of sheaves $\Omega_X[[z]]_\omega$ is not two-periodic. The relation between $\Omega_X[[z]]_\omega$ and $\Omega_{X,\omega,\mathrm{per}}$ has been discussed in [**9**, 1.3.23]. Consider the diagram

$$\mathcal{D}\colon \Omega(X)[[z]]_\omega \stackrel{\frac{d}{dz}}{\leftarrow} \Omega(X)[[z]]_\omega \stackrel{\frac{d}{dz}}{\leftarrow} \Omega(X)[[z]]_\omega \stackrel{\frac{d}{dz}}{\leftarrow} \cdots . \tag{1.1.3}$$

Then there exists an isomorphism

$$\Omega_{X,\omega,\mathrm{per}} \cong \operatorname{holim} \mathcal{D} \ . \tag{1.1.4}$$

1.1.4. — As mentioned above, the isomorphism (1.1.2) depends on the choice of a connection on the gerbe G. Moreover, the diagram $\mathcal{D}$ depends on these choices via ω. In order to construct a natural two-periodic cohomology one must find a natural replacement of the operation $\frac{d}{dz}$ which acts on the left-hand side $Rf_*f^*\underline{\mathbb{R}}_{\mathbf{X}}$ of (1.1.2). It is the first goal of this paper to carry this out properly.

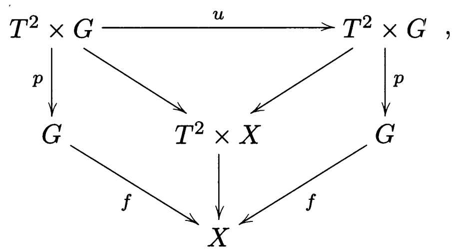
1.1.5. — One can do this construction in the framework of smooth stacks developed in [**9**]. But for the present paper we choose the setting of topological stacks. Only in Subsection 2.3 we work in smooth stacks and discuss the connection with [**9**]. In Section 6 we develop some aspects of the theory of locally compact stacks and the sheaf theory in this context. For the purpose of this introduction we freely use notions and constructions from this theory. We hope that the ideas are understandable by analogy with the usual case of sheaf theory on locally compact spaces.

1.1.6. — Let $G \to X$ be a $U(1)$-banded gerbe over a locally compact stack. The main object of the present paper is a periodization functor

$$P_G : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

which is functorial in $G \to X$, and where $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ and $D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ denote the bounded below and unbounded derived categories of sheaves of abelian groups on the site $\mathbf{X}$ of the stack X. A simple construction of the isomorphism class of $P_G(F)$ is given in Definition 2.4.2. The functorial version is much more complicated. Its construction is completed in Definition 3.4.5.

1.1.7. — Let us sketch the construction of P_G. Recall that gerbes with band $U(1)$ over a locally compact stack Y are classified by $H^3(Y;\mathbb{Z})$, and automorphisms of a given $U(1)$-gerbe are classified by $H^2(Y;\mathbb{Z})$ [**14**]. We consider the diagram



where the automorphism u of gerbes over $T^2 \times X$ is classified by $\mathrm{or}_{T^2} \times 1 \in H^2(T^2 \times X;\mathbb{Z})$, and where or_{T^2} denotes the orientation class of the two-torus. We define a natural transformation

$$D\colon Rf_*f^* \to Rf_*f^*\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

of degree -2 as the composition

$$D\colon Rf_*f^* \xrightarrow{\text{units}} Rf^*Rp_*Ru_*u^*p^*f^* \xrightarrow{fpu=fp} Rf_*Rp_*p^*f^* \xrightarrow{\int_p} Rf_*f^* \ ,$$

where $\int_p\colon Rp_*p^* \to \mathrm{id}$ is the integration map of the oriented T^2-bundle $T^2 \times G \to G$.

For $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ we form the diagram

$$\mathcal{J}_G(F)\colon Rf_*f^*(F) \xleftarrow{D} Rf_*f^*(F)[2] \xleftarrow{D} Rf_*f^*(F)[4] \xleftarrow{D} \cdots$$

in $D(\mathrm{Sh_{Ab}}\mathbf{X})$.

Definition 1.1.5. — *We define the periodization $P_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})$ of F by*

$$P_G(F) := \operatorname{holim} \mathcal{J}_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X}) \ .$$

Note that this introduction is meant as a sketch. In particular, one has to be aware of the fact that the notion of holim in a triagulated category is ambiguous and has to be used with great care, as will be explained below and in the body of the paper. At present, the above definition only fixes the isomorphism class of $P_G(F)$.

1.1.8. — The same construction can be applied in the case of smooth stacks X. It is an immediate consequence of Theorem 2.3.2 that there exists an isomorphism of the diagrams $S_G(\underline{\mathbb{R}}_{\mathbf{X}})$ and $\mathcal{D}$ (see (1.1.3)). Equation (1.1.4) implies the following result.

Corollary 1.1.6. — *If X is a smooth manifold, then there exists an isomorphism*

$$P_G(\underline{\mathbb{R}}_{\mathbf{X}}) \cong \Omega_{X,\omega,\mathrm{per}}$$

in $D(\mathrm{Sh_{Ab}}\mathbf{X})$. In particular we have an isomorphism of two-periodic cohomology groups $H^*_{dR}(X,\omega) \cong H^*(X; P_G(\underline{\mathbb{R}}_{\mathbf{X}}))$.

The existence of this isomorphism played the role of a design criterion for the construction of the periodization functor P_G.

1.1.9. — The operation $D\colon Rf_*f^*(F) \to Rf_*f^*(F)$ is a well-defined morphism in the derived category. In particular, we get a well-defined diagram $\mathcal{J}_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})^{\mathbb{N}^{\mathrm{op}}}$, where we consider the ordered set $\mathbb{N}$ as a category. This determines the isomorphism class of the object $P_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})$. We actually want to define a periodization *functor*

$$P_G\colon D^+(\mathrm{Sh_{Ab}}\mathbf{X}) \to D(\mathrm{Sh_{Ab}}\mathbf{X}) \ ,$$

which also depends functorially on the gerbe $G \to X$. These functorial properties are required in our applications to T-duality, or if one wants to formulate a statement about the naturality of a Chern character from G-twisted K-theory with values in the periodic twisted cohomology $H^*(X; P_G(\underline{\mathbb{R}}_{\mathbf{X}}))$.

In order to define $P_G(F)$ in a functorial way we must refine the diagram $\mathcal{J}_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})^{\mathbb{N}^{\mathrm{op}}}$ to a diagram in $D((\mathrm{Sh_{Ab}}\mathbf{X})^{\mathbb{N}^{\mathrm{op}}})$. This is the technical heart of the present paper. The details of this construction are contained in Section 3 and will be completed in Definition 3.4.5. Along the way, we have to use the enhancement of the category of sheaves to bounded below complexes of flasque sheaves.

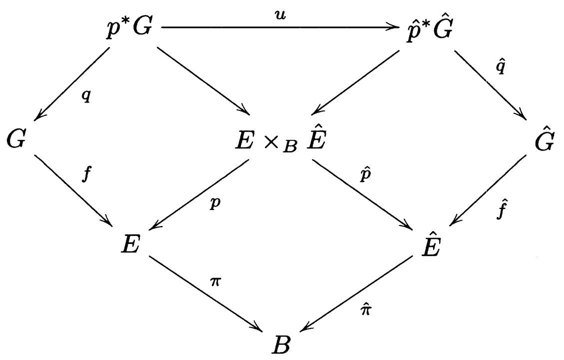
1.1.10. — The periodization functor P_G can be applied to arbitrary objects in $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$. In Proposition 2.5.1 we calculate examples which indicate some interesting arithmetic features of this functor.

1.2. T-duality

1.2.1. — Topological T-duality is a concept which models the underlying topology of mirror symmetry in algebraic geometry or T-duality in string theory. We refer to [**6**] for a more detailed discussion of the literature. In the present paper we introduce the concept of T-duality for pairs (E, G) of a $U(1)$-principal bundle $E \to B$ over a topological stack B together with a topological gerbe $G \to E$ with band $U(1)$ using the notion of a T-duality diagram.

1.2.2. — Consider a diagram

(1.2.1)



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where $\pi, \hat{\pi}$ are $U(1)$-principal bundles, and $f, \hat{f}$ are gerbes with band $U(1)$. In 4.1.3 we describe the isomorphism class of the universal T-duality diagram over the classifying stack $\mathcal{B}U(1)$.

Definition 1.2.2 **(Definition 4.1.3).** — *The diagram* (1.2.1) *is a T-duality diagram, if it is locally isomorphic to the universal T-duality diagram.*

The pair $(\hat{G}, \hat{E})$ is then called a T-dual of (E, G).

1.2.3. — In Lemma 4.1.5 we will check that this generalizes the concept of T-duality (for $U(1)$-bundles) from the classical situation of principal bundles in the category of spaces [**6, 8**] and the slightly more general situation of such bundles in orbispaces [**8**] to arbitrary $U(1)$-actions. The situation of semi-free actions is discussed (in a completely different way) in [**24**]. It is an interesting open problem to relate his approach to the approach used here.

1.2.4. — One of the main themes of topological T-duality is the T-duality transformation in twisted cohomology theories. In [**8**] we observed that if the T-duality transformation is an isomorphism, then the corresponding twisted cohomology theory must be two-periodic.

This applies e.g. to twisted K-theory. In fact, one can argue that twisted K-theory is the universal twisted cohomology theory for which the T-duality transformation is an isomorphism [(1)].

1.2.5. — Our construction of P_G is designed such that the corresponding T-duality transformation is an isomorphism. To this end we define the periodic G-twisted cohomology of E with coefficients in π^*F, $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{B})$, by

$$H^*_{\mathrm{per}}(E, G; \pi^*F) := H^*(E; P_G(\pi^*F)) \ .$$

In this case the T-duality transformation

$$T\colon H^*_{\mathrm{per}}(E, G; \pi^*F) \to H^*_{\mathrm{per}}(\hat{E}, \hat{G}; \hat{\pi}^*F)$$

is induced by the composition

$$\begin{array}{rcl} R\pi_* P_G(\pi^*F) & \stackrel{\mathrm{unit}}{\to} & R\pi_* Rp_* p^* P_G(\pi^*F) \\ & \cong & R\pi_* Rp_* P_{p^*G}(p^*\pi^*F) \\ & \stackrel{u^*}{\cong} & R\pi_* Rp_* P_{\hat{p}^*\hat{G}}(p^*\pi^*F) \\ & \stackrel{\pi p = \hat{\pi}\hat{p}}{\to} & R\hat{\pi}_* R\hat{p}_* P_{\hat{p}^*\hat{G}}(\hat{p}^*\hat{\pi}^*F) \\ & \stackrel{\cong}{\to} & R\hat{\pi}_* R\hat{p}_* \hat{p}^* P_{\hat{G}}(\hat{\pi}^*F) \\ & \stackrel{\int_{\hat{p}}}{\to} & R\hat{\pi}_* P_{\hat{G}}(\hat{\pi}^*(F)) \ . \end{array}$$

Note that here we use the functoriality of the periodization in an essential way.

***Theorem 1.2.3* (Theorem 4.3.7).** — *The T-duality transformation in twisted periodic cohomology is an isomorphism.*

1.2.6. — If $G \to X$ is a gerbe over a nice non-singular space X, then $H^*_{\mathrm{per}}(X, G; \mathbb{R}_{\mathbf{X}})$ is the correct target of a Chern character from twisted K-theory. If X is a topological stack with non-trivial automorphisms of points, then this is no longer correct. At the moment we do understand the special case of orbispaces. In [**10**, Sec. 1.3] we give a detailed motivation for the introduction of the twisted delocalized cohomology.

(1) We thank M. Hopkins for pointing out a proof of this fact.

Let $G \to X$ be a topological gerbe with band $U(1)$ over an orbispace X. In [**10**, Definition 3.4] we show that it gives rise to a sheaf $\mathcal{L} \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{LX}$, where LX is the loop orbispace of X.

The G-twisted delocalized periodic cohomology of X (with complex coefficients) is defined as (see [**10**, Definition 3.5])

$$H^*_{\mathrm{deloc,per}}(X, G) := H^*(LX; P_{G_L}(\mathcal{L})) ,$$

where $G_L \to LX$ is defined by the pull-back

$$\begin{array}{ccc} G_L & \longrightarrow & G \\ \downarrow & & \downarrow \\ LX & \longrightarrow & X \end{array} .$$

Let us now consider a T-duality diagram (1.2.1) over an orbispace B. Then we define a T-duality transformation

$$T\colon H^*_{\mathrm{deloc,per}}(E, G) \to H^*_{\mathrm{deloc,per}}(\hat{E}, \hat{G})$$

by a modification of the construction 1.2.5.

***Theorem 1.2.4* (Theorem 5.4.2).** — *The T-duality transformation in twisted delocalized periodic cohomology is an isomorphism.*

So the situation with twisted delocalized periodic cohomology is better than with orbispace K-theory. At the moment we do not know a proof that the T-duality transformation in twisted orbifold K-theory is an isomorphism (see the corresponding comments in [**8**]). Using the fact that the Chern character is an isomorphism, our result implies that the T-duality transformation in twisted orbifold and orbispace K-theory is an isomorphism after complexification.

1.3. Duality for sheaves on locally compact stacks

1.3.1. — In Section 6 of the present paper we develop some features of a sheaf theory for locally compact stacks. Our main results are the construction of the basic setup, of the functor $f^!$, and the integration $\int_f$ for oriented fiber bundles. Section 6 not only provides the technical background for the applications of sheaf theory in the previous sections, but also contains some additional material of independent interest (in particular the results connected with $f^!$).

1.3.2. — A presheaf F of sets on a topological space X associates to each open subset $U \subseteq X$ a set of sections $F(U)$, and to every inclusion $V \to U$ of open subsets a functorial restriction map $F(U) \to F(V)$, $s \mapsto s_{|V}$. In short, a presheaf is a contravariant functor from the category (X) of open subsets of X to sets. A presheaf is a sheaf if it has the following two properties:

(1) If $s, t \in F(U)$ are two sections and there exists an open covering (U_i) of U such that $s_{|U_i} = t_{|U_i}$ for all i, then $s = t$.
(2) If (U_i) is an open covering of U and (s_i) is a collection of sections $s_i \in F(U_i)$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all pairs i, j, then there exists a section $s \in F(U)$ such that $s_{|U_i} = s_i$ for all i.

The notion of a sheaf is thus determined by the Grothendieck topology on (X) given by the collections of open coverings of open subsets. We will call (X) the small site associated to X.

If X is a topological stack, then the open substacks form a two-category which does not give the appropriate setting for sheaf theory on X. For example, if G is a finite group, then the quotient stack $[*/G]$ is quite non-trivial but does not have proper open substacks. On the other hand its identity one-morphism has the two-automorphism group G, and in a non-trivial theory sheaves should reflect the two-automorphisms.

1.3.3. — For applications to twisted cohomology a setting for sheaf theory on smooth stacks has been introduced in [**9**]. In the present paper we develop a similar theory for topological stacks. There are various choices to be made in order to define the site of a stack in topological spaces. The sheaf theories associated to these choices will have many features in common, but will differ in others. The main goal of the present paper is the construction of the periodization functor P_G associated to a $U(1)$-banded gerbe $G \to X$. One of the main ingredients of the construction is an integration $\int_f$ for oriented fiber bundles f with a closed topological manifold as fiber. In order to define the integration map we need a projection formula which expresses a compatibility of the pull-back and push-forward operations with tensor products, see Lemma 6.2.11. Already for the projection formula in ordinary sheaf theory one needs local compactness assumptions. For this reason we decided to work generally with locally compact stacks and spaces though much of the theory would go through under more general or different assumptions.

1.3.4. — A stack in topological spaces is topological if it admits an atlas $A \to X$. From the atlas we can derive a groupoid $A \times_X A \rightrightarrows A$ which represents X in an appropriate sense. The stack is called locally compact if one can find an atlas $A \to X$ such that the resulting groupoid is locally compact (i.e. A and $A \times_X A$ are locally compact spaces).

The site $\mathbf{X}$ associated to a locally compact stack is the category of locally compact spaces $(U \to X)$ over X such that the morphisms are morphisms of spaces over X (i.e. pairs of a morphism between the spaces and a two-morphism filling the obvious triangle.) We require that the structure morphism $U \to X$ has local sections. The topology on $\mathbf{X}$ is again given by the collections of coverings by open subsets of the objects $(U \to X)$. For many constructions and calculations the restriction functors from sheaves on $\mathbf{X}$ to sheaves on (U) play a distinguished role. They are used to build the connection between operations with sheaves on the stack X and corresponding classical operations in sheaf theory on the spaces U.

1.3.5. — For the theory of stacks in topological spaces in general we refer to [**14**], [**10**], [**22**]. Some special aspects of locally compact stacks are discussed in Subsection 6.1 of the present paper.

In our treatment of sheaf theory on the site $\mathbf{X}$ we give a description of the closed monoidal structure on the categories of sheaves and presheaves of abelian groups $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ and $\mathrm{Pr}_{\mathrm{Ab}}\mathbf{X}$ on $\mathbf{X}$. The interplay between sheaves and presheaves will be important when we study the compatibility of the monoidal structures with the functors

$$f^* : \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y} \leftrightarrows \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} : f_*$$

associated to a morphism of locally compact stacks $f : X \to Y$. In general these functors do not come from a morphisms of sites but are constructed in an ad-hoc manner. Because of this we must check under which conditions properties expected from the classical theory carry over to the present case.

The derived versions of these functors on the bounded below and unbounded derived categories $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ and $D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ will play an important role in the present paper. In order to deal with the unbounded derived category we use an approach via model categories.

1.3.6. — Besides the development of the basic set up which we will not discuss further in the introduction let us now explain the two main results which may be of independent interest.

***Theorem 1.3.1* (Theorem 6.3.2).** — *If $f : X \to Y$ is a proper representable map between locally compact stacks such that f_* has finite cohomological dimension, then the functor $Rf_* : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ has a right-adjoint, i.e. we have an adjoint pair*

$$Rf_* : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \leftrightarrows D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) : f^! \ . \tag{1.3.2}$$

We think that one could prove a more general theorem stating the existence of a right adjoint of a functor $Rf_!$ where $f_!$ is the push-forward with proper support along an arbitrary map between locally compact stacks such that $f_!$ has finite cohomological dimension, though we have not checked all details.

This theorem generalizes a well-known result ([**26**], [**17**, Ch. 3]) in ordinary sheaf theory. Its importance is due to the classical calculation

$$f^!(F) \cong f^*(F)[n] \tag{1.3.3}$$

(compare [**17**, Prop.3.3.2]) for $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}(Y))$, if $f : X \to Y$ is an oriented locally trivial bundle of closed connected topological n-dimensional manifolds on a locally compact space Y. If we would know such an isomorphism in the present case (for sheaves on the sites $\mathbf{X}, \mathbf{Y}$ and stacks X, Y), then we could define the integration map as the composition

$$\int_f : Rf_*f^*(F) \xrightarrow{\sim} Rf_*f^!(F)[-n] \stackrel{\text{counit}}{\to} F[-n] \ ,$$

where the last map is the co-unit of the adjunction (1.3.2).

Unfortunately, at the moment we are not able to calculate $f^!(F)$ in any interesting example. However, we can construct the integration map in a direct manner avoiding the knowledge of (1.3.3).

Some elements of the theory developed here are formally similar to the work [**23**] on sheaves on the lisse étale site of an Artin stack. In this framework in [**18**] a functor $f^!$ was introduced between derived categories of constructible sheaves. On the one hand the methods seem to be completely different. On the other hand this functor has the expected behavior for smooth maps, i.e. it satisfies a relation like (1.3.3). At the moment we do not see even a formal relation between the construction of [**18**] with the construction in the present paper which could be exploited for a calculation of $f^!(F)$.

1.3.7. — The following Theorem is the result of Subsection 6.4.

Theorem 1.3.4. — *If the map $f : X \to Y$ of locally compact stacks is an oriented locally trivial fiber bundle with a closed connected topological n-dimensional manifold as fiber, then there exists an integration map, a natural transformation of functors*

$$\int_f : Rf_*f^* \to \mathrm{id}[-n] : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

which has the expected compatibility with pull-back and compositions.

In Subsection 6.5 we extend the push-forward and pull-back operations to the unbounded derived categories and construct the integration map in this setting.

CHAPTER 2

GERBES AND PERIODIZATION

2.1. Sheaves on the locally compact site of a stack

2.1.1. — Let Top denote the site of topological spaces. The topology is generated by covering families $\mathrm{cov}_{\mathrm{Top}}(A)$ of the objects $A \in \mathrm{Top}$, where $\mathrm{cov}_{\mathrm{Top}}(A)$ is the set of coverings by collections of open subsets.

A stack will be a stack on the site Top. Spaces are considered as stacks through the Yoneda embedding.

A map $A \to X$ from a space A to a stack X which is surjective, representable, and has local sections is called an atlas. We refer to 6.1.2 for definitions and more details about stacks in topological spaces.

Definition 2.1.1. — *A topological stack is a stack which admits an atlas.*

Definition 2.1.2. — *A topological space is locally compact if it is Hausdorff and every point admits a compact neighborhood. A stack is called locally compact if it admits an atlas $A \to X$ such that A and $A \times_X A$ are locally compact.*

If X is a locally compact stack, then the site of X is the subcategory $\mathrm{Top}_{\mathrm{lc}}/X$ of locally compact spaces over X such that the structure map $A \to X$ has local sections. The topology is induced from Top. We denote this site by $\mathbf{X}$ or $\mathrm{Site}(X)$. See 6.1.6 for more details.

2.1.2. — As will be explained in 6.1.9, a morphism of locally compact stacks $f\colon X \to Y$ gives rise to an adjoint pair of functors

$$f^*\colon \mathrm{Sh}\mathbf{Y} \leftrightarrows \mathrm{Sh}\mathbf{X} : f_* \ .$$

The functor f_* is left-exact on the categories of sheaves of abelian groups and admits a right-derived

$$Rf_*\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$$

between the bounded below derived categories, compare 6.1.9.

2.1.3. — Let M be some space.

Definition 2.1.3. — *A map between topological stacks $f : X \to Y$ is a locally trivial fiber bundle with fiber M if for every space $U \to X$ the pull-back $U \times_Y X \to U$ is a locally trivial fiber bundle of spaces with fiber M.*

Assume that M is a closed connected and orientable n-dimensional topological manifold.

Definition 2.1.4. — *Let $f: X \to Y$ be a map of locally compact stacks which is a locally trivial fiber bundle with fiber M. It is called orientable if there exists an isomorphism $R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}) \cong \underline{\mathbb{Z}}_{\mathbf{Y}}$. An orientation of f is a choice of such an isomorphism.*

2.1.4. — Let $f: X \to Y$ be a locally trivial oriented fiber bundle with n-dimensional fiber M over a locally compact stack Y. Under these assumptions we can generalize the integration map (see [**17**, Sec. 3.3])

Theorem 2.1.5 **(Definition 6.4.6).** — *If $f: X \to Y$ be a locally trivial oriented fiber bundle over a locally compact stack with fiber a closed topological manifold of dimension n, then we have an integration map, i.e. a natural transformation of functors*

$$\int_f : Rf_* \circ f^* \to \mathrm{id}: D^+(\mathrm{Sh_{Ab}}\mathbf{Y}) \to D^+(\mathrm{Sh_{Ab}}\mathbf{Y})$$

of degree $-n$.

2.1.5. — We consider a map of locally compact stacks $f: X \to Y$ which is a locally trivial oriented fiber bundle with fiber a closed topological manifold of dimension n. Furthermore let $U \to X$ be a morphism of locally compact stacks which has local sections. Then we form the Cartesian [(1)] diagram

$$\begin{array}{ccc} V & \xrightarrow{v} & X \\ \downarrow g & & \downarrow f \\ U & \xrightarrow{u} & Y. \end{array}$$

Note that $g : V \to U$ is again a locally trivial oriented fiber bundle with fiber a closed topological manifold of dimension n. The orientation of f (which gives the marked

(1) In the present paper by a Cartesian diagram in the two-category of stacks we mean a 2-Cartesian diagram. In particular, the square commutes up to a 2-isomorphism which we often omit to write in order to simplify the notation. More generally, when we talk about a commutative diagram in stacks, then we mean a diagram of 1-morphisms together with a collection of 2-isomorphism filling all faces in a compatible way, and again we will usually not write the 2-isomorphisms explicitly.

isomorphism below) induces an orientation of g by

$$R^n g_*(\mathbb{Z}_{\mathbf{V}}) \cong R^n g_* v^*(\mathbb{Z}_{\mathbf{X}}) \overset{(6.1.15)}{\cong} u^* R^n f_*(\mathbb{Z}_{\mathbf{X}}) \overset{!}{\cong} u^*(\mathbb{Z}_{\mathbf{Y}}) \cong \mathbb{Z}_{\mathbf{U}} \ .$$

Lemma 2.1.6. — *The following diagrams commute*

$$(2.1.7)\quad \begin{array}{ccc} u^* \circ Rf_* \circ f^* & \xrightarrow{\cong} & Rg_* \circ v^* \circ f^* \\ \big\downarrow{\scriptstyle u^* \int_f} & & \big\downarrow{\scriptstyle \cong} \\ u^* & \xleftarrow[\int_g]{} & Rg_* \circ g^* \circ u^* \end{array} \qquad \begin{array}{ccc} Ru_* \circ Rg_* \circ g^* & \xrightarrow{\cong} & Rf_* \circ Rv_* \circ g^* \\ \big\downarrow{\scriptstyle Ru_* \int_g} & & \big\downarrow{\scriptstyle \cong} \\ Ru_* & \xleftarrow[\int_f Ru_*]{} & Rf_* \circ f^* \circ Ru_* \end{array} \ .$$

Proof. — Commutativity of the first diagram follows immediately from the stronger (because valid in the derived category of unbounded complexes) Lemma 6.5.31. Commutativity of the second diagram is proved in Lemma 6.5.31, but only for the bounded below derived category. □

2.2. Algebraic structures on the cohomology of a gerbe

2.2.1. — Let X be a locally compact stack and $f\colon G \to X$ be a topological gerbe with band $U(1)$. Then G is a locally compact stack. Indeed, we can choose an atlas $A \to X$ such that A and $A \times_X A$ are locally compact, and there exists a section

$$\begin{array}{ccc} & \nearrow & G \\ & & \big\downarrow \\ A & \longrightarrow & X \end{array} \ .$$

Then $A \to G$ is an atlas and $A \times_G A \to A \times_X A$ is a locally trivial $U(1)$-bundle. In particular, $A \times_G A$ is a locally compact space.

2.2.2. — By T^2 we denote the two-dimensional torus. We fix an orientation of T^2. We consider the pull-back $\mathrm{pr}_2^* G \cong T^2 \times G \to T^2 \times X$. The isomorphism classes of automorphisms of this gerbe are classified by $H^2(T^2 \times X; \mathbb{Z})$. Let

$$\begin{array}{ccc} \mathrm{pr}_2^* G & \xrightarrow{\ \phi\ } & \mathrm{pr}_2^* G \\ & \searrow \quad \swarrow & \\ & T^2 \times X & \end{array}$$

be an automorphism classified by $\mathrm{or}_{T^2} \times 1_X \in H^2(T^2 \times X; \mathbb{Z})$. We consider the diagram

(2.2.1)
$$\begin{array}{ccccc}
\mathrm{pr}_2^* G & & \xrightarrow{\phi} & & \mathrm{pr}_2^* G \\
\downarrow p & \searrow & & \swarrow & \downarrow p \\
G & & T^2 \times X & & G \\
& {}_f\searrow & \downarrow & \swarrow_f & \\
& & X & &
\end{array} .$$

Notice that ϕ is unique up to a non-canonical 2-isomorphism. In the present paper we prefer a more canonical choice. We will fix the morphism ϕ once and for all in the special case that X is a point and $G = \mathcal{B}U(1)$, i.e. we fix a diagram

$$\begin{array}{ccccc}
T^2 \times \mathcal{B}U(1) & & \xrightarrow{\phi_{\mathrm{univ}}} & & T^2 \times \mathcal{B}U(1) \\
\downarrow & \searrow & & \swarrow & \downarrow \\
\mathcal{B}U(1) & & T^2 & & \mathcal{B}U(1) \\
& \searrow & \downarrow & \swarrow & \\
& & * & &
\end{array} .$$

If $G \to X$ is a topological gerbe with band $U(1)$, then we obtain the induced diagram by taking products

$$\begin{array}{ccccc}
G \times T^2 \times \mathcal{B}U(1) & & \xrightarrow{\mathrm{id}_G \times \phi_{\mathrm{univ}}} & & G \times T^2 \times \mathcal{B}U(1) \\
\downarrow & \searrow & & \swarrow & \downarrow \\
G \times \mathcal{B}U(1) & & X \times T^2 & & G \times \mathcal{B}U(1) \\
& \searrow & \downarrow & \swarrow & \\
& & X & &
\end{array} .$$

We now replace the products $\mathcal{B}U(1) \times G$ by the tensor product of gerbes as explained in [**11**, 6.1.9] and identify $\mathcal{B}U(1) \otimes G$ with G using the canonical isomorphism in order to get

$$\begin{array}{ccccc}
\mathrm{pr}_2^* G & & \xrightarrow{\phi} & & \mathrm{pr}_2^* G \\
\downarrow p & \searrow & & \swarrow & \downarrow p \\
G & & T^2 \times X & & G \\
& {}_f\searrow & \downarrow & \swarrow_f & \\
& & X & &
\end{array} .$$

In this way we have constructed a 2-functor from the 2-category of $U(1)$-banded gerbes over X to the 2-category of diagrams of the form (2.2.1). By taking prefered models for the products we can, if we want, assume a strict equality $f \circ p \circ \phi_G = f \circ p$.

2.2.3. — Observe that the map of locally compact stacks $p\colon \mathrm{pr}_2^*G \to G$ is a locally trivial oriented fiber bundle with fiber T^2. Therefore we have the integration map (see 2.1.5)

$$\int_p : Rp_* \circ p^* \to \mathrm{id} \ .$$

***Definition* 2.2.2.** — *We define a natural endo-transformation D_G of the functor*

$$Rf_* \circ f^*\colon D^+(\mathrm{Sh_{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh_{Ab}}\mathbf{X})$$

of degree -2 which associates to $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ the morphism

$$Rf_* \circ f^*(F) \xrightarrow{\text{units}} Rf_* \circ Rp_* \circ R\phi_* \circ \phi^* \circ p^* \circ f^*(F)$$
$$\xrightarrow{f\circ p\circ\phi = f\circ p} Rf_* \circ Rp_* \circ p^* \circ f^*(F) \xrightarrow{\int_p} Rf_* \circ f^*(F) \ .$$

2.2.4. — It follows from Lemma 2.1.6 that D_G is compatible with pull-back diagrams. In fact, consider a Cartesian diagram

$$\begin{array}{ccc} G' & \longrightarrow & G \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array} \ .$$

Using the canonical construction explained in 2.2.2 we extend this to a morphism between diagrams of the form (2.2.1). Then we have the commutative diagram

$$\begin{array}{ccc} g^* \circ Rf_* \circ f^* & \xrightarrow{\sim} & Rf'_* \circ (f')^* \circ g^* \\ \Big\downarrow{\scriptstyle g^*D_G} & & \Big\downarrow{\scriptstyle D_{G'} \circ g^*} \\ g^* \circ Rf_* \circ f^* & \xrightarrow{\sim} & Rf'_* \circ (f')^* \circ g^* \end{array} \ .$$

2.2.5. — We compute the action of D_G in the case of the trivial gerbe $f : G \to *$ and the sheaf $\underline{F} \in \mathrm{Sh_{Ab}Site}(*)$ represented by a discrete abelian group F. Note that $Rf_* \circ f^*(\underline{F})$ is an object of $D^+(\mathrm{Sh_{Ab}Site}(*))$. We get an object $Rf_* \circ f^*(\underline{F})(*) \in D^+(\mathrm{Ab})$ by evaluation at the object $(* \to *) \in \mathrm{Site}(*)$.

***Lemma* 2.2.3.** — *There exists an isomorphism*

$$H^*(Rf_* \circ f^*(\underline{F})(*)) \cong F \otimes \mathbb{Z}[[z]] \ ,$$

where $\deg(z) = 2$. *On cohomology the transformation D_G is given by $D_G = \mathrm{id} \otimes \frac{d}{dz}$.*

Proof. — We choose a lift $* \to G$. Forming iterated fiber products we get a simplicial space

$$\cdots * \times_G * \times_G * \times_G * \to * \times_G * \times_G * \to * \times_G * \to * \ .$$

Note that $* \times_G * \cong U(1)$. One checks that the simplicial space is equivalent to the simplicial space $BU(1)^{\cdot}$, the classifying space of the group $U(1)$,

$$U(1) \times U(1) \times U(1) \to U(1) \times U(1) \to U(1) \to * \ .$$

Let $(U \to *) \in \mathrm{Site}(*)$. If $H \in \mathrm{Sh_{Ab}}\mathbf{G}$, then we consider an injective resolution $0 \to H \to I^{\cdot}$. The evaluation $I^{\cdot}(U \times BU(1)^{\cdot})$ gives a cosimplicial complex, and after normalization, a double complex. Its total complex represents $Rf_*(H)(U \to *)$ (see [**9**, Lemma 2.41] for a proof of the corresponding statement in the smooth context). We calculate the cohomology of $Rf_*(H)(U \to *)$ using the associated spectral sequence. Its second page has the form

$$E_2^{p,q} \cong H^p(U \times BU(1)^q; H) \ .$$

We now specialize to the sheaf $H = f^*(\underline{F}) \cong \underline{F}_G$, where F is a discrete abelian group, and $U = *$. In this case the spectral sequence is the usual spectral sequence which calculates the cohomology of the realization of the simplicial space $BU(1)^{\cdot}$ with coefficients in F. Note that $H^*(BU(1);\mathbb{Z}) \cong \mathbb{Z}[[z]]$ as rings with $\deg(z) = 2$. Since it is torsion free as an abelian group we get

$$H^*(R^*f_* \circ f^*(\underline{F})(*)) \cong F \otimes H^*(BU(1);\underline{\mathbb{Z}}) \cong F \otimes \mathbb{Z}[[z]] \ .$$

In a similar manner we calculate $Rf_* \circ Rp_* \circ p^* \circ f^*(\underline{F})(*)$. Its cohomology is $H^*(T^2 \times BU(1); F)$, hence we have

$$H^*(Rf_* \circ Rp_* \circ p^* \circ f^*(\underline{F})(*)) \cong F \otimes H^*(T^2 \times BU(1);\mathbb{Z}) \cong F \otimes \Lambda(u,v) \otimes \mathbb{Z}[[z]] \ ,$$

where $u, v \in H^1(T^2, \mathbb{Z})$ are the canonical generators.

For every topological group Γ we have a natural map $\Gamma \to \Omega(B\Gamma)$. By adjointness we get a map $c : U(1) \times \Gamma \to U(1) \wedge \Gamma \to B\Gamma$. We will need a simplicial model $c^{\cdot}$ of this map. We consider the standard simplicial model $\mathbb{S}^{\cdot}$ of $U(1)$ with two non-degenerate simplices, one in degree 0, and one in degree 1. Then $\mathbb{S}^{\cdot} \times \Gamma$ is a simplicial model of $U(1) \times \Gamma$. It suffices to describe the map $c^{\cdot}$ on the non-degenerate part of $\mathbb{S}^{\cdot} \times \Gamma$. The component c^0 maps $\mathbb{S}^0 \times \Gamma$ to the base point $*$ of $B\Gamma^{\cdot}$. The component c^1 is the natural identification of the non-degenerate copy of $\Gamma \subset \mathbb{S}^1 \times \Gamma$ with $\Gamma \cong B\Gamma^1$.

We now specialize to the case $\Gamma = U(1)$. We get a map $c : T^2 \cong U(1) \times U(1) \to BU(1)$, or on the simplicial level, a map $c^{\cdot} : \mathbb{S}^{\cdot} \times U(1) \to BU(1)^{\cdot}$. We have $H^*(BU(1);\mathbb{Z}) \cong \mathbb{Z}[[z]]$ with z of degree 2, and one checks that $uv = c^*(z) \in H^2(T^2;\mathbb{Z})$ (after choosing an appropriate basis $u, v \in H^1(T^2;\mathbb{Z})$).

Note that $BU(1)^{\cdot}$ is a simplicial abelian group. The discussion above shows that the automorphism $\phi\colon G \to G$ in (2.2.1) with $X = *$ and classified by $uv \in H^2(T^2;\mathbb{Z})$ can be arranged so that it induces an automorphism of bundles of $BU(1)^{\cdot}$-torsors

$$(2.2.4)\qquad \begin{array}{ccc} \mathbb{S}^{\cdot} \times U(1) \times BU(1)^{\cdot} & \xrightarrow[\phi^{\cdot}]{(t,x)\mapsto(t,c^{\cdot}(t)x)} & \mathbb{S}^{\cdot} \times U(1) \times BU(1)^{\cdot} \\ & \searrow \quad \swarrow & \\ & \mathbb{S}^{\cdot} \times U(1) & \end{array} .$$

Under this isomorphism the action of

$$(2.2.5)\qquad \phi^*\colon H^*(Rf_* \circ Rp_* \circ p^* \circ f^*(\underline{F})(*)) \to H^*(Rf_* \circ Rp_* \circ p^* \circ f^*(\underline{F})(*))$$

is induced by $z \mapsto z + uv$, $u \mapsto u$, $v \mapsto v$. In order to see this note that $m^*(z) = z_1 + z_2$, where $m : BU(1) \times BU(1) \to BU(1)$ is the multiplication, and $H^*(BU(1) \times BU(1);\mathbb{Z}) \cong \mathbb{Z}[[z_1, z_2]]$. After realization the map $\phi^{\cdot}$ leads to the composition

$$T^2 \times BU(1) \overset{(\mathrm{id}_{T^2},c)\times \mathrm{id}}{\to} T^2 \times BU(1) \times BU(1) \overset{\mathrm{id}_{T^2}\times m}{\to} T^2 \times BU(1)$$

which maps

$$z \overset{(\mathrm{id}_{T^2}\times m)^*}{\mapsto} z_1 + z_2 \overset{((\mathrm{id}_{T^2},c)\times \mathrm{id})^*}{\mapsto} uv + z .$$

In cohomology of the evaluations at the point the integration map

$$\int_p : Rf_* \circ Rp_* \circ p^* \circ f^*(\underline{F}) \to Rf_* \circ f^*(\underline{F})$$

induces the map $F \otimes \Lambda(u, v) \otimes \mathbb{Z}[[z]] \to F \otimes \mathbb{Z}[[z]]$ which takes the coefficient at uv. This implies the assertions of Lemma 2.2.3. □

2.3. Identification of the transformation D_G in the smooth case

2.3.1. — In this subsection we work in the context of [**9**] of manifolds and smooth stacks. It can be considered as a supplement to [**9**] concerning the transformation D_G introduced in Definition 2.2.2 which can be defined in the smooth context in a parallel manner.

If X is a smooth stack, then Ω_X denotes the sheaf of de Rham complexes on X. It associates to $(U \to X) \in \mathbf{X}$ the de Rham complex $\Omega_X(U \to X) := \Omega(U)$ of the manifold U. Note that in this subsection $\mathbf{X}$ denotes the site of a smooth stack introduced in [**9**].

If $\omega \in \Omega^3_X(X)$ is a closed 3-form, then we form the sheaf of twisted de Rham complexes $\Omega_X[[z]]_\omega$. Its evaluation at $(U \to X) \in \mathbf{X}$ is the complex $\Omega_X[[z]]_\omega(U \to X) := \Omega(U)[[z]] \cong \Omega(U) \otimes_{\mathbb{Z}} \mathbb{Z}[[z]]$ with differential $d_{dR} + \omega\frac{d}{dz}$. In this formula the form ω acts by wedge multiplication with the pull-back of ω to U.

Let $f\colon G \to X$ be a gerbe with band $U(1)$ over a *smooth manifold* X. The choice of a gerbe connection determines a closed 3-form $\omega \in \Omega^3_X(X)$ which represents the Dixmier-Douady class of the gerbe. By [**9**, Theorem 1.1] we have an isomorphism

$$Rf_*f^*\underline{\mathbb{R}}_{\mathbf{X}} \xrightarrow{\sim} \Omega_X[[z]]_\omega \tag{2.3.1}$$

in the derived category $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$.

2.3.2.

Theorem 2.3.2. — *We have a commutative diagram*

$$\begin{array}{ccc} Rf_*f^*\underline{\mathbb{R}}_{\mathbf{X}} & \xleftarrow[(2.3.1)]{\cong} & \Omega_X[[z]]_\omega \\ \big\downarrow{\scriptstyle D_G} & & \big\downarrow{\scriptstyle \frac{d}{dz}} \\ Rf_*f^*\underline{\mathbb{R}}_{\mathbf{X}} & \xleftarrow[(2.3.1)]{\cong} & \Omega_X[[z]]_\omega . \end{array}$$

Proof. — The isomorphism (2.3.1) was constructed in [**9**, Section 3] using a particular model of $Rf_*f^*(\underline{\mathbb{R}}_{\mathbf{X}})$. We first recall its construction. Let $A \to G$ be an atlas. For $(U \to X) \in \mathbf{X}$ we form the simplicial object $(A^{\cdot}_U \to G) \in \mathbf{G}^{\Delta^{\mathrm{op}}}$ with

$$A^n_U := \underbrace{A \times_G \cdots \times_G A}_{n+1 \text{ factors}} \times_X U \to G \ .$$

The boundaries and degenerations are given by the projections and diagonals as usual.

If $F \in C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{G})$ is a bounded below complex of presheaves, then we form the simplicial complex of presheaves $(U \to X) \mapsto F(A^{\cdot}_U \to G)$. We let $C_A(F) \in C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{X})$ denote the presheaf of associated total complexes. Sometimes we will write $C^{m,n}_A(F)$ for the summand of bidegree (m,n), where the first entry m denotes the cosimplicial degree.

If F is a complex of flabby sheaves, then by [**9**, Lemma 2.41] we have a natural isomorphism $Rf_*(F) \cong C_A(F)$. Here we use in particular that the functor C_A preserves sheaves.

Note that the resolution $\underline{\mathbb{R}}_G \to \Omega_G$ of the constant sheaf with value $\mathbb{R}$ by the sheaf of de Rham complexes is a flabby resolution (see [**9**, Subsection 3.1]). Therefore we have a natural isomorphism $Rf_*(\underline{\mathbb{R}}_G) \cong C_A(\Omega_G)$.

We choose an atlas $A \to X$ given by the disjoint union of a collection of open subsets of X such that there exists a lift in

$$\begin{array}{ccc} & & G \\ & \nearrow & \big\downarrow{\scriptstyle f} \\ A & \longrightarrow & X \end{array} \ .$$

This lift is an atlas $A \to G$ of G. We furthermore choose a connection datum $(\alpha,\beta) \in \Omega^1(A \times_G A) \times \Omega^2(A)$. The one-form α is a connection of the $U(1)$-principal bundle $A \times_G A \to A \times_X A$. It is related with the two-form β by $d_{dR}\alpha = \delta\beta$. This equation implies that $\delta d_{dR}\beta = 0$ so that $d_{dR}\beta$ assembles to a uniquely determined closed form $\omega \in \Omega^3_X(X)$ (compare [**9**, Section 3.2]). The 3-form ω represents the Dixmier-Douady class of the gerbe $G \to X$ and will be used for twisting the de Rham complex.

The isomorphism (2.3.1) is given by an explicit quasi-isomorphism

$$\Omega_X[[z]]_\omega \to C_A(\Omega_G)\ . \tag{2.3.3}$$

Note that $\Omega_X[[z]]_\omega$ and $C_A(\Omega_G)$ are sheaves of associative DG-algebras central over the sheaf of DG-algebras Ω_X, and that z generates $\Omega_X[[z]]_\omega$. The quasi-isomorphism (2.3.3) is the unique morphism of sheaves of associative DG-algebras, central over Ω_X, with

$$z \mapsto (\alpha,\beta) \in C^{1,1}_A(\Omega_G)(X) \oplus C^{0,2}_A(\Omega_G)(X)\ .$$

For more details we refer to [**9**, Subsection 3.2]

2.3.3. — For $i = 1,\dots,n$ there are $U(1)$-principal bundle structures

$$p_i : \underbrace{A \times_G \cdots \times_G A}_{n+1 \text{ factors}} \to \underbrace{A \times_G \cdots \times_G A}_{i \text{ factors}} \times_X \underbrace{A \times_G \cdots \times_G A}_{n-i+1 \text{ factors}}\ .$$

Furthermore, we have embeddings

$$j_i : \underbrace{A \times_G \cdots \times_G A}_{n \text{ factors}} \to \underbrace{A \times_G \cdots \times_G A}_{i \text{ factors}} \times_X \underbrace{A \times_G \cdots \times_G A}_{n-i+1 \text{ factors}}$$

given by

$$j_i := \underbrace{\mathrm{id}_A \cdots \times \mathrm{id}_A}_{i-1 \text{ factors}} \times \Delta_A \times \underbrace{\mathrm{id}_A \cdots \times \mathrm{id}_A}_{n-i \text{ factors}}\ ,$$

where $\Delta : A \to A \times_X A$ is the diagonal.

If $(U \to X) \in \mathbf{X}$, then the maps p_i and j_i induce similar maps on the product $\cdots \times_X U$ of these manifolds over X with U which we denote by the same symbols. For $i = 1,\dots,n$ we define the map of degree -1

$$v_i \colon \Omega(A^n_U) \to \Omega(A^{n-1}_U)$$

as the composition of the integration over the fiber of p_i with the pull-back along j_i, i.e. $v_i := j_i^* \circ \int_{p_i}$. Since the construction of v_i is natural with respect to U we can view v_i as a morphism of sheaves $C^{n,m}_A(\Omega_G) \to C^{n-1,m-1}_A(\Omega_G)$. We define the family of morphisms

$$D_n := \sum_{i=1}^n (-1)^i v_i : C^{n,*}_A(\Omega_G) \to C^{n-1,*-1}_A(\Omega_G)$$

and let $D : C_A(\Omega_G) \to C_A(\Omega_G)$ be the endomorphism of sheaves of degree -2 given by D_n in bidegree $(n,*)$.

2.3.4.

Lemma 2.3.4. — *The map $D : C_A(\Omega_G) \to C_A(\Omega_G)$ is a derivation of Ω_X-modules.*

Proof. — Note that v_j commutes with the de Rham differential. Moreover, if

$$q_k : \underbrace{A \times_G \cdots \times_G A}_{n+1 \text{ factors}} \to \underbrace{A \times_G \cdots \times_G A}_{n \text{ factors}}$$

is the projection which leaves out the k-th factor ($k = 0, \ldots, n$), then we have the relations

$$\begin{aligned} v_j q_k^* &= q_{k-1}^* v_j, \qquad && j < k \\ v_j q_k^* &= q_k^* v_{j-1}, && j > k+1 \\ v_j q_k^* &= 0, && j = k, k+1. \end{aligned}$$

Observe that in the last case q_k factors over the bundle which is used for the integration in the definition of v_k or v_{k+1}, and the composition of a pullback along a bundle projection followed by an integration along the same bundle projection vanishes. These relations imply by a direct calculation that D is a chain map for the Čech-de Rham differential of $C_A(\Omega_G)$.

Moreover, it follows immediately from the definition of D that it is Ω_X-linear (even Ω_A-linear).

It is again a straightforward calculation to verify that D is a derivation for the associative product on $C_A(\Omega_G)$ (compare [**9**, 2.4.9] for the product structure). □

2.3.5.

Lemma 2.3.5. — *We have a commutative diagram*

$$\begin{array}{ccc} \Omega_X[[z]]_\omega & \xrightarrow{(2.3.3)} & C_A(\Omega_G) \\ \downarrow{\scriptstyle \frac{d}{dz}} & & \downarrow{\scriptstyle D} \\ \Omega_X[[z]]_\omega & \xrightarrow{(2.3.3)} & C_A(\Omega_G). \end{array}$$

Proof. — Since α is the connection one-form of a $U(1)$-connection on the total space of the $U(1)$-principal bundle $p_1 : A \times_G A \to A \times_X A$ we have $\int_{p_1} \alpha = 1$. Consequently, $D(\alpha, \beta) = 1$. This implies the assertion, since D and $\frac{d}{dz}$ are Ω_X-linear derivation, and z generates $\Omega_X[[z]]_\omega$. □

In view of Lemma 2.3.5, in order to finish the proof of Theorem 2.3.2 is suffices to show that the operation D coincides with the operation of $\int_p \circ \phi^* \circ p^*$ on $C_A(\Omega_G)$.

2.3.6. — Let $M^\cdot$ be a simplicial manifold and consider the bundle $U(1)\times M^\cdot \to M^\cdot$. We describe the integration map

$$\int : \Omega(U(1)\times M^\cdot) \to \Omega(M^\cdot)$$

in the simplicial picture, i.e. as a map

$$\int : \Omega(\mathbb{S}^\cdot \times M^\cdot) \to \Omega(M^\cdot)\ .$$

For $n \geq 1$ the manifolds $\mathbb{S}^n \times M^n$ consists of n copies $\sigma_1(M^n),\dots,\sigma_n(M^n)$ of M^n which correspond to the points of $\mathbb{S}^n$ which are degenerations of the non-degenerated point of $\mathbb{S}^1$ (where the index measures which 1-simplex in the boundary is non-degenerate), and an additional copy of M^n corresponding the point of $\mathbb{S}^n$ which is the degeneration of the point in $\mathbb{S}^0$. For $k=1,\dots,n+1$ let $j_k : M^n \to \mathbb{S}^{n+1}\times M^{n+1}$ be the map $M^n \to \sigma_k(M^{n+1}) \subset \mathbb{S}^{n+1}\times M^{n+1}$, which corresponds the kth degeneration $[n+1]\to[n]$. We now define a chain map of total complexes

$$\int : \Omega(\mathbb{S}^\cdot \times M^\cdot) \to \Omega(M^\cdot)$$

of degree -1 which is given by

$$\sum_{k=1}^{n+1}(-1)^k j_k^* : \Omega(\mathbb{S}^{n+1}\times M^{n+1}) \to \Omega(M^n)\ , \tag{2.3.6}$$

and is zero on $\Omega(\mathbb{S}^0\times M^0)$. This map realizes the integration in the simplicial picture.

2.3.7. — For $(U\to X)\in \mathbf{X}$ the automorphism of gerbes $\phi : T^2\times G \to T^2 \times G$ induces an automorphism of simplicial sets

$$\phi^\cdot : \mathbb{S}^\cdot \times U(1)\times A_U^\cdot \to \mathbb{S}^\cdot \times U(1)\times A_U^\cdot$$

which we now describe explicitly by an extension of the special case (2.2.4) to general base spaces.

If $t\in \mathbb{S}^n\times U(1)$ belongs to $U(1)\cong \sigma_k(U(1)) \subset \mathbb{S}^n\times U(1)$, $k=1,\dots,n$, then $\phi^\cdot(t,a) = (t, m_k(t,a))$, where $m_k : U(1)\times A_U^n \to A_U^n$ is the action of $U(1)$ on the principal fibration p_k. If $t\in\mathbb{S}^n\times U(1)$ belongs to the degeneration of $\mathbb{S}^0\times U(1)$, then $\phi^\cdot(t,a)=(t,a)$. This formula provides a simplicial description of the action of

$$\phi^* : C_{\mathbb{S}^\cdot\times U(1)\times A}(\Omega_G) \to C_{\mathbb{S}^\cdot\times U(1)\times A}(\Omega_G)\ .$$

Combining the description of the integration map (2.3.6) with this formula for the action of ϕ^* it is now straightforward to show the equality of maps

$$D = \int_p \circ\phi^*\circ p^* : C_A(\Omega_G) \to C_A(\Omega_G)\ . \qquad \square$$

2.4. Two-periodization — up to isomorphism

2.4.1. — Let $f\colon G \to X$ be a topological gerbe with band $U(1)$ over a locally compact stack X. In Definition 2.2.2 we have constructed a natural endomorphism $D_G \in \mathrm{End}(Rf_* \circ f^*)$ of degree -2. To any object $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ we associate the inductive system

$$\mathcal{J}_G(F)\colon Rf_* \circ f^*(F) \overset{D_G}{\leftarrow} Rf_* \circ f^*(F)[2] \overset{D_G}{\leftarrow} Rf_* \circ f^*(F)[4] \overset{D_G}{\leftarrow} \cdots \tag{2.4.1}$$

indexed by $\{0, 1, 2, \ldots\}$.

Using the inclusion $D^+(\mathrm{Sh_{Ab}}\mathbf{X}) \to D(\mathrm{Sh_{Ab}}\mathbf{X})$ of the bounded below into the unbounded derived category of sheaves of abelian groups on X we can consider $\mathcal{J}_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})^{\mathbb{N}^{\mathrm{op}}}$, where the ordered set of integers $\mathbb{N}$ is considered as a category.

2.4.2. — Using the triangulated structure of $D(\mathrm{Sh_{Ab}}\mathbf{X})$ one can define for each object $\mathcal{J} \in D(\mathrm{Sh_{Ab}}\mathbf{X})^{\mathbb{N}^{\mathrm{op}}}$ an object $\mathrm{holim}\, \mathcal{J} \in D(\mathrm{Sh_{Ab}}\mathbf{X})$ which is unique up to non-canonical isomorphism (see [**21**]). An explicit construction of this homotopy limit uses the extension of maps in $D(\mathrm{Sh_{Ab}}\mathbf{X})$ to exact triangles by a mapping cylinder construction. In particular, we obtain $\mathrm{holim}\, \mathcal{J}_G(F)$ by the extension to a triangle of the map $1 - \hat{D}$ in

$$\mathrm{holim}\, \mathcal{J}_G(F) \to \prod_{i \geq 0} Rf_* \circ f^*(F)[2i] \xrightarrow{1-\hat{D}} \prod_{i \geq 0} Rf_* \circ f^*(F)[2i] \to \mathrm{holim}\, \mathcal{J}_G(F)[1]\ ,$$

where

$$\hat{D}\colon \prod_{i \geq 0} Rf_* \circ f^*(F)[2i] \to \prod_{i \geq 0} Rf_* \circ f^*(F)[2i]$$

maps the sequence $(x_i)_{i \geq 0}$ to the sequence $(D_G x_{i+1})_{i \geq 0}$.

2.4.3. — We can now define the periodization $P_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})$ of an object $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$.

Definition 2.4.2. — *For $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ we define $P_G(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})$ by*

$$P_G(F) := \mathrm{holim}\, \mathcal{J}_G(F)\ .$$

Note that $P_G(F)$ is well-defined up to non-canonical isomorphism.

2.4.4. — The operator

$$\prod_{i \geq 0} D_G\colon \prod_{i \geq 0} Rf_* \circ f^*(F)[2i] \to \Big(\prod_{i \geq 0} Rf_* \circ f^*(F)[2i]\Big)[-2]$$

commutes with $\hat{D}$ and therefore induces a map $W\colon P_G(F) \to P_G(F)[-2]$ via an extension in the diagram

$$\begin{array}{ccc}
P_G(F) & \xrightarrow{W} & P_G(F)[-2] \\
\downarrow & & \downarrow \\
\prod_{i\geq 0} Rf_* \circ f^*(F)[2i] & \xrightarrow{\prod_{i\geq 0} D_G} & \prod_{i\geq 0} Rf_* \circ f^*(F)[2i])[-2] \\
\downarrow{\scriptstyle 1-\hat{D}} & & \downarrow{\scriptstyle 1-\hat{D}} \\
\prod_{i\geq 0} Rf_* \circ f^*(F)[2i] & \xrightarrow{\prod_{i\geq 0} D_G} & \prod_{i\geq 0} Rf_* \circ f^*(F)[2i])[-2] \\
\downarrow & & \downarrow \\
P_G(F)[1] & \xrightarrow{W} & P_G(F)[1][-2]\ .
\end{array}$$

Note that such an extension exists by the axioms of a triangulated category, but it might not be unique.

The following proposition asserts that $P_G(F)$ is two-periodic.

Proposition 2.4.3. — *The map $W\colon P_G(F) \to P_G(F)[-2]$ is an isomorphism.*

Proof. — For notational convenience, we consider the following general situation. Let $D(A)$ be the unbounded derived category of a Grothendieck abelian category. Note that $\mathrm{Sh}_{\mathrm{Ab}}(\mathbf{X})$ is such a category (see Section 3.3.1). We consider an object $X \in D(A)$ together with a morphism $D\colon X \to X[-2]$. We can assume that D is represented by a map of complexes $D\colon X \to X[-2]$. We obtain the extension $1 - \hat{D}$ to a triangle

$$Y \to \prod_{i\geq 0} X[2i] \xrightarrow{1-\hat{D}} \prod_{i\geq 0} X[2i] \to Y[1] \tag{2.4.4}$$

where $Y := \prod_{i\geq 0} X[2i] \oplus (\prod_{i\geq 0} X[2i])[1]$ with the differential

$$\delta := \begin{pmatrix} d & 1-\hat{D} \\ 0 & -d \end{pmatrix},$$

where d is the differential of X. The induced map $W\colon Y \to Y[-2]$ is given by

$$W := \begin{pmatrix} \prod_{i\geq 0} D & 0 \\ 0 & \prod_{i\geq 0} D \end{pmatrix}.$$

Let

$$E\colon \prod_{i\geq 0} X[2i] \to (\prod_{i\geq 0} X[2i])[2]$$

be the shift $E(x_i)_{i\geq 0} := (x_{i+1})_{i\geq 0}$. Note that E commutes with $1-\hat{D}$, too. Therefore we obtain the extension $S\colon Y \to Y[2]$ in the diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & \prod_{i\geq 0} X[2i] & \xrightarrow{1-\hat{D}} & \prod_{i\geq 0} X[2i] & \longrightarrow & Y[1] \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle S} \\ Y[2] & \longrightarrow & (\prod_{i\geq 0} X[2i])[2] & \xrightarrow{1-\hat{D}} & (\prod_{i\geq 0} X[2i])[2] & \longrightarrow & Y[1][2] \end{array} .$$

by the matrix

$$S := \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} .$$

Proposition 2.4.3 is a consequence of the following Lemma.

Lemma 2.4.5. — *We have the equalities $W \circ S = \mathrm{id} = S \circ W$.*

Proof. — First observe that $\prod_{i\geq 0} D \circ E = \hat{D} = E \circ \prod_{i\geq 0} D$. Therefore $W \circ S = S \circ W = \left(\begin{smallmatrix} \hat{D} & 0 \\ 0 & \hat{D} \end{smallmatrix}\right)$. In order to show that $W \circ S = \mathrm{id}$ we show that the map

$$I := \begin{pmatrix} \hat{D} & 0 \\ 0 & \hat{D} \end{pmatrix} .$$

on Y is homotopic to the identity and therefore is equal to the identity in the derived category. This follows from

$$1 - I = \delta \circ J + J \circ \delta$$

with

$$J := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

□

2.4.5. — We continue with the notation introduced in the proof of Proposition 2.4.3. Applying a homological functor to the triangle (2.4.4) we get the long exact sequence

$$\cdots \to H^*(Y) \to \prod_{i\geq 0} H^*(X[2i]) \to \prod_{i\geq 0} H^*(X[2i]) \to H^*(Y[1]) \to .$$

If we analyze the middle map and compare it with the ordinary definition of limits in abelian categories we get the following result.

Corollary 2.4.6. — *We have an exact sequence:*

$$0 \to \lim_i{}^1 H^*(X[2i])[-1] \to H^*(Y) \to \lim_i H^*(X[2i]) \to 0 .$$

2.4.6. — Note that the construction

$$\mathrm{holim} : D(A)^{\mathbb{N}^{\mathrm{op}}} \to D(A)$$

is not a functor. The construction of the homotopy limit holim (S) for $S \in D(A)^{\mathbb{N}^{\mathrm{op}}}$ via mapping cylinders uses explicit representatives of the maps of the system S and depends non-trivially on this choice.

A homotopy limit functor holim : $D(A^{\mathbb{N}^{\mathrm{op}}}) \to D(A)$ can be defined as the right-derived functor of lim: $A^{\mathbb{N}^{\mathrm{op}}} \to A$. Note that in the domain we take the derived category of the abelian category of $\mathbb{N}^{\mathrm{op}}$-diagrams in A as opposed to $\mathbb{N}^{\mathrm{op}}$-diagrams in the derived category of A. In Section 3 we will use this idea and refine P_G to a periodization functor

$$P_G \colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

which is a triangulated functor and natural in $G \to X$. The main idea is the construction of a refinement of the diagram (2.4.1) to a diagram in $D((\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^{\mathbb{N}^{\mathrm{op}}})$, see 3.4.6 (the details are in fact more complicated).

2.5. Calculations

2.5.1. — In this subsection we calculate $P_G(\underline{F})$ in the special case, where $G \to *$ is the (trivial) $U(1)$-gerbe over the point, and $\underline{F} \in \mathrm{Sh}_{\mathrm{Ab}}\mathrm{Site}(*)$ is the sheaf represented by a discrete abelian group F. We will calculate the abelian group $H^*(*; P_G(\underline{F}))$. This cohomology is two-periodic so that we only have to distinguish the even and the odd-degree case. In the table below $\mathbb{A}_f^{\mathbb{Q}}$ denotes the group of finite adeles of $\mathbb{Q}$, which contains $\mathbb{Q}$ via the diagonal embedding.

Proposition 2.5.1. — *We have the following table for the cohomology $H^*(*; P_G(\underline{F}))$.*

F	$H^{\mathrm{ev}}(*; P_G(\underline{F}))$	$H^{\mathrm{odd}}(*; P_G(\underline{F}))$
$\mathbb{Z}$	0	$\mathbb{A}_f^{\mathbb{Q}}/\mathbb{Q}$
$\mathbb{Q}$	$\mathbb{Q}$	0
$\mathbb{Z}/n\mathbb{Z}$	0	0
$\mathbb{Q}/\mathbb{Z}$	$\mathbb{A}_f^{\mathbb{Q}}$	0

2.5.2. — To prove Proposition 2.5.1, we use the exact sequence 2.4.6 where

$$H^*(X) = H^*(*; Rf_* \circ f^*(\underline{F})) \cong F \otimes \mathbb{Z}[[z]] \cong F[[z]]$$

by Lemma 2.2.3 with z of degree 2. We must discuss the cohomology of the complex

$$0 \to \prod_{i \geq 0} F[[z]][2i] \stackrel{1-\hat{D}}{\to} \prod_{i \geq 0} F[[z]][2i] \to 0 \ ,$$

where $\hat{D}(x_i)_{i\geq 0} = (D_G x_{i+1})_{i\geq 0}$ with $D_G = \frac{d}{dz}$. This means that we have to study the solution theory for the system

$$x_i - \frac{d}{dz}x_{i+1} = a_i \ , \quad i \geq 0 \ , \quad x_i \in F[[z]] \ . \tag{2.5.2}$$

2.5.3. — Let us start with the case $F = \mathbb{Q}$. Since we can divide by arbitrary integers the operator D_G is surjective and we can for any $(a_i)_{i\in\mathbb{N}}$ solve this system inductively. Therefore the cokernel $\lim^1_i \mathbb{Q}[u]$ of $1 - \hat{D}$ is trivial. A solution of the homogeneous system is uniquely determined by the choice of x_0 and the constant terms of the x_i, $i \geq 1$. Note that the constant term of x_i is in degree $-2i$. It follows that

$$H^{\mathrm{ev}}(*; P_G(\underline{\mathbb{Q}})) \cong \mathbb{Q} \ , \quad H^{\mathrm{odd}}(*; P_G(\underline{\mathbb{Q}})) \cong 0 \ .$$

2.5.4. — We now discuss torsion coefficients $F = \mathbb{Z}/n\mathbb{Z}$. Write $x_i = \sum x_{i,k}z^k$, $a_i = \sum a_{i,k}z^k$ with $x_{i,k}, a_{i,k} \in \mathbb{Z}/n\mathbb{Z}$. Then we have to solve

$$\sum_{k=0}^{\infty} x_{i,k}z^k - (k+1)x_{i+1,k+1}z^k = \sum_{k=0}^{\infty} a_{i,k}z^k \qquad \forall i \geq 0.$$

Equating coefficients this system decouples into finite systems

$$\begin{aligned} x_{i,kn} - (kn+1)x_{i+1,kn+1} &= a_{i,kn} \\ x_{i,kn+1} - (kn+2)x_{i+1,kn+2} &= a_{i,kn+1} \\ &\vdots \\ x_{i,kn+n-2} - (kn+n-1)x_{i+1,kn+n-1} &= a_{i,kn+n+2} \\ x_{i,kn+n-1-r} + \underbrace{(kn+n)x_{i+1,kn+n}}_{=0} &= a_{ikn+n-1} \ , \end{aligned}$$

where $i, k \geq 0$. We see that we can always solve this system uniquely by backwards induction. We get

$$H^{\mathrm{ev}}(*; P_G(\underline{\mathbb{Z}/n\mathbb{Z}})) \cong 0 \ , \quad H^{\mathrm{odd}}(*; P_G(\underline{\mathbb{Z}/n\mathbb{Z}})) \cong 0 \ .$$

2.5.5. — Let us now assume that $F = \mathbb{Q}/\mathbb{Z}$. Since this group is divisible we can solve the system (2.5.2) for every $(a_i)_{i\in\mathbb{N}}$. It follows that

$$H^{\mathrm{odd}}(*; P_G(\underline{\mathbb{Q}/\mathbb{Z}})) \cong 0 \ .$$

We now discuss the solution of the homogeneous system in degree 0. We can choose x_0 arbitrary. If we have found x_i for $i = 0, \dots, n-1$, then we must solve $x_{n-1} = nx_n$ in the next step. We see that x_n is well-defined up to the image of $\mathbb{Z}/n\mathbb{Z} \cong n^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. We see that $H^{\mathrm{ev}}(*; P_G(\underline{\mathbb{Q}/\mathbb{Z}}))$ admits a sequence of quotients

$$H^{\mathrm{ev}}(*; P_G(\underline{\mathbb{Q}/\mathbb{Z}})) \to \cdots \to Q^n \to Q^{n-1} \to \cdots \to Q^0$$

where $Q^n \cong \mathbb{Q}/\mathbb{Z}$ and $Q^n \to Q^{n-1}$ is given by multiplication with n for all $n \in \mathbb{N}$. The limit

$$\mathbb{A}_f^{\mathbb{Q}} \cong \lim_{\substack{\longleftarrow \\ n\in\mathbb{N}}} (\mathbb{Q}/n!\mathbb{Z})$$

is the ring $\mathbb{A}_f^{\mathbb{Q}}$ of finite adeles of $\mathbb{Q}$, and $\mathbb{Q} \subset \mathbb{A}_f^{\mathbb{Q}}$ is a subgroup. We thus get

$$H^{\mathrm{ev}}(*; P_G(\underline{\mathbb{Q}/\mathbb{Z}})) \cong \mathbb{A}_f^{\mathbb{Q}} .$$

2.5.6. — Finally assume that $F = \mathbb{Z}$. We must again consider the system (2.5.2) of equations above. Let us discuss this system in degree $2r$. Then the relevant coefficients of x_i and a_i are sequences of integers, and (writing out only these) $dx_{i+1} = (r+i+1)x_{i+1}$. We see that the homogeneous equation has only the trivial solution since otherwise the integer x_0 must be divisible by $n+i+1$ for all $i \geq 0$. Hence

$$H^{\mathrm{ev}}(*; P_G(\underline{\mathbb{Z}})) \cong 0 .$$

In order to calculate $H^{\mathrm{odd}}(*; P_G(\underline{\mathbb{Z}}))$ we consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 .$$

It gives rise to an exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Q}} \to \underline{\mathbb{Q}/\mathbb{Z}} \to 0 .$$

and a long exact cohomology sequence. In Section 3.4 we will construct a functorial version of P_G which is a triangulated functor, and which coincides with the isomorphism class constructed above. Using this functor, we get a triangle

$$P_G(\underline{\mathbb{Z}}) \to P_G(\underline{\mathbb{Q}}) \to P_G(\underline{\mathbb{Q}/\mathbb{Z}}) \to P_G(\underline{\mathbb{Z}})[1]$$

and therefore a long exact cohomology sequence

$$H^*(*; P_G(\underline{\mathbb{Z}})) \to H^*(*; P_G(\underline{\mathbb{Q}})) \to H^*(*; P_G(\underline{\mathbb{Q}/\mathbb{Z}})) \to H^*(*; P_G(\underline{\mathbb{Z}}))[1] .$$

By the calculations for $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ we get exact sequences

$$0 \to \mathbb{Q} \xrightarrow{c} \mathbb{A}_f^{\mathbb{Q}} \to H^{\mathrm{odd}}(*; P_G(\underline{\mathbb{Z}})) \to 0 ,$$

where c is the canonical embedding. Therefore

$$H^{\mathrm{odd}}(*; P_G(\underline{F})) \cong \mathbb{A}_f^{\mathbb{Q}}/\mathbb{Q} .$$

□

CHAPTER 3

FUNCTORIAL PERIODIZATION

3.1. Flabby resolutions

3.1.1. — Let $\mathbf{X}$ be a site, e.g. the site of a locally compact stack. For $U \in \mathbf{X}$ let $\tau := (U_i \to U)_{i\in I} \in \mathrm{cov}_{\mathbf{X}}(U)$ be a covering family. Then we consider $V := \bigsqcup_{i\in I} U_i \to U$. Forming iterated fiber products we obtain a simplicial object $V^\cdot$ in $\mathbf{X}$ with

$$V^n = \underbrace{V \times_U \cdots \times_U V}_{n+1 \text{ factors}} .$$

If $F \in \mathrm{Pr}\mathbf{X}$ is a presheaf on X, then we form the cosimplicial set $C^\cdot(\tau, F) := F(V^\cdot)$.

3.1.2. — If F is a presheaf of abelian groups, then we form the Čech complex $\check{C}(\tau, F)$ which is the chain complex associated to the cosimplicial abelian group $C^\cdot(\tau, F)$.

If F is a sheaf, then $H^0\check{C}(\tau, F) \cong F(U)$. We recall the following definition (see [**25**, Definition 3.5.1]).

Definition 3.1.1 **(see [25, 3.5.1]).** — *A sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is called flabby if for all $U \in \mathbf{X}$ and $\tau \in \mathrm{cov}_{\mathbf{X}}(U)$ we have $H^i\check{C}(\tau, F) \cong 0$ for all $i \geq 1$.*

By [**25**, Cor. 3.5.3] a sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is flabby if and only if $R^k i(F) = 0$ for all $k \geq 1$, where $i : \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Pr}_{\mathrm{Ab}}\mathbf{X}$ is the inclusion of sheaves into presheaves.

As an immediate consequence of the definition a sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is flabby if and only if the restriction F_U of F to the site (U) is flabby for all $(U \to X) \in \mathbf{X}$ (see 6.1.14 for the notation).

3.1.3. — Let now X be a locally compact stack and $\mathbf{X}$ be the site of X. Occasionally, in the present paper we need the stronger notion of a flasque sheaf.

Definition 3.1.2. — *A sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is called flasque if for every $(U \to X) \in \mathbf{X}$ and open subset $V \subseteq U$ the restriction $F(U \to X) \to F(V \to X)$ is surjective.*

In the literature, e.g. in [**17**] or [**5**], this is sometimes used as the definition of flabbiness.

Lemma 3.1.3. — *A flasque sheaf is flabby.*

Proof. — For $U \in \mathbf{X}$ let $\Gamma_U : \mathrm{Sh_{Ab}}\mathbf{X} \to \mathrm{Ab}$ be the section functor $F \mapsto \Gamma_U(F) := F(U)$. For $V \subseteq U$ we have $\Gamma_V(F_U) = \Gamma_V(F)$. A sheaf $F \in \mathrm{Sh_{Ab}}\mathbf{X}$ is flasque by definition if and only if F_U is flasque for all $U \in \mathbf{X}$. But a flasque sheaf is Γ_V-acyclic for every $V \subseteq U$ by [**5**, Ch. 2, Thm. 5.4] (note that in this reference our flasque is called flabby). By [**25**, Cor. 3.5.3] it is flabby in the sense of 3.1.1.

This argument shows that F_U is flabby for all $(U \to X) \in \mathbf{X}$ and implies that F itself is flabby. □

We do not know if the converse of Lemma 3.1.3 is true. Therefore we must be careful when using results from the literature.

3.1.4.

Lemma 3.1.4. — *If $f : X \to Y$ is a representable map of locally compact stacks, then a flabby sheaf is f_*-acyclic.*

Proof. — Let $F \in \mathrm{Sh_{Ab}}\mathbf{X}$ be a flabby sheaf. We must show that $R^k f_*(F) = 0$ for all $k \geq 1$. We have a morphism of sites $f^\sharp : \mathbf{Y} \to \mathbf{X}$, see 6.1.10. The functor ${}^p f_* : \mathrm{Pr}\mathbf{X} \to \mathrm{Pr}\mathbf{Y}$ is given by ${}^p f_* F := F \circ f^\sharp$. It is in particular exact. Therefore we have $Rf_* \cong i^\sharp \circ {}^p f_* \circ Ri$. Since a flabby sheaf is i-acyclic we conclude that $R^k i(F) = 0$ for $k \geq 1$. This implies $R^k f_*(F) = 0$ for $k \geq 1$. □

3.1.5.

Lemma 3.1.5. — *If a morphism $f\colon X \to Y$ of locally compact stacks has local sections, then the functor $f^*\colon \mathrm{Sh_{Ab}}\mathbf{Y} \to \mathrm{Sh_{Ab}}\mathbf{X}$ preserves flabby sheaves.*

Proof. — Let $F \in \mathrm{Sh_{Ab}}\mathbf{Y}$ be flabby. We consider an object $(U \to X) \in \mathbf{X}$ and a covering family $\tau \in \mathrm{cov}_{\mathbf{X}}(U)$. Then we must show that the higher cohomology groups of $\check{C}(\tau, f^*F)$ vanish.

We obtain a covering family $f_\sharp\tau \in \mathrm{cov}_{\mathbf{Y}}(f_\sharp U)$, see 6.1.11. Let $V^\cdot$ be the simplicial object associated to τ as in 3.1.1. Since $f_\sharp$ preserves fiber products in the sense of [**25**, 1.2.2(ii)] we see that $f_\sharp V^\cdot$ is the simplicial object in $\mathbf{Y}$ associated to $f_\sharp\tau$. The rule $f^*F(U) \cong F(f_\sharp U)$ (see again 6.1.11) gives the isomorphism of cosimplicial sets $f^*F(V^\cdot) \cong F(f_\sharp V^\cdot)$ and hence an isomorphism of complexes

$$\check{C}(\tau, f^*F) \cong \check{C}(f_\sharp\tau, F) .$$

Since F is flabby the higher cohomology groups of the right-hand side vanish. □

3.1.6. — We now construct a canonical flabby resolution functor

$$\mathcal{Fl}\colon \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \ , \quad \mathrm{id} \to \mathcal{Fl} \ .$$

It associates to a F a sort of Godement resolution which consists in fact of flasque sheaves.

For a space U let (U) denote the site of open subsets of U with the topology of open coverings. We will first construct flabby resolution functors

$$\mathcal{Fl}_U\colon \mathrm{Sh}_{\mathrm{Ab}}(U) \to C^+(\mathrm{Sh}_{\mathrm{Ab}}(U)) \ , \quad \mathrm{id} \to \mathcal{Fl}_U$$

for all $(U \to X) \in \mathbf{X}$ which are compatible with the morphisms $V \to U$ in $\mathbf{X}$. For $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ we obtain a collection of flabby resolutions $(F_U \to \mathcal{Fl}_U(F_U))_{U\in\mathbf{X}}$, which by 6.1.14 give rise to a resolution $F \to \mathcal{Fl}(F)$. In the following we discuss these steps in detail.

3.1.7. — Let $p_U\colon \hat{U} \to U$ be the identity map, where $\hat{U}$ is the set U with the discrete topology. Let $F \in \mathrm{Sh}_{\mathrm{Ab}}(U)$. We set $\mathcal{Fl}^0_U(F) := (p_U)_* \circ p_U^*(F)$ and let $F \to \mathcal{Fl}^0_U(F)$ be given by the unit $\mathrm{id} \to (p_U)_* \circ p_U^*$.

Lemma 3.1.6. — *The sequence* $0 \to F \to (p_U)_* \circ p_U^* F$ *is exact.*

Proof. — Let $w \in U$. We must show that the induced map on stalks $F_w \to ((p_U)_* \circ p_U^* F)_w$ is injective. This immediately follows from the description

$$((p_U)_* \circ p_U^* F)_w = \mathrm{colim}_{w\in W\subseteq U} \prod_{v\in W} F_v \ . \qquad \square$$

3.1.8. — We now construct $\mathcal{Fl}_U(F)$ inductively. Assume that we have already constructed $\mathcal{Fl}^0_U(F) \to \cdots \to \mathcal{Fl}^k_U(F)$. Then we let

$$\mathcal{Fl}^{k+1}_U(F) := (p_U)_* \circ p_U^*(\mathrm{coker}(\mathcal{Fl}^{k-1}_U(F) \to \mathcal{Fl}^k_U(F))$$

and $\mathcal{Fl}^k_U(F) \to \mathcal{Fl}^{k+1}_U(F)$ be again given by

$$\mathcal{Fl}^k_U(F) \to \mathrm{coker}(\mathcal{Fl}^{k-1}_U(F) \to \mathcal{Fl}^k_U)) \overset{\mathrm{unit}}{\to} \mathcal{Fl}^{k+1}_U(F) \ .$$

In this way we construct an exact complex

$$0 \to F \to \mathcal{Fl}^0_U(F) \to \mathcal{Fl}^1_U(F) \to \cdots \to \mathcal{Fl}^k_U(F) \to \cdots \ .$$

All pieces of the construction are functorial. Hence, the association $F \mapsto \mathcal{Fl}_U(F)$ is functorial in F. The inclusion $F \to \mathcal{Fl}^0_U(F)$ gives the natural transformation $\mathrm{id} \to \mathcal{Fl}_U$.

3.1.9.

Lemma 3.1.7. — *For any sheaf* $F \in \mathrm{Sh}_{\mathrm{Ab}}(U)$ *the sheaf* $(p_U)_* \circ p_U^*(F)$ *is flasque and flabby.*

Proof. — For $W \subseteq U$ we have

$$(p_U)_* \circ p_U^*(F)(W) \cong \prod_{w \in W} F_w \ . \tag{3.1.8}$$

It is now obvious that $(p_U)_* \circ p_U^*(F)(U) \to (p_U)_* \circ p_U^*(F)(W)$ is surjective. A flasque sheaf is flabby by Lemma 3.1.3. □

3.1.10. — We now consider a sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$. For $(U \to \mathbf{X})$ let $F_U \in \mathrm{Sh}_{\mathrm{Ab}}(U)$ denote its restriction to (U). We apply the previous construction to all objects $(U \to X) \in \mathbf{X}$ and the sheaves F_U. Then we get a collection of complexes of sheaves $\mathcal{G}\ell_U(F_U)$ for all $(U \to X) \in \mathbf{X}$. Let $f\colon V \to U$ be a morphism in $\mathbf{X}$. We shall construct a functorial morphism $f^*\mathcal{G}\ell_U(F_U) \to \mathcal{G}\ell_V(F_V)$.

Let $G \in \mathrm{Sh}(U)$, $H \in \mathrm{Sh}(V)$, and $f^*G \to H$ be a morphism of sheaves. We consider the diagram

$$\begin{array}{ccc} \hat{V} & \xrightarrow{\hat{f}} & \hat{U} \\ \downarrow{\scriptstyle p_V} & & \downarrow{\scriptstyle p_U} \\ V & \xrightarrow{f} & U \end{array} \ .$$

It induces the transformation, natural in G and H,

$$\begin{aligned} f^* \circ (p_U)_* \circ p_U^*(G) &\to (p_V)_* \circ \hat{f}^* \circ p_U^*(G) \\ &\cong (p_V)_* \circ p_V^* \circ f^*(G) \\ &\to (p_V)_* \circ p_V^*(H) \end{aligned}$$

We now construct the map $f^*\mathcal{G}\ell_U(F_U) \to \mathcal{G}\ell_V(F_V)$ of complexes inductively. Assume that we have already constructed the morphisms $f^*(\mathcal{G}\ell_U^i(F_U)) \to \mathcal{G}\ell_V^i(F_V)$ for all $i \leq k$ compatible with the differential. Using that f^* is right exact (Lemma 6.1.9), we have an induced morphism

$$f^*\mathrm{coker}(\mathcal{G}\ell_U^{k-1}(F_U) \to \mathcal{G}\ell_U^k(F_U)) \to \mathrm{coker}(\mathcal{G}\ell_V^{k-1}(F_V) \to \mathcal{G}\ell_V^k(F_V)).$$

The construction above gives a morphism $f^*\mathcal{G}\ell_U^{k+1}(F_U) \to \mathcal{G}\ell_V^{k+1}(F_V)$, again compatible with the differential of the complexes.

In this way we get the morphism $f^*\mathcal{G}\ell_U(F_U) \to \mathcal{G}\ell_V(F_V)$. By an inspection of the construction we check that for a second morphism $g\colon W \to V$ in $\mathbf{X}$ the morphisms $g^*f^*\mathcal{G}_U(F_U) \to g^*\mathcal{G}_V(F_V) \to \mathcal{G}_W(F_W)$ and $(f \circ g)^*\mathcal{G}_U(F_U) \to \mathcal{G}_W(F_W)$ coincide.

The collections of resolutions $F_U \to \mathcal{G}\ell_U(F_U)$, $(U \to X) \in \mathbf{X}$, determines a resolution $F \to \mathcal{G}\ell(F)$ in $C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$.

3.1.11.

Lemma 3.1.9. — *The association $F \mapsto (F \to \mathcal{F}\!\ell(F))$ is a functorial flabby resolution.*

Proof. — The local constructions $F_U \mapsto \mathcal{F}\!\ell_U(F_U)$ are functorial in F_U. The connecting maps $f^*\mathcal{F}\!\ell_U(F_U) \to \mathcal{F}\!\ell_V(F_V)$ are compatible with this functoriality. It follows that the construction $F \to \mathcal{F}\!\ell(F)$ is functorial in F.

The restrictions $\mathrm{Sh}\mathbf{X} \to \mathrm{Sh}(U)$ detect flabbiness and exact sequences (see 6.1.14). Therefore the local statements 3.1.6 and 3.1.7 imply that the sequence $0 \to F \to \mathcal{F}\!\ell(F)$ is a quasi-isomorphism, and that the sheaves $\mathcal{F}\!\ell^k(F)$ are flabby for all $k \geq 0$. □

Definition 3.1.10. — *We call $F \to \mathcal{F}\!\ell(F)$ the functorial flabby resolution of F.*

Note that it actually produces resolutions by flasque sheaves.

3.1.12. — Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a map of locally compact stacks which has local sections. Let $\mathcal{F}\!\ell_{\mathbf{X}}$ and $\mathcal{F}\!\ell_{\mathbf{Y}}$ denote the flabby resolution functors for $\mathbf{X}$ and $\mathbf{Y}$ according to Definition 3.1.10.

Lemma 3.1.11. — *We have a natural isomorphism of functors $f^* \circ \mathcal{F}\!\ell_{\mathbf{Y}} \cong \mathcal{F}\!\ell_{\mathbf{X}} \circ f^*$.*

Proof. — For $(U \to X) \in \mathbf{X}$ we have by 6.1.11 a natural isomorphism $f^*F_U \cong F_{f_\sharp U}$. It gives natural isomorphisms $\mathcal{F}\!\ell_U((f^*F)_U) \cong \mathcal{F}\!\ell_{f_\sharp U}(F_{f_\sharp U})$ and thus $\mathcal{F}\!\ell_{\mathbf{X}}(f^*F)_U \cong (f^*\mathcal{F}\!\ell_{\mathbf{Y}})_U$. Finally this collection of isomorphisms gives a natural isomorphism

$$\mathcal{F}\!\ell_{\mathbf{X}}(f^*F) \cong f^*\mathcal{F}\!\ell_{\mathbf{Y}}(F)\ .$$ □

3.1.13.

Lemma 3.1.12. — *The functorial flabby resolution functor preserves flatness.*

Proof. — Consider a space U, $p : \hat{U} \to U$ as above and a flat sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}(U)$. Then $\mathrm{coker}(F \to p_*p^*(F))$ is flat as shown in the proof of [**17**, Lemma 3.1.4]. This implies inductively that the sheaves $\mathcal{F}\!\ell^k_U(F)$ are flat for all $k \geq 0$. The result for the functorial flabby resolution functor on $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ now follows from the fact that the restriction functors $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}(U)$ detect flatness (see 6.2.6). □

3.1.14. — We can extend the flabby resolution functor 3.1.10 to a quasi-isomorphism preserving functor

$$\mathcal{F}\!\ell\colon C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

by applying $\mathcal{F}\!\ell$ to a complex term-wise and forming the total complex of the resulting double complex.

3.2. A model for the push-forward

3.2.1. — Let $f\colon G \to X$ be a morphism of locally compact stacks which has local sections. Following [**9**, Sec. 2.4] we construct a nice model for the functor $Rf_* \circ f^*\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$. We choose an atlas $a\colon A \to G$. Then by Proposition 6.1.1 the composition $f \circ a\colon A \to G \to X$ is representable. Then we can define the functor

$$^{p}C_A\colon C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{G}) \to C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{X})$$

as in [**9**, Sec. 2.4] (the subscript ^{p} indicates that it acts between categories of presheaves).

3.2.2. — We recall the definition $^{p}C_A$. For $(U \to X)$ consider the Cartesian diagram

$$\begin{array}{ccc} G_U & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & X \end{array} .$$

Then for $F \in \mathrm{Pr}_{\mathrm{Ab}}\mathbf{G}$ we have

$$^{p}C_A^k(F)(U \to X) = F((\underbrace{A \times_G \cdots \times_G A}_{k+1 \text{ factors}}) \times_G G_U \to G) . \tag{3.2.1}$$

The differential $^{p}C_A^k(F)(U \to X) \to {}^{p}C_A^{k+1}(F)(U \to X)$ is induced as usual as an alternating sum by the projections

$$(\underbrace{A \times_G \cdots \times_G A}_{k+2 \text{ factors}}) \to (\underbrace{A \times_G \cdots \times_G A}_{k+1 \text{ factors}}) .$$

We extend the functor $^{p}C_A$ to sheaves by the formula

$$C_A := i^\sharp \circ {}^{p}C_A \circ i .$$

3.2.3. — The functor

$$i^\sharp\colon C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{X}) \to C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

is exact by 6.1.8. The functor $^{p}C_A$ is exact, see [**9**, 2.4.8]. Since flabby sheaves are i-acyclic the functor $i \circ \mathcal{F}l : C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{X})$ preserves quasi-isomorphisms.

Therefore the composition

$$i^\sharp \circ {}^{p}C_A \circ i \circ \mathcal{F}l = C_A \circ \mathcal{F}l : C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \to C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

preserves quasi-isomorphisms and descends to the homotopy categories (1)

$$C_A \circ \mathcal{F}l\colon hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \to hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) .$$

(1) By abuse of notation we use the same symbol

After identification of the homotopy categories with the derived categories we have by [**9**, 2.41] that

$$C_A \circ \mathcal{H} \cong Rf_*\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \ .$$

3.2.4. — Since $f\colon G \to X$ has local sections the functor f^* is exact. It therefore descends to

$$f^*\colon hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \ .$$

The composition

$$C_A \circ \mathcal{H} \circ f^*\colon hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

thus represents

$$Rf_* \circ f^*\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \ .$$

3.2.5. — We now study the dependence of C_A on the choice of the atlas $A \to G$. Let us consider a diagram

(3.2.2)
$$\begin{array}{ccc} A' & \xrightarrow{\ \phi\ } & A \\ & {\scriptstyle a'}\searrow \quad \swarrow{\scriptstyle a} & \\ & G & \end{array} \ ,$$

where a' satisfies the same assumptions as a (see 3.2.1). The map ϕ induces maps

$$\begin{array}{ccc} (A' \times_G \cdots \times_G A') \times_G G_U & \xrightarrow{\ \phi^{k+1} \times \mathrm{id}_{G_U}\ } & (A \times_G \cdots \times_G A) \times_G G_U \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

and therefore

$$\begin{aligned} {}^pC^k_A(F)(U \to X) &= F((\underbrace{A \times_G \cdots \times_G A}_{k+1 \text{ factors}}) \times_G G_U \to G) \\ &\to F((\underbrace{A' \times_G \cdots \times_G A'}_{k+1 \text{ factors}}) \times_G G_U \to G) \\ &= {}^pC^k_{A'}(F)(U \to X) \ . \end{aligned}$$

This map is natural in F and preserves the cosimplicial structures. In other words, the diagram (3.2.2) induces a natural transformation

$${}^pC_\phi\colon {}^pC_A \to {}^pC_{A'} \ .$$

Composing with $i^\sharp$ and $i \circ \mathcal{H}$ we get a morphism of functors

$$C_\phi\colon C_A \circ \mathcal{H} \to C_{A'} \circ \mathcal{H}\colon hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \to hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \ .$$

Both $C_A \circ \mathcal{G}\ell$ and $C_{A'} \circ \mathcal{G}\ell$ represent Rf_*. Using the explicit constructions and the proof of [**9**, Lemma 2.36] one checks that the diagram

$$\begin{array}{ccc} H^0(C_A \circ \mathcal{G}\ell)(F) & \xrightarrow{H^0(C_\phi)} & H^0(C_{A'} \circ \mathcal{G}\ell)(F) \\ & \searrow \quad \swarrow & \\ & f_*(F) & \end{array}$$

commutes. Therefore, on the level of derived categories, $C_\phi : C_A \circ \mathcal{G}\ell \to C_{A'} \circ \mathcal{G}\ell$ is the canonical isomorphism between two realizations of Rf_*.

3.2.6. — Let $q\colon H \to G$ be a representable morphism with local sections. Consider the pullback diagram

$$\begin{array}{ccc} B & \xrightarrow{b} & H \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{a} & G \\ & & \downarrow{\scriptstyle f} \\ & & X \end{array} .$$

Then $b : B \to H$ is an atlas, and we can form the functor $C_B\colon C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{H}) \to C^+(\mathrm{Pr}_{\mathrm{Ab}}\mathbf{X})$.

Observe that

$$B \times_H \cdots \times_H B \cong (A \times_G \cdots \times_G A) \times_G H \; .$$

For $(U \to X)$ consider the diagram

$$\begin{array}{ccc} H_U & \longrightarrow & H \\ \downarrow & & \downarrow{\scriptstyle q} \\ G_U & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & X \end{array} .$$

Observe further that

$$(B \times_H \cdots \times_H B) \times_H H_U \cong (A \times_G \cdots \times_G A) \times_G G_U \times_G H \; .$$

For a presheaf $F \in \mathrm{Pr}\mathbf{H}$ and $(V \to G) \in \mathbf{G}$ we have ${}^{p}q_{*}(F)(V) = F(V \times_{G} H)$. We now have the following identity

$$\begin{aligned}
{}^{p}C_{A}^{k} \circ {}^{p}q_{*}(F)(U \to X) &\cong {}^{p}q_{*}(F)(\underbrace{(A \times_{G} \cdots \times_{G} A)}_{k+1 \text{ factors}} \times_{G} G_{U} \to G) \\
&\cong F(((\underbrace{A \times_{G} \cdots \times_{G} A}_{k+1 \text{ factors}}) \times_{G} G_{U}) \times_{G} H \to H) \\
&\cong F((\underbrace{B \times_{H} \cdots \times_{H} B}_{k+1 \text{ factors}}) \times_{H} H_{U} \to H) \\
&\cong {}^{p}C_{B}^{k}(F)(U \to X)
\end{aligned}$$

This isomorphism is functorial in F and induces a natural isomorphism

$${}^{p}C_{A} \circ {}^{p}q_{*} \cong {}^{p}C_{q^{*}A} \ ,$$

where we write $q^{*}A := B$.

The functor ${}^{p}q_{*}$ preserves sheaves [**9**, Lemma 2.13]. Therefore we get the identity

$$i \circ i^{\sharp} \circ {}^{p}q_{*} \circ i = {}^{p}q_{*} \circ i$$

and thus an isomorphism

$$C_{A} \circ q_{*} \cong i^{\sharp} \circ {}^{p}C_{A} \circ i \circ i^{\sharp} \circ {}^{p}q_{*} \circ i \cong i^{\sharp} \circ {}^{p}C_{A} \circ {}^{p}q_{*} \circ i \cong i^{\sharp} \circ {}^{p}C_{q^{*}A} \circ i \cong C_{q^{*}A} \ . \tag{3.2.3}$$

3.2.7. — Consider a Cartesian diagram

$$\begin{array}{ccc}
H & \xrightarrow{v} & G \\
\downarrow{\scriptstyle g} & & \downarrow \\
Y & \xrightarrow{u} & X
\end{array}$$

where u has local sections. We extend the diagram to

$$\begin{array}{ccc}
B & \longrightarrow & A \\
\downarrow & & \downarrow \\
H & \xrightarrow{v} & G \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{u} & X
\end{array} \ .$$

The map $B \to H$ is again an atlas.

Lemma 3.2.4. — *We have a natural isomorphism of functors*

$$u^{*} \circ C_{A} \cong C_{B} \circ v^{*} \ .$$

Proof. — We first find a natural isomorphism

$$^{p}u^{*} \circ {}^{p}C_{A} \cong {}^{p}C_{B} \circ {}^{p}v^{*}.$$

Let $(U \to Y) \in \mathbf{Y}$ and $F \in \mathrm{Pr}_{\mathrm{Ab}}\mathbf{G}$. Then we have

$$^{p}u^{*} \circ {}^{p}C_{A}(F)(U) \cong {}^{p}C_{A}(F)(u_{\sharp}U)\ .$$

We have a diagram

$$\begin{array}{ccccc} H_U \cong G_{u_\sharp U} & \longrightarrow & H & \xrightarrow{v} & G \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ U & \longrightarrow & Y & \xrightarrow{u} & X \end{array}\ .$$

We calculate

$$\begin{aligned} (A \times_G \cdots \times_G A) \times_G G_{u_\sharp U} &\cong (A \times_G \cdots \times_G A) \times_G H \times_H G_{u_\sharp U} \\ &\cong v_\sharp(B \times_H \cdots \times_H B) \times_H H_U \end{aligned}$$

This implies that

$$\begin{aligned} {}^{p}u^{*} \circ C_A(F)(U) &\cong {}^{p}C_A(F)(u_\sharp U) \\ &\cong F((A \times_G \cdots \times_G A) \times_G G_{u_\sharp U}) \\ &\cong F(v_\sharp((B \times_H \cdots \times_H B) \times_H H_U)) \\ &\cong ({}^{p}v^{*}F)((B \times_H \cdots \times_H B) \times_H H_U) \\ &\cong {}^{p}C_B \circ {}^{p}v^{*}(F)(U) \end{aligned}$$

Since u and v have local sections, by 6.1.11 the functors ${}^{p}u^{*}$ and ${}^{p}v^{*}$ commute with $i \circ i^{\sharp}$, and this isomorphism induces

$$u^{*} \circ C_A \cong C_B \circ v^{*}$$

(compare with the calculation (3.2.3)). □

3.2.8. — The isomorphisms of Lemma 3.2.4 and Lemma 3.1.11 induce an isomorphism

$$u^{*} \circ C_A \circ \mathscr{H} \cong C_B \circ u^{*} \circ \mathscr{H} \cong C_B \circ \mathscr{H} \circ v^{*}\ . \tag{3.2.5}$$

On the other hand, by Lemma 6.1.12 we have an isomorphism

$$u^{*} \circ Rf_{*} \cong Rg_{*} \circ v^{*}\ .$$

Lemma 3.2.6. — *The following diagram of natural isomorphisms of functors*

$$D^{+}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \to D^{+}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{H})$$

commutes.

$$\begin{array}{ccc} u^* \circ C_A \circ \mathcal{H}\ell & \xrightarrow{\cong} & C_B \circ \mathcal{H}\ell \circ v^* \\ \downarrow \cong & & \downarrow \cong \\ u^* \circ Rf_* & \xrightarrow{\cong} & Rg_* \circ v^* \end{array}$$

Proof. — It is easy to check that this commutativity holds true on the level of zeroth cohomology sheaves. Since all functors are the derived versions of their zeroth cohomology functors the required commutativity follows. □

Corollary 3.2.7. — *The following diagram of natural isomorphisms commutes*

$$\begin{array}{ccc} u^* \circ C_A \circ \mathcal{H}\ell \circ f^* & \xrightarrow{\cong} & C_B \circ \mathcal{H}\ell \circ g^* \circ u^* \\ \downarrow \cong & & \downarrow \cong \\ u^* \circ Rf_* \circ f^* & \xrightarrow{\cong} & Rg_* \circ g^* \circ u^* \end{array}$$

3.3. Zig-zag diagrams and limits

3.3.1. — We define the unbounded derived category $D(\mathcal{A})$ of an abelian category as the homotopy category $hC(\mathcal{A})$ of complexes (with no restrictions) in $\mathcal{A}$.

Definition 3.3.1. — *An abelian category $\mathcal{A}$ with the following properties*

(1) *$\mathcal{A}$ is cocomplete,*
(2) *filtered colimits are exact,*
(3) *$\mathcal{A}$ has a generator, i.e. there is an object Z such that for every object F with proper subobject $F' \subset F$, $\mathrm{Hom}(Z, F') \to \mathrm{Hom}(Z, F)$ is not surjective.*

is called a Grothendieck abelian category.

In this section, we will consider a Grothendieck category in which countable products exist, e.g. a complete Grothendieck category. The category $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ of sheaves of abelian groups on a site $\mathbf{X}$ is a complete Grothendieck abelian category [**25**, Chapter I, Thm. 3.2.1].

Lemma 3.3.2. — *If Z is a small category and $\mathcal{A}$ is a Grothendieck abelian category in which countable products exists, then the diagram category $\mathcal{A}^Z$ is again a Grothendieck abelian category in which countable products exist.*

This is proved in [**25**, 1.4.3].

3.3.2. — We consider the category $C(\mathcal{A})$ of complexes in a Grothendieck abelian category $\mathcal{A}$. It is known that $C(\mathcal{A})$ has a model category structure (see [**16**, Theorem 2.2] where this fact is attributed to Joyal, [**15**, Thm. 2.3.12] for the example of the category of modules over a ring, and [**2**] for a proof in general). This model structure is given by the following data:

(1) The weak equivalences are the quasi-isomorphisms.
(2) The cofibrations are the degree-wise injections.
(3) The fibrations are defined by the right lifting property.

By $hC(\mathcal{A})$ we denote the homotopy category of $C(\mathcal{A})$. The category $hC(\mathcal{A})$ is triangulated with the shift functor $T\colon hC(\mathcal{A}) \to hC(\mathcal{A})$ given by the shift of complexes $T(X) = X[1]$. The class of distinguished triangles is generated by the mapping cone sequences on $C(\mathcal{A})$,

$$\cdots \to A \xrightarrow{f} B \to C(f) \to T(A) \cdots .$$

The extension of a morphism in $[f] \in hC(\mathcal{A})$ with chosen representative $f \in C(\mathcal{A})$ to a triangle can thus naturally be defined using the mapping cone $C(f)$.

3.3.3. — Let $\mathcal{A}$ be a Grothendieck abelian category, and consider a small category Z. Then we have an equivalence $C(\mathcal{A})^Z \cong C(\mathcal{A}^Z)$. Because $\mathcal{A}^Z$ is a Grothendieck category by Lemma 3.3.2, we can equip the category of Z-diagrams $C(\mathcal{A})^Z$ with the injective model category structure. By translation of 3.3.2 we get the following description.

(1) The weak equivalences are the level-wise quasi-isomorphisms.
(2) The cofibrations are the level-wise injections.
(3) The fibrations are defined by the right lifting property.

3.3.4. — We consider the category U pictured by

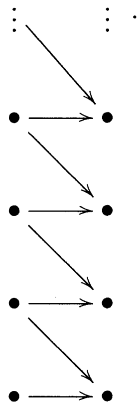
$$\begin{array}{ccccccc} \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet \\ \uparrow & & & & & & \downarrow \\ \bullet & & & & & & \bullet \end{array}$$

We let $\mathcal{D}(\mathcal{A}) \subset C^+(\mathcal{A})^U$ be the subcategory of objects of the form

(3.3.3)
$$\begin{array}{ccccccc} Y_0 & \xleftarrow[\sim]{} & Y_1 & \longrightarrow & Y_2 & \xleftarrow[\sim]{} & Y_3 \\ \uparrow & & & & & & \downarrow \\ X & & & & & & X[-2] \end{array}$$

with bounded below complexes Y_i, X. A morphism in the category $\mathcal{D}(\mathcal{A})$ is given by maps $Y_i \to Y_i'$, $i = 0, 1, 2, 3$, and $X \to X'$ which are compatible with the structure maps. A quasi-isomorphism in this category is a morphism which is a quasi-isomorphism level-wise.

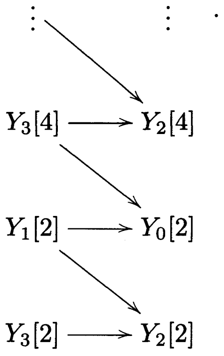
3.3.5. — We let Z be the category pictured by



Let $C(\mathscr{A})^Z$ be the category of Z-diagrams of complexes in $\mathscr{A}$. We define a functor

$$R_1\colon \mathscr{D}(\mathscr{A}) \to C(\mathscr{A})^Z$$

which maps the diagram (3.3.3) to the Z-diagram



The maps are induced by the shifted maps of the diagram (3.3.3), and the composition $Y_3[2k+2] \to X[2k] \to Y_0[2k]$. The functor R_1 preserves quasi-isomorphisms, since those are defined level-wise.

3.3.6. — We now define a triangulated functor

$$\lim\colon h(C(\mathscr{A})^Z) \to hC(\mathscr{A})$$

by a direct construction on the level of complexes. Consider a Z-diagram $X \in C(\mathscr{C})^Z$

$$\begin{array}{ccc} C_3 & \xrightarrow{c_3} & B_3 \; . \\ & \searrow^{d_3} & \\ C_2 & \xrightarrow{c_2} & B_2 \\ & \searrow^{d_2} & \\ C_1 & \xrightarrow{c_1} & B_1 \\ & \searrow^{d_1} & \\ C_0 & \xrightarrow{c_0} & B_0 \end{array}$$

We define the morphism in $C(\mathscr{C})$

$$\phi_X \colon \prod_{i\geq 0} C_i \to \prod_{i\geq 0} B_i$$

which maps $(x_i)_{i\geq 0}$ to $(c_i(x_i) - d_{i+1}(x_{i+1}))_{i\geq 0}$. Then we define $\lim(X)$ as a shifted cone of ϕ_X:

$$\lim(X) := C(\phi_X)[-1] \in C(\mathscr{C}) \; .$$

Since quasi-isomorphisms in $C(\mathscr{C})^Z$ are defined level-wise, the functorial construction $X \to \lim X$ preserves quasi-isomorphisms and thus descends to a functor

$$\lim \colon h(C(\mathscr{C})^Z) \to hC(\mathscr{C}) \; .$$

Note that lim commutes with the shift and sum, so that it is a triangulated functor.

3.3.7. — We now consider the composition $\lim \circ R_1 \colon \mathscr{D}(\mathscr{C}) \to hC(\mathscr{C})$. The composition of the maps (or their inverses, respectively) in the diagram (3.3.3) gives rise to a morphism $D[-2] \colon X \to X[-2]$ in $hC(\mathscr{C})$. We consider the sequence

$$X^\bullet \colon X \overset{D}{\leftarrow} X[2] \overset{D[2]}{\leftarrow} X[4] \leftarrow \cdots \; . \tag{3.3.4}$$

in $hC(\mathscr{C})$. As already explained in 2.4, for such a diagram in the triangulated category $hC(\mathscr{C})$ the homotopy limit $\operatorname{holim}(X^\bullet) \in hC(\mathscr{C})$ is a well-defined isomorphism class of objects. It is given by the mapping cone shifted by -1 of the morphism

$$\prod_{i\geq 0} X[2i] \to \prod_{i\geq 0} X[2i]$$

which maps $(x_i)_{i\geq 0}$ to $(x_i - D[2i]x_{i+1})_{i\geq 0}$ (see [**21**, Sec. 1.6]).

Lemma 3.3.5. — *For a diagram $W \in \mathscr{D}(\mathscr{C})$ of the form* (3.3.3) *we have a non-canonical isomorphism*

$$\operatorname{holim}(X^\bullet) \cong \lim \circ R_1(W) \; .$$

Proof. — We use the dual statement of [**21**, Lemma 1.7.1]. For $i \geq 1$ let $C_{2i-1} := Y_3[2i]$, $C_{2i} := Y_1[2i]$, $B_{2i-1} := Y_2[2i]$ and $B_{2i} := Y_0[2i]$. Note that we have morphisms $v_i \colon C_i \to B_i$ in $C(\mathcal{C})$ which become isomorphisms in $hC(\mathcal{C})$. Moreover, we have maps $w_{2i} \colon C_{2i} \to B_{2i-1}$ coming from the map $Y_1 \to Y_2$ of (3.3.3), and morphisms $w_{2i+1} \colon C_{2i+1} \to B_{2i}$ coming from $Y_3[2] \to X \to Y_0$ of (3.3.3). We consider the diagram in $hC(\mathcal{C})$, using the invertibility of v_i in $hC(\mathcal{C})$,

$$\begin{array}{ccc} \prod_{i\geq 1} C_i & \xrightarrow[\phi_{R_1(W)}]{\prod v_i - \prod w_i} & \prod_{i\geq 1} B_i \\ \downarrow \mathrm{id} & & \downarrow \prod_{i\geq 1} v_i^{-1} \\ \prod_{i\geq 1} C_i & \longrightarrow & \prod_{i\geq 1} C_i \, , \end{array}$$

whose vertical maps are isomorphism. By definition, the mapping cone of the upper horizontal map is $\lim \circ R_1(W)$. Because the vertical maps are isomorphisms in $hC(\mathcal{C})$, this is isomorphic to the mapping cone of the lower horizontal map, which gives the homotopy limit of the sequence

$$Y_3[2] \leftarrow Y_1[2] \leftarrow Y_3[4] \leftarrow Y_1[4] \leftarrow Y_3[6] \cdots .$$

We can expand this sequence to

$$\begin{aligned} (3.3.6) \quad X \leftarrow Y_3[2] \leftarrow Y_2[2] \leftarrow Y_1[2] \leftarrow Y_0[2] \leftarrow X[2] \\ \leftarrow Y_3[4] \leftarrow Y_2[4] \leftarrow Y_1[4] \leftarrow Y_0[4] \leftarrow X[4] \leftarrow Y_3[6] \cdots , \end{aligned}$$

and because the sequence (3.3.4) is just another contraction of (3.3.6), by [**21**, Lemma 1.7.1] its homotopy limit holim $(X^\bullet)$ is then also isomorphic to $\lim \circ R_1(W)$. □

3.4. The functorial periodization

3.4.1. — Let X be a locally compact stack. Define $C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) \subseteq C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ to be the full subcategory of bounded below complexes of flat sheaves.

Lemma 3.4.1. — *This inclusion induces an equivalence of homotopy categories*

$$hC^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) \xrightarrow{\sim} hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) .$$

Proof. — We first construct a functorial flat resolution functor

$$R : \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to C^b(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) .$$

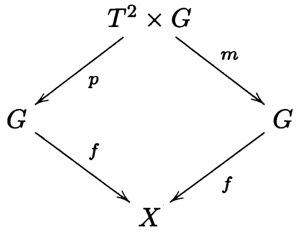
Note that a torsion free sheaf is flat. If $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$, then let $\hat{F} \in \mathrm{Pr}\mathbf{X}$ denote the underlying presheaf of sets. Let $\mathbb{Z}\hat{F} \in \mathrm{Pr}_{\mathrm{Ab}}\mathbf{X}$ be the presheaf of free abelian groups generated by $\hat{F}$, and $\mathbb{Z}F := i^\sharp \mathbb{Z}\hat{F}$ be its sheafification. Then we have a natural evaluation $\mathbb{Z}\hat{F} \to F$, which extends uniquely to $e : \mathbb{Z}F \to F$ since F is a sheaf. We define $R(F)$ to be the complex $\ker(e) \to \mathbb{Z}F$, where $\mathbb{Z}F$ is in degree zero. The natural map $R(F) \to F$ is a quasi-isomorphism. Moreover, $\mathbb{Z}F$ and its subsheaf $\ker(e)$ are torsion-free, hence flat.

We extend R to a functor $R : C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X})$ by applying R objectwise and taking the total complex of the resulting double complex.

The inclusion $C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) \to C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ and $R : C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X})$ induce mutually inverse functors of the homotopy categories. □

3.4.2. — Let $f : G \to X$ be a topological gerbe with band $U(1)$ over a locally compact stack. Recall the associated geometry introduced in 2.2.1. Using the functorial version we get the diagram

(3.4.2)



which 2-functorially depends on the gerbe $G \to X$. The map $p : T^2 \times G \to G$ is the projection onto the second factor, and $m := p \circ \phi$.

3.4.3. — Observe that p is a trivial oriented fiber bundle with fiber T^2. Let

$$0 \to \underline{\mathbb{Z}}_{\mathrm{Site}(T^2 \times G)} \to \mathcal{H}(\underline{\mathbb{Z}}_{\mathrm{Site}(T^2 \times G)})$$

be the functorial flat and flabby resolution of $\underline{\mathbb{Z}}_{\mathbf{G}}$ constructed in 3.1.10, see also 3.1.12 for flatness. By

$$K^{\cdot} : 0 \to K^0 \to K^1 \to K^2 \to 0$$

we denote the truncation of $\mathcal{H}(\underline{\mathbb{Z}}_{\mathrm{Site}(T^2 \times G)})$ after the second term, i.e. with

$$K^2 := \ker(\mathcal{H}^2(\underline{\mathbb{Z}}_{\mathrm{Site}(T^2 \times G)}) \to \mathcal{H}^3(\underline{\mathbb{Z}}_{\mathrm{Site}(T^2 \times G)})) \ .$$

The complex $K^{\cdot}$ is still a flat and p_*-acyclic resolution of $\underline{\mathbb{Z}}_{\mathrm{Site}(T^2 \times G)}$ (Lemma 6.3.3). Let

$$T : C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathrm{Site}(T^2 \times G)) \to C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathrm{Site}(T^2 \times G))$$

be the functor given on objects by

$$T_{K^{\cdot}}(F) := F \otimes K^{\cdot} \ .$$

3.4.4. — We consider the commutative diagram 3.4.2. Since $f \circ p \cong f \circ m$ (recall that we actually can assume equality) we have by Lemma 6.6.8 and Corollary 6.6.9 isomorphisms of functors $m^* \circ f^* \cong p^* \circ f^*$ and $f_* \circ m_* \cong f_* \circ p_*$. We fix an atlas $A \to G$ and define $X : C^+(\mathrm{Sh}^{\mathrm{flat}}\mathbf{X}) \to C^+(\mathrm{Sh}\mathbf{X})$ by

$$X := C_A \circ f^* \circ \mathcal{H} \ .$$

Since f has local sections we have $f^* \circ \mathcal{H} \cong \mathcal{H} \circ f^*$ by Lemma 3.1.11. It now follows from 3.2.4 that $X \cong C_A \circ \mathcal{H} \circ f^*$ preserves quasi-isomorphisms. It therefore descends to the homotopy categories and induces the functor $Rf_* \circ f^*$ as composition

$$D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{G}) \overset{Lemma\ 3.4.1}{\cong} hC^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{G}) \xrightarrow{X} hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \cong D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) .$$

3.4.5. — We further form $B := m^*A \times_{T^2 \times G} p^*A$. It comes with a natural morphism $B \to m^*A$ over $T^2 \times G$ which induces a transformation $C_{m^*A} \to C_B$. Using the unit $\mathrm{id} \to m_* \circ m^*$, the inclusion $\mathrm{id} \to T_{K^\cdot}$, and the isomorphisms $m^* \circ f^* \cong p^* \circ f^*$, and using that by 3.2.6 $C_A \circ m_* \cong C_{m^*A}$, we define a natural transformation

$$\begin{aligned} X &= C_A \circ f^* \circ \mathcal{H} \\ &\to C_A \circ m_* \circ m^* \circ f^* \circ \mathcal{H} \\ &\to C_A \circ m_* \circ T_{K^\cdot} \circ m^* \circ f^* \circ \mathcal{H} \\ &\cong C_{m^*A} \circ T_{K^\cdot} \circ m^* \circ f^* \circ \mathcal{H} \\ &\cong C_{m^*A} \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{H} \\ &\to C_{m^*A} \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{H} \\ &\to C_B \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{H} \\ &=: Y_0 \end{aligned}$$

Using the other projection $B \to p^*A$ we define

$$\begin{aligned} Y_1 &:= C_{p^*A} \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \\ &\xrightarrow{\sim} C_B \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \\ &\xrightarrow{\sim} C_B \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{H} \\ &= Y_0 . \end{aligned}$$

Using the identity $C_{p^*A} \cong C_A \circ p_*$ we define

$$\begin{aligned} Y_1 &= C_{p^*A} \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \\ &\cong C_A \circ p_* \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \\ &\to C_A \circ \mathcal{H} \circ p_* \circ \mathcal{H} \circ T_{K^\cdot} \circ p^* \circ f^* \\ &=: Y_2 \end{aligned}$$

Note that $p_* \circ T_K$ is an exact functor by Lemma 6.3.6 and calculates Rp_* by Corollary 6.4.4. Since $p_* \circ \mathcal{H} \circ T_K$ represents the same functor, the map $p_* \circ T_K \to p_* \circ \mathcal{H} \circ T_K$ induces a quasi-isomorphism which is preserved by $C_A \circ \mathcal{H}$. The natural transformation $T_{p_*K^\cdot} \xrightarrow{\sim} p_* \circ T_{K^\cdot} \circ p^*$ is an isomorphism, if applied to complexes of flat sheaves by 6.2.11. By Lemma 6.1.11 the pull-back f^* preserves flatness.

These two facts explain the quasi-isomorphisms in

$$\begin{aligned} Y_3 &:= C_A \circ \mathcal{Fl} \circ T_{p_* K^\cdot} \circ f^* \\ &\xrightarrow{\sim} C_A \circ \mathcal{Fl} \circ p_* \circ T_{K^\cdot} \circ p^* \circ f^* \\ &\xrightarrow{\sim} C_A \circ \mathcal{Fl} \circ p_* \circ \mathcal{Fl} \circ T_{K^\cdot} \circ p^* \circ f^* \\ &= Y_2 \ . \end{aligned}$$

Using the projection $T_{p_*K} \stackrel{[-2]}{\rightarrow}$ id of (6.5.8) we define the natural transformation

$$\begin{aligned} Y_3 &= C_A \circ \mathcal{Fl} \circ T_{p_* K^\cdot} \circ f^* \\ &\to C_A \circ \mathcal{Fl} \circ f^*[-2] \\ &\cong C_A \circ f^* \circ \mathcal{Fl}[-2] \\ &= X[-2] \ . \end{aligned} \tag{3.4.3}$$

Observe that all functors Y_i preserve quasi-isomorphisms, using that f^*, p^*, $C_A \circ \mathcal{Fl}$, $p_* \circ T_K$ (and by Lemma 6.2.11 therefore also T_{p_*K}) do so.

3.4.6. — The construction 3.4.4, 3.4.5 gives a quasi-isomorphism preserving functor

$$R_0 \colon C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) \to \mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

(see 3.3.4 for the definition of the target). By composition with the functor R_1 (see 3.3.5) we get a functor

$$R := R_1 \circ R_0 \colon C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) \to C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^{\mathcal{Z}} \ .$$

It preserves quasi-isomorphisms and therefore descends to (again using Lemma 3.4.1)

$$R \colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to h(C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^{\mathcal{Z}}) \ .$$

3.4.7. — The construction of the functor R_0 explicitly depends on the choice of an atlas $A \to G$. These choices form a subcategory $\mathcal{Z} \subset$ Stacks$/G$. The choice of $A \to G$ enters the definition via the functor C_A. For the moment let us indicate the dependence on A in the notation and write R_0^A for the functor R_0 defined with the choice A.

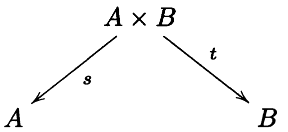
Observe, that $A \to m^*A$, $A \to p^*A$ and $A \to m^*A \times_{T^2 \times G} p^*A$ are functors Stacks$/G \to$ Stacks$/(T^2 \times G)$. The construction 3.2.5 shows that for a given $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ the association $A \to R_0^A(F)$ extends to a functor

$$R_0^{\cdots}(F) \colon \mathcal{Z}^{\mathrm{op}} \to \mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \ .$$

The components $X \cong C_A \circ \mathcal{Fl} \circ f^*$ and $Y_i \cong C_? \circ \mathcal{Fl} \circ \cdots$ (where $? \in \{A, p^*A, m^*A, m^*A \times_{T^2 \times G} p^*A\}$) all involve a flabby resolution functor in front of C_*. If $A \to A'$ is a morphism in $\mathcal{Z}$, then the transformation $C_{A'} \circ \mathcal{Fl} \to C_A \circ \mathcal{Fl}$ (or the similar transformations for the other subscripts) produce quasi-isomorphisms by 3.2.5.

It follows that the functor $R_0^{\cdot\cdot}(F)\colon \mathcal{Z}^{\mathrm{op}} \to \mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ maps all morphisms to quasi-isomorphisms. We now consider the composition $R^{\cdot\cdot}(F) := R_1 \circ R_0^{\cdot\cdot}(F)\colon \mathcal{Z}^{\mathrm{op}} \to h(C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^{\mathcal{Z}})$.

For two objects $A, B \in \mathcal{Z}$ we consider the diagram

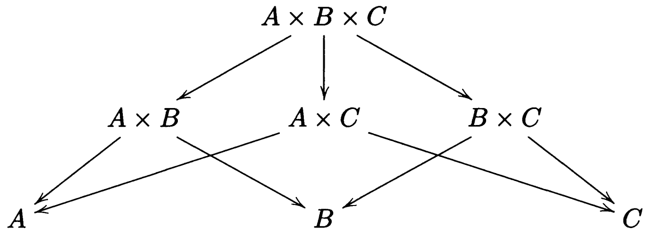


,

where the fiber product is taken in Stacks$/G$. We consider the isomorphism

$$R(A,B) := R^t \circ (R^s)^{-1}\colon R^A(F) \to R^B(F)$$

in $h(C(\mathrm{Sh}_{\mathrm{Ab}}(X))^{\mathcal{Z}})$. Using the commutativity of the squares in the diagram



we check that

$$R(A,B) \circ R(B,C) = R(A,C)\ .$$

This has the following consequence:.

Lemma 3.4.4. — *The functor* $R\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to hC((\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^{\mathcal{Z}})$ *is independent of the choice of the atlas* $A \to G$ *up to canonical isomorphism.*

Consider an automorphism $\phi\colon A \to A$ in $\mathcal{Z}$ and observe that it induces the identity on the level of cohomology, i.e. $H^*(R^\phi) = \mathrm{id}$. It is an interesting question whether R^ϕ is the identity.

3.4.8.

Definition 3.4.5. — *We define the periodization functor*

$$P_G := \lim \circ R\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to h(C((\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^{\mathcal{Z}})) \to hC(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})\ .$$

By Lemma 3.4.4 it is well defined up to canonical isomorphism.

3.4.9. — Let $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$. By 3.2.4 $X(F) = C_A \circ f^* \circ \mathcal{Fl}(F)$ represents $Rf_* \circ f^*(F)$. The composition $D[-2]\colon X \to X[-2]$ of the maps (or their inverses, respectively) in the diagram $R_0^A(F) \in \mathcal{D}(\mathrm{Sh_{Ab}}\mathbf{X})$ represents the map $D_G\colon Rf_* \circ f^*(F) \to Rf_* \circ f^*(F)[-2]$ defined in Definition 2.2.2. By Lemma 3.3.5 we see that $P_G(F)$ (according to 3.4.5) is isomorphic to our former Definition 2.4.2 of the isomorphism class $P_G(F)$.

3.5. Properties of the periodization functor

3.5.1. — The domain and the target of P_G are triangulated categories. Distinguished triangles in both categories are all triangles which are isomorphic to mapping cone sequences

$$\cdots \to C(f)[-1] \to X \xrightarrow{f} Y \to C(f) \to \cdots .$$

Lemma 3.5.1. — *The functor $P_G\colon D^+(\mathrm{Sh_{Ab}}\mathbf{X}) \to hC(\mathrm{Sh_{Ab}}\mathbf{X})$ is triangulated.*

Proof. — We must show that it is additive, preserves the shift, and maps distinguished triangles to distinguished triangles. It follows from the explicit constructions that the functors lim and R_1 are additive and preserve the shift. The functorial flabby resolution $\mathcal{Fl}$ on sheaves is additive. On complexes of sheaves it is defined as the level-wise application of the flabby resolution functor composed with the total complex construction. Therefore it also commutes with the shift. All other functors involved in the construction of R_0 (e.g. C_A, q^*, $T_{K^\cdot}$) are additive and commute with the shift, too.

Since the distinguished triangles in $D^+(\mathrm{Sh_{Ab}}\mathbf{X})$, $h(C(\mathrm{Sh_{Ab}}\mathbf{X})^Z)$, and $hC(\mathrm{Sh_{Ab}}\mathbf{X})$ are defined as triangles which are isomorphic to mapping cone sequences, and the latter only depend on the additive structure and the shift, we see that lim and R preserve triangles. □

3.5.2.

Lemma 3.5.2. — *For $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ the object $P_G(F) \in hC(\mathrm{Sh_{Ab}}\mathbf{X})$ is two-periodic.*

Proof. — The isomorphism $P_G(F)[2] \to P_G(F)$ is given by the isomorphism W in 2.4.3. □

The two-periodicity will be analyzed in more detail in Subsection 3.6.

3.5.3. — Let $u\colon Y \to X$ be a map of topological stacks which admits local sections. Then we consider a Cartesian diagram

$$\begin{array}{ccc} H & \xrightarrow{v} & G \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & X. \end{array} \tag{3.5.3}$$

Lemma 3.5.4. — *The diagram* (3.5.3) *induces an isomorphism* $u^* \circ P_G \xrightarrow{\sim} P_H \circ u^*$.

Proof. — By taking the pull-back of (3.4.2) along u we get the extension of the Cartesian diagram above to

$$\begin{array}{ccc} T^2 \times H & \xrightarrow{w} & T^2 \times G \\ \Downarrow{\scriptstyle n,q} & & \Downarrow{\scriptstyle m,p} \\ H & \xrightarrow{v} & G \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & X \end{array} \;.$$

Note that there is no 2-isomorphism between n and q or m and p, respectively. Since u has local sections the functor $u^*\colon \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$ is exact by Lemma 6.1.11. It therefore extends to functors $u^*\colon \mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to \mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ and $u^*\colon C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^Z \to C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})^Z$ which both preserve quasi-isomorphisms. We therefore also have corresponding functors on the derived categories which will all be denoted by u^*. In the following we are going to show that there are natural isomorphisms

(1) $u^* \circ R_1 \cong R_1 \circ u^*$
(2) $u^* \circ \lim \cong \lim \circ u^*$
(3) $u^* \circ R_0 \cong R_0 \circ u^*$

of functors on the level of homotopy categories.

In fact it follows from an inspection of the construction of R_1 that already $u^* \circ R_1 \cong R_1 \circ u^*$ on the level of functors $\mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})^Z$, i.e. before descending to the homotopy category. Assertion (1) follows.

Since $u^*\colon C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})^Z \to C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})^Z$ preserves products and mapping cones we again have $u^* \circ \lim \cong \lim \circ u^*$ before going to the homotopy categories. This implies (2).

In order to see (3), using v we construct a canonical isomorphism

$$u^* \circ R_0^A \cong R_0^C \circ u^*\colon C^+(\mathrm{Sh}^{\mathrm{flat}}_{\mathrm{Ab}}\mathbf{X}) \to \mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \;,$$

where we indicate the dependence of the functor R_0 on the choices by a superscript as in 3.4.7. The atlas $C \to H$ is given by the diagram

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ H & \xrightarrow{v} & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & X \end{array} ,$$

where the upper square is also Cartesian.

The isomorphism (3) is induced by a collection of isomorphisms indexed by the objects of the diagram U (3.3.4) which induce a morphism of diagrams in $h\mathcal{D}(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$.

First we have

$$\begin{aligned} u^* \circ X &= u^* \circ C_A \circ f^* \circ \mathcal{H} \\ &\cong C_C \circ v^* \circ f^* \circ \mathcal{H} \\ &\cong C_C \circ g^* \circ u^* \circ \mathcal{H} \\ &\cong C_C \circ g^* \circ \mathcal{H} \circ u^* \\ &= X \circ u^* \end{aligned} \tag{3.5.5}$$

where we use Lemma 3.2.4, $v^* \circ f^* \cong g^* \circ u^*$ (see Lemma 6.6.9) and the fact that the flabby resolution functor commutes with the pull-back by u, since u has local sections (Lemma 3.1.11).

Let $D := n^*C \times_{T^2 \times H} q^*C$. We write $K^{\cdot}_{T^2\times G}$ for the complex formerly denoted by $K^{\cdot}$.

Next we observe that there is a canonical isomorphism $w^* K^{\cdot}_{T^2\times G} \cong K^{\cdot}_{T^2\times H}$. In fact $K^{\cdot}_{T^2\times G}$ and $K^{\cdot}_{T^2\times H}$ are given by truncations of the complexes $\mathcal{H}(\underline{\mathbb{Z}}_{\mathrm{Site}(T^2\times G)})$ and $\mathcal{H}(\underline{\mathbb{Z}}_{\mathrm{Site}(T^2\times H)})$. The isomorphism is induced by the fact that w^* commutes with the flabby resolution functor, and the isomorphism

$$w^* \underline{\mathbb{Z}}_{\mathrm{Site}(T^2\times G)} \cong \underline{\mathbb{Z}}_{\mathrm{Site}(T^2\times H)} .$$

This implies by Lemma 6.2.5 that $w^* \circ T_{K^{\cdot}_{T^2\times G}} \cong T_{K^{\cdot}_{T^2\times H}} \circ w^*$. In order to increase readability of the formulas we will omit the double subscript from now on and write $T_{K^{\cdot}}$ for both functors. Using this observation, Lemma 3.2.4, and the other previously used isomorphisms, we get

$$
\begin{aligned}
u^* \circ Y_0 &\cong u^* \circ C_B \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{F}\ell \\
&\cong C_D \circ w^* \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{F}\ell \\
&\cong C_D \circ \mathcal{F}\ell \circ w^* \circ T_{K^\cdot} \circ p^* \circ f^* \circ \mathcal{F}\ell \\
&\cong C_D \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ w^* \circ p^* \circ f^* \circ \mathcal{F}\ell \\
&\cong C_D \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ q^* \circ v^* \circ f^* \circ \mathcal{F}\ell \\
&\cong C_D \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ q^* \circ g^* \circ u^* \circ \mathcal{F}\ell \\
&\cong C_D \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ q^* \circ g^* \circ \mathcal{F}\ell \circ u^* \\
&\cong Y_0 \circ u^*
\end{aligned}
$$

In a similar manner we get

$$
\begin{aligned}
u^* \circ Y_1 &\cong u^* \circ C_{p^*A} \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ p^* \circ f^* \\
&\cong C_{q^*C} \circ w^* \circ \mathcal{F}\ell \circ T_{K^\cdot} \circ p^* \circ f^* \\
&\;\vdots \\
&\cong Y_1 \circ u^* \\
u^* \circ Y_2 &\cong Y_2 \circ u^* \\
u^* \circ Y_3 &\cong Y_3 \circ u^*
\end{aligned}
$$

For these isomorphisms, we use in particular Lemma 6.1.12 to get $v^*p_* \cong q_*w^*$, and moreover Lemma 6.2.5 to get the chain of isomorphisms

$$v^*(F \otimes p_*K) \cong v^*F \otimes v^*p_*K \cong v^*F \otimes q_*w^*K \cong v^*F \otimes q_*K \cong T_{q_*K}(v^*F),$$

which gives the isomorphism $v^* \circ T_{p_*K} \cong T_{q_*K} \circ v^*$.

By a tedious check of the commutativity of many little squares we see that these maps indeed define an isomorphism of functors $u^* \circ R_0^A \cong R_0^C \circ v^*$. As an example of these checks, let us indicate some details of the argument for the map $Y_3 \to X[-2]$. For $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ we have the maps $\phi : Y_3(F) \to X[-2](F)$ and $\psi : Y_3(u^*F) \to X[-2](u^*F)$ given by (3.4.3). We must show that

$$
\begin{array}{ccc}
u^*Y_3(F) & \xrightarrow{\cong} & Y_3(u^*F) \\
\big\downarrow{\scriptstyle u^*\phi} & & \big\downarrow{\scriptstyle \psi} \\
u^*X[-2](F) & \xrightarrow{\cong} & X[-2](u^*F)
\end{array}
$$

commutes. This indeed follows from the sequence of commutative diagrams

$$
(3.5.6)\qquad
\begin{array}{ccccccc}
u^*Y_3 & = & u^*C_A\mathcal{H}\ell T_{p_*K}f^* & \xrightarrow{T_{p_*K}\xrightarrow{[2]}\mathrm{id}} & u^*C_A\mathcal{H}\ell f^*[-2] & = & u^*X[-2] \\
 & & \downarrow\cong & & \downarrow\cong & & \\
 & & C_Bv^*\mathcal{H}\ell T_{p_*K}f^* & \xrightarrow{T_{p_*K}\xrightarrow{[2]}\mathrm{id}} & C_Bv^*\mathcal{H}\ell f^*[-2] & & \\
 & & \downarrow\cong & & \downarrow\cong & & \\
 & & C_B\mathcal{H}\ell v^*T_{p_*K}f^* & \xrightarrow{T_{p_*K}\xrightarrow{[2]}\mathrm{id}} & C_B\mathcal{H}\ell v^*f^*[-2] & & \\
 & & \downarrow\cong & & \downarrow\cong & & \\
Y_3u^* & = & C_B\mathcal{H}\ell T_{q_*K}g^*u^* & \xrightarrow{T_{q_*K}\xrightarrow{[2]}\mathrm{id}} & C_B\mathcal{H}\ell g^*u^*[-2] & = & X[-2]u^*
\end{array}
$$

where for the last we use that w preserves the orientation of the fiber T^2. □

The following statement directly follows from the constructions.

Lemma 3.5.7. — *The isomorphism of Lemma 3.5.4 behaves functorially under compositions of diagrams of the form* (3.5.3).

3.5.4. — Let $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$. Recall that $P_G(F)$ is the homotopy limit of a Z-diagram consisting of sheaves $Y_0[2i]$, $Y_1[2i]$, $Y_2[2i]$, $Y_3[2i]$. For all $i \geq 0$ we construct an evaluation transformation

$$e_i\colon P_G(F) \to Rf_* \circ f^*(F)[2i]$$

as the composition of the canonical map from the limit to $Y_3[2i+2]$ with the structure map to $X[2i]$ and the identification $X[2i](F) \cong Rf_* \circ f^*[2i](F)$. To be precise we consider $Rf_*f^*(F) \in D(\mathrm{Sh_{Ab}}\mathbf{X})$ via the inclusion $D^+(\mathrm{Sh_{Ab}}\mathbf{X}) \to D(\mathrm{Sh_{Ab}}\mathbf{X})$. In the situation of 3.5.3 an inspection of the proof of Lemma 3.5.4 together with Corollary 3.2.7 shows that we have a commutative diagram in $D(\mathrm{Sh_{Ab}}\mathbf{X})$

$$
(3.5.8)\qquad
\begin{array}{ccc}
u^*P_G(F) & \xrightarrow[v^*]{\cong} & P_H(u^*F) \\
\downarrow{\scriptstyle u^*e_i} & & \downarrow{\scriptstyle e_i} \\
u^*Rf_*f^*(F)[2i] & \xrightarrow[v^*]{\cong} & Rg_*g^*(u^*F)[2i]
\end{array}\ .
$$

Note, however, that the morphism in the bottom line is only defined on $D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ (or equivalently on its image in $D(\mathrm{Sh_{Ab}}\mathbf{X})$), and we do not know whether we can extend it to the full unbounded derived category. Fortunately, we do not have to do this for the purposes of the present paper.

3.5.5. — Consider the special case of the diagram (3.5.3) where $Y = X$, $u = \mathrm{id}_X$, $H = G$, and v is an automorphism of the gerbe G. Lemma 3.5.4 provides an automorphism $v^* : P_G \to P_G$ of periodization functors.

3.5.6. — Let us illustrate this automorphism by an example. We consider the trivial $U(1)$-gerbe $G \to S^2$ over S^2 and let $\phi \in \mathrm{Aut}(G/S^2)$ be classified by $1 \in H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$. It induces an automorphism of the cohomology $H^*(S^2; P_G(\underline{F}_{S^2}))$, where $\underline{F}_{S^2}$ is the sheaf represented by a discrete abelian group F. We have a Cartesian diagram

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{B}U(1) \\ \big\downarrow{\scriptstyle g} & & \big\downarrow \\ S^2 & \xrightarrow{f} & * \end{array} .$$

Since $f^*\underline{F}_* \cong \underline{F}_{S^2}$ we have

$$\begin{array}{rcl} H^*(S^2; P_G(\underline{F}_{S^2})) & \cong & H^*(S^2; P_G(f^*\underline{F}_*)) \\ & \overset{\text{Lemma 3.5.4}}{\cong} & H^*(S^2; f^*P_{\mathcal{B}U(1)}(\underline{F}_*)) \\ & \overset{\text{Lemma 6.2.13}}{\cong} & H^*(S^2;\underline{\mathbb{Z}}) \otimes H^*(*; P_{\mathcal{B}U(1)}(\underline{F}_*)) \\ & \cong & \mathbb{Z}[w]/(w^2) \otimes H^*(*; P_{\mathcal{B}U(1)}(\underline{F}_*)) , \end{array}$$

where $H^*(*; P_{\mathcal{B}U(1)}(\underline{F}_*))$ has been calculated in examples in Proposition 2.5.1. If $F = \mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$, then $H^{\mathrm{ev}}(*; P_{\mathcal{B}U(1)}(\underline{F}_*)) \cong \mathbb{Q}$ or $\cdots \cong \mathbb{A}^{\mathbb{Q}}_f$, respectively. If $F = \mathbb{Z}$, then $H^{\mathrm{odd}}(*; P_{\mathcal{B}U(1)}(\underline{\mathbb{Z}}_*)) \cong \mathbb{A}^{\mathbb{Q}}_f/\mathbb{Q}$.

Lemma 3.5.9. — *In all these cases the action of ϕ^* is given by*

$$\phi^*(1 \otimes \lambda + w \otimes \mu) = 1 \otimes \lambda + w \otimes (\lambda + \mu) ,$$

where $\lambda, \mu \in \mathbb{Q}$, $\mathbb{A}^{\mathbb{Q}}_f$, or $\mathbb{A}^{\mathbb{Q}}_f/\mathbb{Q}$, respectively.

Proof. — We will use the description of $H^*(S^2, P_G(\underline{F}_{S^2}))$ given in Corollary 2.4.6. In Lemma 2.2.3 we have calculated the automorphism on $H^*(S^2, Rg_*g^*\underline{F}_{S^2}) \cong F[w][[z]]/(w^2)$ induced by the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g} & \\ & S^2 & \end{array} .$$

It is given by $z \mapsto z + w$, $w \mapsto w$. The operation induced by D_G is $\frac{d}{dz}$, and the periodized cohomology is given as the kernel (in the cases $F = \mathbb{Q}$ and $F = \mathbb{Q}/\mathbb{Z}$) or cokernel (in the case $F = \mathbb{Z}$) of $\prod_{i\geq 0} \mathrm{id}[2i] - \prod_{i\geq 0} D_G[2i]$ on $\prod_{i\geq 0} F[w][[z]]/(w^2)[2i]$. Recall from 2.5.3 that the class $a \in H^0(S^2, P_G(\underline{\mathbb{Q}}_{S^2})) \cong \mathbb{Q}[w]/(w^2)$ is represented by $(a, az, az^2/2, \ldots, az^k/k! \ldots)$, which is mapped by ϕ^* to $(a, a(w+z), a(w+z)^2/2, \ldots)$.

We must read off a representative of this class in the form above. If $a = w$ then $w(w+z)^k/k! = wz^k/k!$ and therefore $\phi^* w = w$. On the other hand, if $a = 1$, then $a(w+z)^k/k! = z^k/k! + wz^{k-1}/(k-1)!$, so that $\phi^*(1) = 1 + w$.

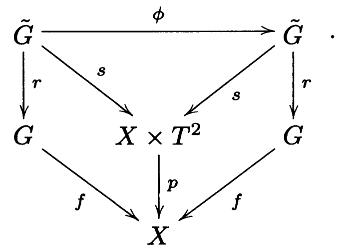
Exactly the same argument applies if $F = \mathbb{Q}/\mathbb{Z}$. Finally, the cohomology with coefficients $F = \mathbb{Z}$ is the cokernel (up to shift of degree) of the map induced by the inclusion $\mathbb{Q} \hookrightarrow \mathbb{A}_f^{\mathbb{Q}}$, which implies the assertion also for $F = \mathbb{Z}$. □

3.6. Periodicity

3.6.1. — We consider a topological $U(1)$-gerbe $f\colon G \to X$ over a locally compact stack. Let $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$. In Lemma 3.5.2 we have argued that $P_G(F) \in D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ is two-periodic. The periodicity is implemented by a certain isomorphism $W : P_G(F)[2] \to P_G(F)$ which may depend on additional choices, see also the discussion in 2.4.4. In the present subsection we show that there is a canonical two-periodicity isomorphism.

3.6.2. — The gerbe $G \to X$ gives rise in a 2-functorial way to the diagram (see 2.2.1 for details)

(3.6.1)



This diagram induces the desired periodization isomorphism as the following composition of natural transformations

$$(3.6.2)\quad W\colon P_G(F) \overset{\text{unit}}{\to} Rp_*p^*P_G(F) \overset{\text{Lemma 3.5.4}}{\to} Rp_*P_{\tilde{G}}(p^*F)$$

$$\xrightarrow{\phi^*} Rp_*P_{\tilde{G}}(p^*F) \cong Rp_*p^*P_G(F) \xrightarrow{\int_p} P_G(F)[-2]\ .$$

Proposition 3.6.3. — *The transformation* (3.6.2)

$$W\colon P_G(F) \to P_G(F)[-2]$$

is a canonical choice for the isomorphism in Proposition 2.4.3.

3.6.3. — To start the proof of Proposition 3.6.3, recall the definition

$$D_G\colon Rf_*f^*(F) \to Rf_*f^*(F)[-2]$$

as the composition

$$Rf_*f^*(F) \xrightarrow{\text{unit}} Rf_*Rr_*R\phi_*\phi^*r^*f^*(F) \overset{!}{\cong} Rf_*Rr_*r^*f^*(F) \xrightarrow{\int_r} Rf_*f^*(F)[-2] ,$$

where at the marked isomorphism "!" we use the natural isomorphisms 6.6.13 and 6.6.9 associated to the identity $f \circ r = f \circ r \circ \phi$

Recall from 3.5.4 the definition of the natural evaluation transformation $e_i \colon P_G(F) \to Rf_*f^*(F)[2i]$ for all $i \geq 0$.

Lemma 3.6.4. — *The following diagram commutes:*

$$\begin{array}{ccc} P_G(F) & \xrightarrow{W} & P_G(F) \\ \downarrow{\scriptstyle e_{i+1}} & & \downarrow{\scriptstyle e_i} \\ Rf_*f^*(F)[2i+2] & \xrightarrow{D_G} & Rf_*f^*(F)[2i] \end{array} .$$

Proof. — We split this square in parts. First we observe that in $D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$

$$\begin{array}{ccccc} P_G(F) & \xrightarrow{\text{unit}} & Rp_*p^*P_G(F) & \xrightarrow[\cong]{Rp_*r^*} & Rp_*P_{\tilde{G}}(p^*F) \\ \downarrow{\scriptstyle e_{i+1}} & & \downarrow{\scriptstyle Rp_*p^*e_{i+1}} & & \downarrow{\scriptstyle Rp_*e_{i+1}} \\ Rf_*f^*(F)[2i+2] & \xrightarrow{\text{unit}} & Rp_*p^*Rf_*f^*(F)[2i+2] & \xrightarrow[\cong]{Rp_*r^*} & Rp_*Rs_*s^*p^*(F)[2i+2] \\ \downarrow{\scriptstyle =} & & & & \downarrow{\scriptstyle \cong} \\ Rf_*f^*(F)[2i+2] & \xrightarrow{Rf_*f^*\text{unit}} & Rf_*f^*Rp_*p^*(F) & \xrightarrow{\cong} & Rf_*Rr_*r^*f^*(F)[2i+2] \end{array}$$

commutes (use Lemma 6.1.12 for the upper left and the lower and 3.5.4 for the upper right rectangle).

In the next step we observe that

$$\begin{array}{ccccc} Rp_*P_{\tilde{G}}(p^*F) & \xrightarrow{\text{id}} & Rp_*P_{\tilde{G}}(p^*F) & \xrightarrow[\cong]{Rp_*\phi^*} & Rp_*P_{\tilde{G}}(p^*F) \\ \downarrow{\scriptstyle Rp_*e_{i+1}} & & & & \downarrow{\scriptstyle Rp_*e_{i+1}} \\ Rp_*Rs_*s^*p^*(F)[2i+2] & \xrightarrow{\text{unit}} & Rp_*Rs_*R\phi_*\phi^*s^*p^*(F)[2i+2] & \xrightarrow{\cong} & Rp_*Rs_*s^*p^*(F)[2i+2] \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ Rf_*Rr_*r^*f^*(F)[2i+2] & \xrightarrow{\text{unit}} & Rf_*Rr_*R\phi_*\phi^*r^*f^*(F)[2i+2] & \xrightarrow{\cong} & Rf_*Rr_*r^*f^*(F)[2i+2] \end{array}$$

commutes, where we use for the upper rectangle again 3.5.4, and $p \circ s \circ \phi = p \circ s$, $p \circ s = f \circ r$, $f \circ r \circ \phi = f \circ r$ and Lemma 6.1.12 for the remaining squares.

In the last step we observe the commutativity of

$$\begin{array}{ccccc}
Rp_*P_{\tilde{G}}(p^*F) & \xrightarrow[\cong]{(r^*)^{-1}} & Rp_*p^*P_G(F) & \xrightarrow{\int_p} & P_G(F)[-2] \overset{T^{-2}}{\cong} P_G(F) \\
\downarrow{\scriptstyle Rp_*e_{i+1}} & & \downarrow{\scriptstyle Rp_*p^*e_{i+1}} & & \downarrow{\scriptstyle T^{-2}e_{i+1}\cong e_i} \\
Rp_*Rs_*s^*p^*(F)[2i+2] & \xrightarrow[\cong]{(r^*)^{-1}} & Rp_*p^*Rf_*f^*(F)[2i+2] & \xrightarrow{\int_p} & Rf_*f^*(F)[2i] \\
\downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle =} \\
Rf_*Rr_*r^*f^*(F)[2i+2] & & \xrightarrow{Rf_*(\int_r)} & & Rf_*f^*(F)[2i].
\end{array}$$

Again, for the commutativity of the upper left rectangle we use (3.5.8) of 3.5.4. For the upper right corner we use the fact that $\int_p$ is a natural transformation between the functors Rp_*p^* and id on $D(\mathrm{Sh_{Ab}}\mathbf{X})$. For the lower rectangle we use Lemma 6.5.31. □

3.6.4. — We now finish the proof of Proposition 3.6.3. We have an exact triangle

$$\cdots \to P_G(F) \overset{\prod_{i\geq 0} e_i}{\to} \prod_{i\geq 0} Rf_*f^*(F)[2i] \overset{\alpha}{\to} \prod_{i\geq 0} Rf_*f^*(F)[2i] \overset{[1]}{\to} \cdots$$

where (using the language of elements) the map α is given by

$$\alpha(x_i)_{i\geq 0} = (x_i - D_G x_{i+1})_{i\geq 0}.$$

By Lemma 3.6.4 we have a morphism of exact triangles

$$\begin{array}{ccccc}
P_G(F) & \xrightarrow{\prod_{i\geq 0} e_i} & \prod_{i\geq 0} Rf_*f^*(F)[2i] & \xrightarrow{\alpha} & \prod_{i\geq 0} Rf_*f^*(F)[2i] \\
\downarrow{\scriptstyle W} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\
P_G(F)[-2] & \xrightarrow{\prod_{i\geq 0} e_i} & \prod_{i\geq 0} Rf_*f^*(F)[2i-2] & \xrightarrow{\alpha} & \prod_{i\geq 0} Rf_*f^*(F)[2i-2],
\end{array}$$

where the map β is given by $\beta(x_i)_{i\geq 0} := (D_G x_i)_{i\geq 0}$. In Lemma 2.4.5 we have shown that W is an isomorphism. □

CHAPTER 4

T-DUALITY

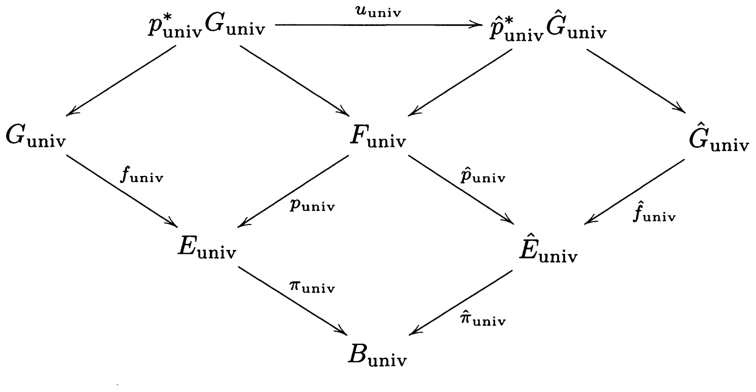
4.1. The universal T-duality diagram

4.1.1. — Topological T-duality intends to model the underlying topology of string theoretic T-duality on the level of targets and quantum field theory. In the special case of targets modeled by a gerbe on top of a T^n-principal bundle over a space, topological T-duality is by now a well-defined mathematical concept, see [**11**], [**6**] and the literature cited therein. In the case of T-principal bundles it was extended to orbifolds in [**8**]. In the present paper we propose a definition of T-duality in the case of T-bundles over arbitrary stacks. This framework includes arbitrary T-actions on spaces. The special case of an almost free action (i.e. every orbit is either free or a fixed point) has been treated with completely different methods in [**24**].

4.1.2. — The notion of a T-duality diagram has first been introduced in [**6**]. In the present paper we first produce a universal T-duality diagram over the stack $\mathcal{B}U(1) = [*/U(1)]$. Then we proceed to define a T-duality diagram over a general stack as one which is locally isomorphic to the universal one.

4.1.3. — The universal T-duality diagram is a diagram of stacks

(4.1.1)



.

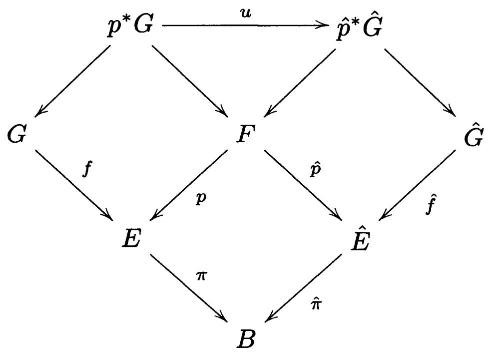
In the following we explain the stacks and the maps.

— $B_{\text{univ}} := \mathcal{B}U(1)$
— $E_{\text{univ}} := *$ and π_{univ} is the map which classifies the trivial $U(1)$-bundle over the point $*$.
— $G_{\text{univ}} := \mathcal{B}U(1)$, and f_{univ} is the unique map.
— $\hat{E}_{\text{univ}} := \mathcal{B}U(1) \times U(1)$, and $\hat{\pi}_{\text{univ}}$ is the projection onto the first factor.
— $\hat{f}_{\text{univ}} \colon \hat{G}_{\text{univ}} \to \hat{E}_{\text{univ}}$ is a gerbe with band $U(1)$ classified by $z \otimes v \in H^2(\mathcal{B}U(1);\mathbb{Z}) \otimes H^1(U(1);\mathbb{Z}) \cong H^3(\mathcal{B}U(1) \times U(1);\mathbb{Z})$, where $z \in H^2(\mathcal{B}U(1);\mathbb{Z})$ and $v \in H^1(U(1);\mathbb{Z})$ are the standard generators.
— $F_{\text{univ}} := E_{\text{univ}} \times_{B_{\text{univ}}} \hat{E}_{\text{univ}} \cong U(1)$, and $p_{\text{univ}}, \hat{p}_{\text{univ}}$ are the canonical projections.
— Since $H^2(F_{\text{univ}};\mathbb{Z}) \cong 0 \cong H^3(F_{\text{univ}};\mathbb{Z})$, the pull-back $\hat{p}^*_{\text{univ}}\hat{G}_{\text{univ}}$ can be identified with the trivial gerbe $p^*_{\text{univ}}G_{\text{univ}} \cong U(1) \times \mathcal{B}U(1)$ by a unique isomorphism class of maps represented by u_{univ}.

Let us fix once and for all a universal T-duality diagram (i.e. a choice of u_{univ} in its isomorphism class and 2-isomorphisms filling the faces).

4.1.4. — Let B be a topological stack and consider a diagram

(4.1.2)



of topological stacks where the squares are Cartesian, $f \colon G \to E$ and $\hat{f} \colon \hat{G} \to \hat{E}$ are topological $U(1)$-gerbes, and u is an isomorphism of gerbes over F.

An isomorphism between two such diagrams over B is first of all a large commutative diagram in stacks, but we furthermore require that the horizontal morphisms are morphisms of $U(1)$-banded gerbes in all places where this condition makes sense.

Definition 4.1.3. — *The diagram* (4.1.2) *is called a T-duality diagram if for every object $(U \to B) \in \mathbf{B}$ there exists a covering $(U_i \to U)_{i \in I} \in \operatorname{cov}_{\mathbf{B}}(U)$ such that for all $i \in I$ the pull-back of the diagram* (4.1.2) *along the map $U_i \to U \to B$ is isomorphic to the pull-back of the universal T-duality diagram* (4.1.1) *along a map $U_i \to B_{\text{univ}}$.*

4.1.5. — In the following we describe the concept of T-duality. Let B be a topological stack. A pair (E, G) over B consists of a T-principal bundle $\pi\colon E \to B$ and a $U(1)$-gerbe $f\colon G \to E$.

Definition 4.1.4. — *We say that a pair (E, G) admits a T-dual, if it appears as a part of a T-duality diagram 4.1.2. In this case the pair $(\hat{E}, \hat{G})$ is called a T-dual of (E, G).*

This is our proposal for the mathematical concept of T-duality for pairs of T-principal bundles and gerbes. Using the T^n-bundle variant of the universal T-duality diagram one can easily generalize this definition to the higher-dimensional case. But note that, in contrast to the case of one-dimensional fibers, a unique isomorphism u_{univ} does not exist for T^n if one uses the exactly parallel setup. This explains why suitable modifications are necessary in [**6**]. In particular, the universal base space is not simply the n-fold product of copies of B_{univ} used in the one-dimensional case.

4.1.6. — In the following we show that the concept of topological T-duality as defined above really coincides with the former definitions.

Lemma 4.1.5. — *Definitions 4.1.3 and 4.1.4 reduce to the notion of T-duality as used in* [**6**], [**7**], *if B is a locally acyclic space.*

Proof. — By Definition 4.1.3 a T-duality triple over a space B is given by the following data:

(1) locally trivial $U(1)$-principal bundles $E, \hat{E}$ over B,
(2) $U(1)$-banded gerbes G, $\hat{G}$ over E or $\hat{E}$, respectively,
(3) an isomorphism u between the pullbacks of G and $\hat{G}$ to the correspondence space $E \times_B \hat{E}$.

Every point $b \in B$ admits an acyclic neighborhood $b \in U \subseteq B$. The bundles E and $\hat{E}$ are trivial over U, i.e. we have $E_{|U} \cong U \times U(1) \cong \hat{E}_{|U}$. Since $H^3(U \times U(1); \mathbb{Z}) \cong 0$, the restrictions of the gerbes $G_{|E_{|U}}$ and $\hat{G}_{|\hat{E}_{|U}}$ are trivial, too. The Definition 4.1.3 requires that the isomorphism of trivial gerbes $u_{|E_{|U} \times_U \hat{E}_{|U}}$ is classified by the generator of $H^2(E_{|U} \times_U \hat{E}_{|U}; \mathbb{Z})$ (note that $E_{|U} \times_U \hat{E}_{|U} \cong U \times U(1) \times U(1)$). This reformulation of the definition of a T-duality triple over a locally acyclic space B is exactly the definition of a T-duality triple in [**6**].

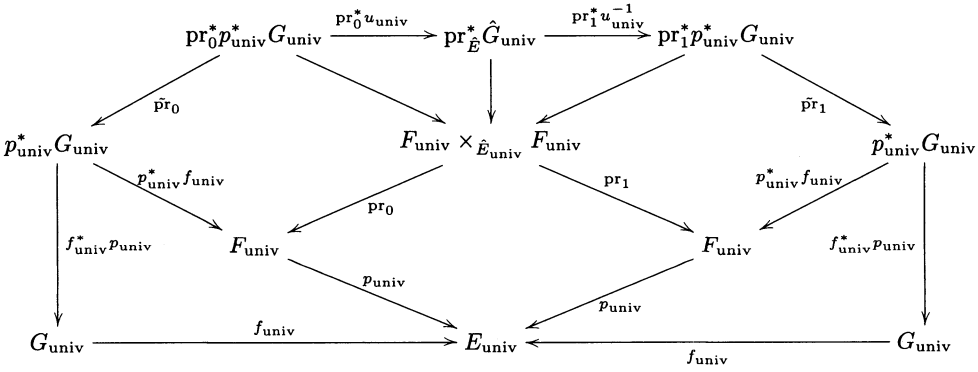
In the approach of [**7**] to T-duality we start with a pair (E, G). We characterize T-dual pairs by topological conditions. We then analyze the classifying space of pairs and observe that the universal pair has a unique T-dual pair which gives rise to the T-duality transformation.

It turns out that the classifying space of pairs in [**7**] is equivalent to the classifying space of T-duality triples in [**6**], and that the universal pair and its dual are parts of the universal T-duality triple. This shows that the approaches of [**7**] and [**6**] are equivalent. □

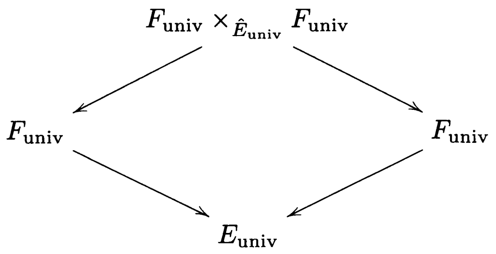
4.2. T-duality and periodization diagrams

4.2.1. — Recall that the construction of the periodization functor P_G was based on the diagrams introduced in 2.2.1. In the present subsection we relate these diagrams to T-duality.

4.2.2. — The double of the universal T-duality diagram (4.1.1) is (by definition) the big universal periodization diagram
(4.2.1)



Note that all squares are Cartesian, with the exception of the central square



which does not commute. The same remark applies to similar diagrams we introduce later.

4.2.3. — We form the diagram (1)

$$(4.2.2) \qquad \mathrm{pr}_0^* p_{\mathrm{univ}}^* G_{\mathrm{univ}} \underset{m_{\mathrm{univ}}}{\overset{q_{\mathrm{univ}}}{\rightrightarrows}} G_{\mathrm{univ}} \xrightarrow{f_{\mathrm{univ}}} E_{\mathrm{univ}} ,$$

where

$$m_{\mathrm{univ}} := f_{\mathrm{univ}}^* p_{\mathrm{univ}} \circ \tilde{\mathrm{pr}}_1 \circ \mathrm{pr}_1^* u_{\mathrm{univ}}^{-1} \circ \mathrm{pr}_0^* u_{\mathrm{univ}} , \quad q_{\mathrm{univ}} := f_{\mathrm{univ}}^* p_{\mathrm{univ}} \circ \tilde{\mathrm{pr}}_0 .$$

Definition 4.2.3. — *The diagram* (4.2.2) *is called the small universal periodization diagram.*

4.2.4. — Let $f\colon G \to X$ be a topological gerbe with band $U(1)$ over a stack X. Then we consider the pull-back of the small universal periodization diagram to X via the projection $r\colon X \to E_{\mathrm{univ}} \cong *$. We form the tensor product with the gerbe G (see [**11**, 6.1.9] for some details on such tensor products) and obtain the diagram

$$(4.2.4) \qquad \tilde{H} \underset{m}{\overset{q}{\rightrightarrows}} H \xrightarrow{f} X ,$$

where

$$\tilde{H} := \mathrm{pr}_X^* G \otimes \mathrm{pr}^*_{F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}}} \mathrm{pr}_0^* p_{\mathrm{univ}}^* G_{\mathrm{univ}} , \quad H := G \otimes r^* G_{\mathrm{univ}} ,$$

$$\mathrm{pr}_X \colon X \times F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}} \to X ,$$

$$\mathrm{pr}_{F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}}} \colon X \times F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}} \to F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}}$$

are the projections, and m, q are induced by the corresponding universal maps m_{univ} or q_{univ}, respectively.

Definition 4.2.5. — *The diagram* (4.2.4) *is called the small periodization diagram of* $G \to X$.

In fact we have defined a 2-functor from gerbes/X to a 2-category of such small periodization diagrams. Using the fact that $G_{\mathrm{univ}} = \mathcal{B}U(1)$ we have a canonical identification $H \cong G$. Furthermore, $F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}} \cong T^2$, and we can identify $\tilde{H} \to X \times F_{\mathrm{univ}} \times_{\hat{E}_{\mathrm{univ}}} F_{\mathrm{univ}}$ with $G \times T^2 \to X \times T^2$.

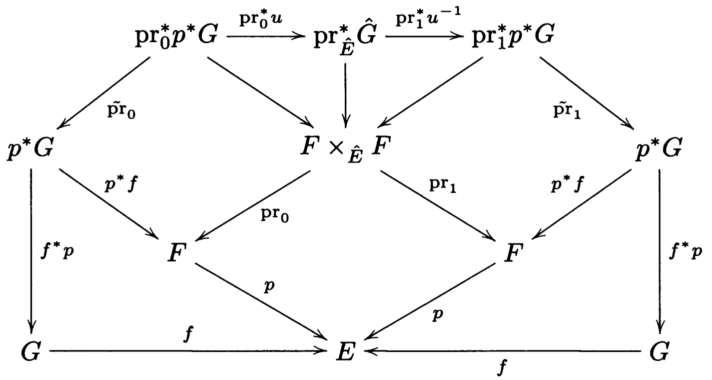
Lemma 4.2.6. — *With these identifications the small periodization diagram* (4.2.4) *is isomorphic to the diagram* (3.4.2) *used in the definition of* P_G.

Proof. — This follows directly from the definitions of these maps. □

(1) This diagram does not commute. It is a short-hand for a square of the form (3.4.2) with a 2-isomorphism between $f_{\mathrm{univ}} \circ q_{\mathrm{univ}}$ and $f_{\mathrm{univ}} \circ m_{\mathrm{univ}}$. We will adopt a similar convention for other diagrams written in this short-hand form below.

4.2.5. — The T-duality diagram (4.1.2) gives rise to the big double T-duality diagram

(4.2.7)



Note that the middle square does not commute. We have

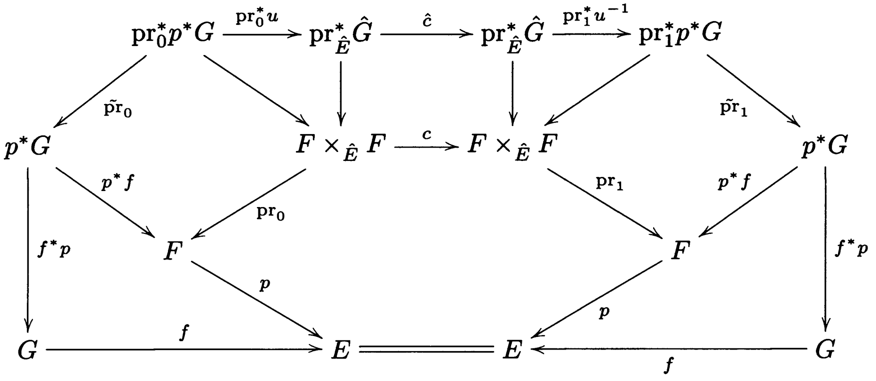
$$F\times_{\hat{E}}F\cong(E\times_B\hat{E})\times_{\hat{E}}(\hat{E}\times_B E)\cong E\times_B\hat{E}\times_B E\xleftarrow{\sim}E\times_B\hat{E}\times U(1)\ ,$$

where the last arrow is given by $(e,\hat{e},eu)\leftarrow(e,\hat{e},u)$. Under this identification $\mathrm{pr}_0(e,\hat{e},u)=(e,\hat{e})$ and $\mathrm{pr}_1(e,\hat{e},u)=(eu,\hat{e})$. We can correct this non-commutativity as follows. Let $c:F\times_{\hat{E}}F\to F\times_{\hat{E}}F$ be the isomorphism, which under the above identification is given by $c(e,\hat{e},u):=(eu^{-1},\hat{e},u)$. Note that $\mathrm{pr}_1\circ c=\mathrm{pr}_0$. Furthermore note that $\mathrm{pr}_{\hat{E}}=\mathrm{pr}_{\hat{E}}\circ c:F\times_{\hat{E}}F\to\hat{E}$. Therefore we get a canonical morphism $\hat{c}$ satisfying $\overline{\mathrm{pr}_{\hat{E}}}=\overline{\mathrm{pr}_{\hat{E}}}\circ\hat{c}$ in the diagram

$$\begin{array}{ccccc}
\mathrm{pr}_{\hat{E}}^*\hat{G} & \xrightarrow{\hat{c}} & \mathrm{pr}_{\hat{E}}^*\hat{G} & \xrightarrow{\overline{\mathrm{pr}_{\hat{E}}}} & \hat{G} \\
\downarrow & & \downarrow & & \downarrow \\
F\times_{\hat{E}}F & \xrightarrow{c} & F\times_{\hat{E}}F & \xrightarrow{\mathrm{pr}_{\hat{E}}} & \hat{E}
\end{array}\ .$$

If we plug this in the big double T-duality diagram, then we get the big commutative T-duality diagram diagram

(4.2.8)



From this we derive the diagram

$$(4.2.9) \qquad \mathrm{pr}_0^*p^*G \overset{q_T}{\underset{m_T}{\rightrightarrows}} G \xrightarrow{f} E \ ,$$

where

$$q_T := f^*p \circ \tilde{\mathrm{pr}}_0 \ , \quad m_T := f^*p \circ \tilde{\mathrm{pr}}_1 \circ \mathrm{pr}_1^*u^{-1} \circ \hat{c} \circ \mathrm{pr}_0^*u \ .$$

Definition 4.2.10. — *The diagram* (4.2.9) *is called the small double T-duality diagram associated to* (4.1.2).

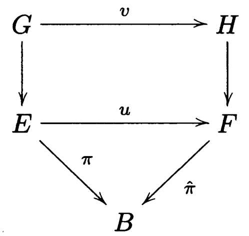
4.2.6. — The following fact is an immediate consequence of the definitions.

Proposition 4.2.11. — *The small double T-duality diagram* (4.2.9) *is locally isomorphic to the small periodization diagram* (4.2.4) *of $G \to E$.*

4.3. Twisted cohomology and the T-duality transformation

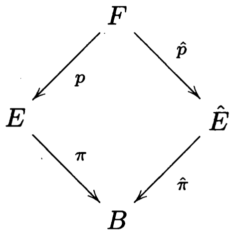
4.3.1. — Let E be a topological stack. In order to write out operations on twisted cohomology effectively we introduce some notation for operations on $D^+(\mathrm{Sh_{Ab}}E)$ or $D(\mathrm{Sh_{Ab}}E)$. If $p : F \to E$ is a map of topological stacks, then we let $\mathfrak{p}^* : \mathrm{id} \to Rp_*p^*$ denote the unit. If p is an oriented fiber bundle, then we let $\mathfrak{p}_! : Rp_*p^* \to \mathrm{id}$ denote the integration map. If $\pi : E \to B$ is a second map, then we write $\pi_*\mathfrak{p}^*$, $\pi_*\mathfrak{p}_!$ or simply also $\mathfrak{p}^*$ and $\mathfrak{p}_!$ for the induced transformations $R\pi_*\pi^* \to R\pi_*Rp_*p^*\pi^*$ and $R\pi_*Rp_*p^*\pi^* \to R\pi_*\pi^*$.

If



is a diagram with $U(1)$-gerbes $H \to F$ and $G \to E$ such that the square is Cartesian, then we write $P(v)$ for the transformation $u^* \circ P_H \to P_G \circ u^*$, and we use the same symbol for the induced transformation $R\pi_* u^* P_H \hat{\pi}^* \to R\pi_* P_G u^* \hat{\pi}^*$.

In a commutative diagram



we will use the symbol $\mathfrak{I}$ or, if necessary, $\mathfrak{I}_{\pi\circ p = \hat{\pi}\circ\hat{p}}$ in order to denote the transformation

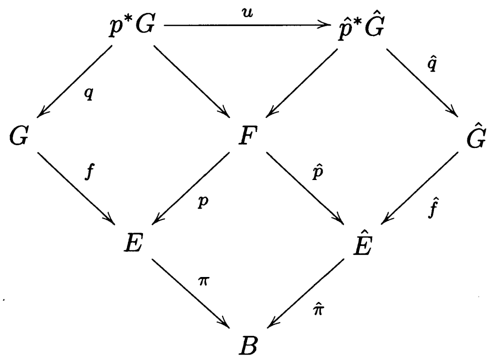
$$R\pi_* Rp^* p^* \pi^* \xrightarrow{\sim} R\hat{\pi}_* R\hat{p}_* \hat{p}^* \hat{\pi}^* \ .$$

4.3.2. — We consider a topological gerbe $f\colon G \to E$ with band $U(1)$ over a locally compact stack. In **[9]** we define the G-twisted cohomology of E with coefficients in $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{E})$ by

$$H^*(E, G; F) := H^*(E; Rf_* f^*(F)) \ .$$

4.3.3. — Assume now that $f\colon G \to E$ is a part of a T-duality diagram

(4.3.1)



.

Then we define the transformation

$$J := \hat{\mathfrak{q}}_! \circ \mathfrak{J} \circ (\mathfrak{u}^{-1})^* \circ \mathfrak{q}^* : R\pi_* Rf_* f^* \pi^* \to R\hat{\pi}_* R\hat{f}_* \hat{f}^* \hat{\pi}^* . \tag{4.3.2}$$

Note that here $\mathfrak{J} = \mathfrak{J}_{\pi f q u^{-1} = \hat{\pi} f \hat{q}}$.

Consider a sheaf $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{B})$. Note that, by definition, $H^*(E, G; \pi^* F) = H^*(B; R\pi_* Rf_* f^* \pi^* F)$.

Definition 4.3.3. — *For $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{E})$ the T-duality transformation is defined as the map*

$$T\colon H^*(E, G; \pi^* F) \to H^{*-1}(\hat{E}, \hat{G}; \hat{\pi}^* F)$$

induced by the natural transformation (4.3.2).

4.3.4. — Let us calculate the effect of the T-duality transformation in a simple example. There is a unique isomorphism class of T-duality diagrams over the point $B = *$. In this case $E = U(1)$ and $G = U(1) \times \mathcal{B}U(1)$. We consider a discrete abelian group F. Then we have

$$H^*(E, G; \pi^* \underline{F}_B) \cong \mathbb{Z}[[z]][v]/(v^2) \otimes F \ , H^*(\hat{E}, \hat{G}; \hat{\pi}^* \underline{F}_B) \cong \mathbb{Z}[[z]][\hat{v}]/(\hat{v}^2) \otimes F \ ,$$

where $\deg(v) = 1 = \deg(\hat{v})$ and $\deg(z) = 2$.

To explicitly calculate the effect of T in this case, observe that the cohomology of $Rf_* Rq_* q^* f^* \underline{F}$ is $\mathbb{Z}[[z]] \otimes \Lambda(v, \hat{v}) \otimes F$ with v and $\hat{v}$ the generators corresponding to the two S^1-factors E and $\hat{E}$ in F. The automorphism u induces in cohomology, i.e. on $\mathbb{Z}[[z]] \otimes \Lambda(v, \hat{v}) \otimes F$, the algebra homomorphism given by $z \mapsto z + v\hat{v}$, $v \mapsto v$, $\hat{v} \mapsto \hat{v}$. It follows that

$$T(z^n \otimes f) = \int_{F/\hat{E}} (z^n \otimes f + n z^{n-1} v\hat{v} \otimes f) = n z^{n-1} \hat{v} \otimes f$$

$$T(z^n v \otimes f) = \int_{F/\hat{E}} z^n v \otimes f = z^n \otimes f.$$

We see that the T-duality transformation is not an isomorphism.

4.3.5. — Our main motivation for introducing the periodization functor is the construction of twisted sheaf cohomology which admits a T-duality *isomorphism*. Let $G \to E$ be a topological gerbe with band $U(1)$ over a locally compact stack E.

Definition 4.3.4. — *We define the periodic G-twisted cohomology of E with coefficients in $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{E})$ by*

$$H^*_{\mathrm{per}}(E, G; F) := H^*(E; P_G(F)) \ .$$

Note that here we use the sheaf theory operations for the unbounded derived category, see Subsection 6.5 for details.

4.3.6. — Assume again that $f\colon G \to E$ is part of a T-duality diagram (4.3.1). We define a natural transformation

$$J\colon R\pi_* \circ P_G \circ \pi^* \to R\hat{\pi}_* \circ P_{\hat{G}} \circ \hat{\pi}^* \tag{4.3.5}$$

by

$$J := \hat{\mathfrak{p}}_! \circ \mathfrak{J} \circ P(u)^{-1} \circ \mathfrak{p}^* \ .$$

Consider a sheaf $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{B})$. Note that by definition $H^*_{\mathrm{per}}(E,G;\pi^*F) = H^*(B, R\pi_*P_G(\pi^*(F)))$.

Definition 4.3.6. — *For $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{E})$ the T-duality transformation in periodic twisted cohomology*

$$T\colon H^*_{\mathrm{per}}(E,G;\pi^*F) \to H^{*-1}_{\mathrm{per}}(\hat{E},\hat{G};\hat{\pi}^*F)$$

is the map induced by the natural transformation (4.3.5).

4.3.7. — As an illustration let us calculate the action of the T-duality transformation in the example started in 4.3.4. The sequence $\mathcal{J}_G(\underline{F})$ for $F = \mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ either has trivial lim or trivial $\lim^1$. Therefore in this special case the morphism T calculated in 4.3.4 defines uniquely an endomorphism of $H^*_{\mathrm{per}}(E,G;\pi^*\underline{F}_B)$ (we identify $E \cong \hat{E}$). For example if $F = \mathbb{Q}$, then we read off directly from 4.3.4 that (with $H^0_{\mathrm{per}}(E,G;\pi^*\underline{\mathbb{Q}}) \cong \mathbb{Q}[v]/v^2$) the T-duality morphism is

$$T\colon \mathbb{Q}[v]/v^2 \to \mathbb{Q}[v]/v^2\ , \quad T(v) = 1\ , \quad T(1) = v\ .$$

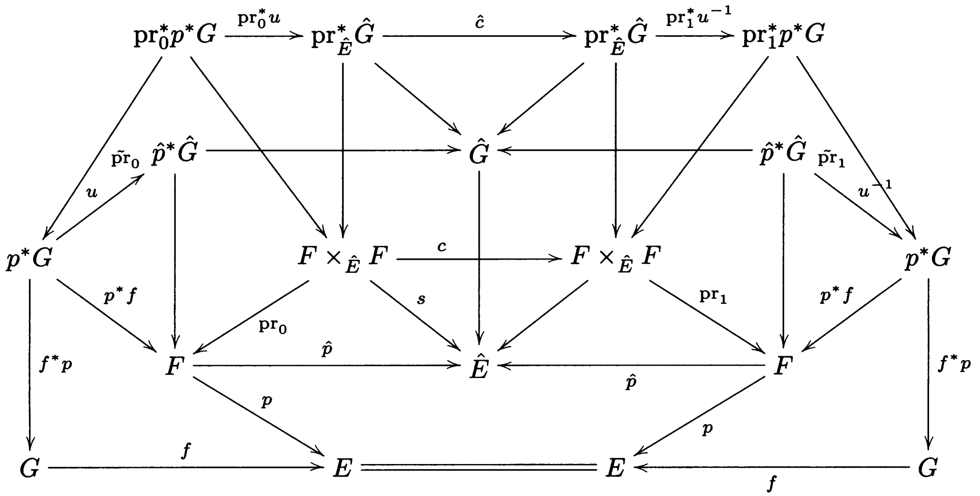
In particular, we see in this example that now we get an isomorphism.

4.3.8. — In the remainder of the present subsection we show the following theorem.

Theorem 4.3.7. — *The T-duality transformation in twisted periodic cohomology 4.3.6 is an isomorphism.*

Proof. — The opposite of the T-duality diagram (4.3.1) is obtained by reflecting it in the middle vertical, and by replacing u by its inverse. We let $T'\colon H^*_{\mathrm{per}}(\hat{E},\hat{G};\hat{\pi}^*F) \to H^{*-1}_{\mathrm{per}}(E,G;\pi^*F)$ be the associated T-duality transformation.

Both, the T-duality diagram and its opposite can be recognized as subdiagrams of the (slightly extended) big commutative T-duality diagram
(4.3.8)



We now calculate the composition $T' \circ T$. The compatibility of the integration with pull-back in the Cartesian diagram

$$\begin{array}{ccc} F & \xleftarrow{\mathrm{pr}_0} & F \times_{\hat{E}} F \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \hat{E} & \xleftarrow[\hat{p}]{} & F \end{array}$$

is employed in the equality marked by ! below. The equality $\hat{p} \circ \mathrm{pr}_0 \circ c^{-1} = \hat{p} \circ \mathrm{pr}_0$ is used in the equality !!. Finally we use $\mathrm{pr}_0 \circ c = \mathrm{pr}_1$ at !!!. We have

$$\begin{aligned}
J' \circ J &= \mathfrak{p}_! \circ \mathfrak{J} \circ P(u) \circ \hat{\mathfrak{p}}^* \circ \hat{\mathfrak{p}}_! \circ \mathfrak{J} \circ P(u)^{-1} \circ \mathfrak{p}^* \\
&\overset{!}{=} \mathfrak{p}_! \circ \mathfrak{J} \circ P(u) \circ \mathfrak{pr}_{1!} \circ \mathfrak{J} \circ \mathfrak{pr}_0{}^* \circ \mathfrak{J} \circ P(u)^{-1} \circ \mathfrak{p}^* \\
&\overset{!!}{=} \mathfrak{p}_! \circ \mathfrak{J} \circ P(u) \circ \mathfrak{pr}_{1!} \circ \mathfrak{J} \circ P(\hat{c}^{-1}) \circ (\mathfrak{c}^{-1})^* \circ \mathfrak{pr}_0{}^* \circ \mathfrak{J} \circ P(u)^{-1} \circ \mathfrak{p}^* \\
&\overset{!!!}{=} \mathfrak{p}_! \circ \mathfrak{pr}_{1!} \circ P(\mathrm{pr}_1^* u) \circ P(\hat{c}^{-1}) \circ P(\mathrm{pr}_0^* u)^{-1} \circ \mathfrak{pr}_1{}^* \circ \mathfrak{p}^* \\
&= \mathfrak{p}_! \circ \mathfrak{pr}_{1!} \circ P(\mathrm{pr}_1^* u \circ \hat{c}^{-1} \circ (\mathrm{pr}_0^* u)^{-1}) \circ \mathfrak{pr}_1{}^* \circ \mathfrak{p}^*
\end{aligned}$$

This is exactly the transformation coming from the associated small double T-duality diagram (4.2.9) (actually its mirror). Since this is locally isomorphic to the small periodization diagram we see that locally $J' \circ J$ coincides with $\pi_* W$, where W is as in Proposition 3.6.3. By Proposition 3.6.3 this transformation is an isomorphism on periodic sheaves of the form $R\pi_* P_G(\pi^* F)$. Therefore $T \circ T'$ is an isomorphism.

We can interchange the roles of T and T', hence $T \circ T'$ is an isomorphism, too. This implies the result. □

CHAPTER 5

ORBISPACES

5.1. Twisted periodic delocalized cohomology of orbispaces

5.1.1. — Let us recall some notions related to orbispaces (compare [**10**]). Orbispaces as particular kind of topological stacks have previously been introduced in [**8**, Sec. 2.1] and [**22**, Sec. 19.3]. In the present paper we use the set-up of [**8**] but add the additional condition that an orbifold atlas should be separated. This condition is needed in order to show that the loop stack of an orbifold is again an orbifold.

(1) A topological groupoid $A\colon A^1 \Rightarrow A^0$ is called separated if the identity $\mathbf{1}_A\colon A^0 \to A^1$ of the groupoid is a closed map.
(2) A topological groupoid $A^1 \Rightarrow A^0$ is called proper if $(s,r)\colon A^1 \to A^0 \times A^0$ is a proper map.
(3) A topological groupoid is called étale if the source and range maps $s, r\colon A^1 \to A^0$ are étale.
(4) A proper étale topological groupoid $A^1 \Rightarrow A^0$ is called very proper if there exists a continuous function $\chi\colon A^0 \to [0,1]$ such that
    (a) $r\colon \mathrm{supp}(s^*\chi) \to A^0$ is proper
    (b) $\sum_{y\in A^x} \chi(s(y)) = 1$ for all $x \in A^0$.
(5) A topological stack is called (very) proper (or étale, separated, respectively), if it admits an atlas $A \to X$ such that the topological groupoid $A \times_X A \Rightarrow A$ is (very) proper (or étale, separated, respectively).
(6) An orbispace atlas of a topological stack X is an atlas $A \to X$ such that $A \times_X A \Rightarrow A$ is a very proper étale and separated groupoid.
(7) An orbispace X is a topological stack which admits an orbispace atlas.
(8) If X, Y are orbispaces, then a morphism of orbispaces $X \to Y$ is a representable morphism of stacks.
(9) A locally compact orbispace is an orbispace X which admits an orbispace atlas $A \to X$ such that A is locally compact.

5.1.2. — If X is a stack, then its inertia stack (sometimes called loop stack) LX is defined as the two-categorical equalizer of the diagram

$$X \underset{\mathrm{id}_X}{\overset{\mathrm{id}_X}{\rightrightarrows}} X \ .$$

In [**10**, Sec 2.2] we have introduced an explicit model of LX and studied its properties. The loop stack LX depends 2-functorially on X. Indeed, since $\underline{\mathrm{Hom}}_{\mathrm{Cat}}$ is a strict 2-functor, the loop functor is a strict functor between 2-categories. As already mentioned before, later we will suppress the 2-morphisms in 2-commutative diagrams in 2-categories for better legibility. If X is a topological stack (orbispace), then LX is a topological stack (orbispace), too (see [**10**, Lemma 2.25], [**10**, Lemma 2.33]).

Lemma 5.1.1. — *If X is a locally compact orbispace, then LX is a locally compact orbispace, too.*

Proof. — Let $A \to X$ be a locally compact orbispace atlas of X. Then we have the proper, separated and étale topological groupoid $A \times_X A \Rightarrow A$. Since the source map of this groupoid is étale, the space of morphisms $A \times_X A$ of this groupoid is locally compact, too.

In the proof of Lemma [**10**, Lemma 2.25] we constructed an orbispace atlas $W \to LX$ of LX, where W was given by the pull-back of spaces

$$\begin{array}{ccc} W & \longrightarrow & A \times_X A \\ \big\downarrow{\scriptstyle w} & & \big\downarrow{\scriptstyle (\mathrm{pr}_1, \mathrm{pr}_2)} \\ A & \xrightarrow{\mathrm{diag}} & A \times A \end{array} \ .$$

This implies that W is locally compact. □

5.1.3. — Let $G \to X$ be a topological gerbe with band $U(1)$ over a locally compact orbispace. The truly interesting G-twisted cohomology of X (with complex coefficients) is not the cohomology $H^*_{\mathrm{per}}(X, G; \underline{\mathbb{C}})$ (see 4.3.6), but a more complicated delocalized version $H^*_{\mathrm{deloc,per}}(X, G)$, which we will define below (see [**10**, Sec. 1.3] for an explanation).

As shown in [**10**, Sec. 2.5] the gerbe gives rise to a principal bundle $\tilde{G}^\delta \to LX$ with structure group $U(1)^\delta$ in a functorial way, where $U(1)^\delta$ denotes the group $U(1)$ with the discrete topology. By $\mathcal{L} \in \mathrm{Sh_{Ab}}\mathbf{LX}$ we denote the sheaf of locally constant sections of the associated vector bundle $\tilde{G}^\delta \times_{U(1)^\delta} \mathbb{C} \to LX$.

We define the gerbe $G_L \to LX$ as the pull-back

$$\begin{array}{ccc} G_L & \longrightarrow & G \\ \big\downarrow{\scriptstyle f_L} & & \big\downarrow{\scriptstyle f} \\ LX & \longrightarrow & X \end{array} \ .$$

Definition 5.1.2. — *We define*

$$\mathcal{L}_G := P_{G_L}(\mathcal{L}) \in D(\mathrm{Sh_{Ab}}\mathbf{LX}) .$$

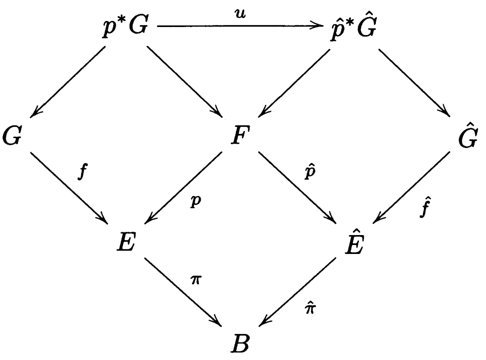
The G-twisted delocalized periodic cohomology of X is defined as

$$H^*_{\mathrm{deloc,per}}(X, G) := H^*(LX; \mathcal{L}_G) .$$

5.2. The T-duality transformation in twisted periodic delocalized cohomology

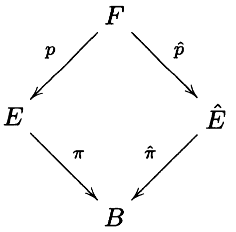
5.2.1. — We consider a T-duality diagram

(5.2.1)

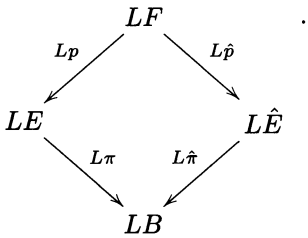


(see Definition 4.1.3), where B is a locally compact orbispace.

We apply the loops functor L: orbispaces $\to$ orbispaces to the subdiagram



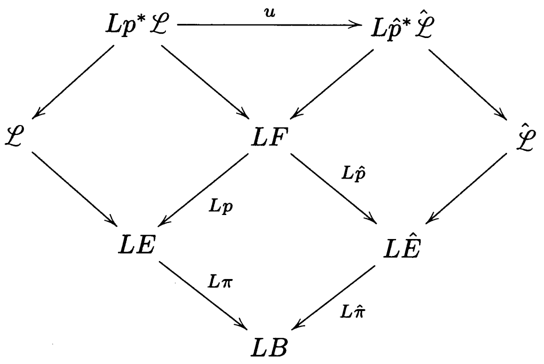
and get



In the first diagram the maps $p, \hat{p}, \pi, \hat{\pi}$ are all $U(1)$-principal bundles. The maps $Lp, L\hat{p}, L\pi, L\hat{\pi}$ are not necessarily surjective. Thus in general the derived diagram of loop stacks is not part of a T-duality diagram. But it is so locally in a certain sense which we will explain in the following.

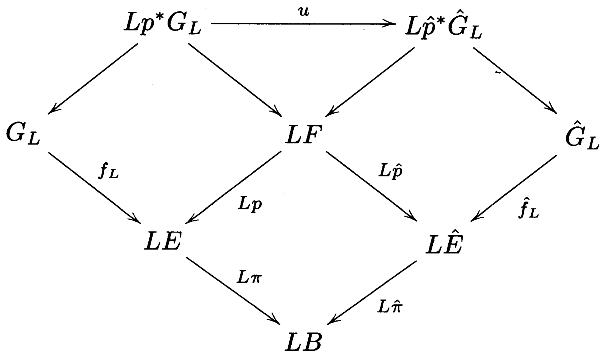
5.2.2. — We can extend the second diagram by the local systems (see 5.1.3)

(5.2.2)



and the pull-backs of gerbes

(5.2.3)



In particular, we have an isomorphism

$$u : Lp^*\mathcal{L}_G \xrightarrow{\sim} L\hat{p}^*\hat{\mathcal{L}}_{\hat{G}} \ . \tag{5.2.4}$$

5.2.3. — Note that $\hat{p}\colon F \to \hat{E}$ is a $U(1)$-principal bundle. In [**10**, Lemma 2.34] we have constructed a map $h\colon L\hat{E} \to U(1)^\delta$ which measures the action of the automorphisms of the points of $\hat{E}$ on the fibers of $\hat{p}$. We get a decomposition into a disjoint union of open substacks

$$L\hat{E} \cong \bigsqcup_{u\in U(1)} L\hat{E}_u \ ,$$

where $L\hat{E}_u := h^{-1}(u)$. Here and in the following we use the simplified notation $h^{-1}(u)$ for the pullback of $h\colon L\hat{E} \to U(1)^\delta$ along the inclusion $i_u\colon * \to U(1)$ with $i_u(*) := u$. By [**10**, Lemma 2.36], the map $L\hat{p}\colon LF \to L\hat{E}$ factors over the inclusion $J : L\hat{E}_1 \to L\hat{E}$, and the corresponding map $L\hat{p}_1\colon LF \to L\hat{E}_1$ is a $U(1)$-principal bundle. The integration

$$\mathfrak{L}\hat{\mathfrak{p}}_{1!}\colon R(L\hat{p}_1)_* \circ L\hat{p}_1^* \to \mathrm{id}$$

is well-defined. The open inclusion J induces a natural transformation $\mathfrak{J}_!\colon RJ_* \circ J^* \to \mathrm{id}$. We can thus define

$$\mathfrak{L}\hat{\mathfrak{p}}_! := \mathfrak{J}_! \circ \mathfrak{L}\hat{\mathfrak{p}}_{1!}\colon RL\hat{p}_* \circ L\hat{p}^* \to \mathrm{id}\ .$$

5.2.4.

Definition 5.2.5. — *The local T-duality transformation associated to the diagram* (5.2.1) *is given by the composition*

$$T_{\mathrm{loc}} := \mathfrak{L}\hat{\mathfrak{p}}_! \circ u \circ \mathfrak{L}\mathfrak{p}^*\colon RL\pi_*\mathscr{L}_G \to RL\hat{\pi}_*\hat{\mathscr{L}}_{\hat{G}}\ ,$$

where u is induced by (5.2.4).

Note that $H^*_{\mathrm{deloc,per}}(E,G) \cong H^*(LB; RL\pi_*\mathscr{L}_G)$. Hence we can make the following definition.

Definition 5.2.6. — *The T-duality transformation in twisted periodic delocalized cohomology associated to the T-duality diagram* (5.2.1) *is the transformation*

$$T\colon H^*_{\mathrm{deloc,per}}(E,G) \to H^*_{\mathrm{deloc,per}}(\hat{E},\hat{G})$$

induced by the local T-duality transformation T_{loc} defined in 5.2.5.

5.3. The geometry of T-duality diagrams over orbispaces

5.3.1. — We consider a T-duality diagram (5.2.1) over a locally compact orbispace. As explained in [**10**, Sec. 2.5] (see also 5.1.3) the gerbe $G \to E$ naturally gives rise to a $U(1)^\delta$-principal bundle $\tilde{G}^\delta \to LE$. Let $g\colon LB_1 \to U(1)^\delta$ be the function which describes the holonomy of the bundle $\tilde{G}^\delta \to LE$ along the fibers of $LE \to LB_1$ (see [**10**, 2.6.3]). In the following we recall from [**10**] a cohomological description of the functions g and h (introduced in 5.2.3).

Let $c_1 \in H^2(B;\mathbb{Z})$ denote the first Chern class of the $U(1)$-principal bundle $\pi\colon E \to B$, and let $d \in H^3(E;\mathbb{Z})$ denote the Dixmier-Douady class of the gerbe $f\colon G \to E$. By integration over the fiber it gives rise to a class $\int_\pi d \in H^2(B;\mathbb{Z})$. In [**10**, 2.4.11] we have shown that a class $\chi \in H^2(B;\mathbb{Z})$ gives rise to a function $\bar{\chi}\colon LB \to U(1)^\delta$ in a natural way.

***Proposition 5.3.1* (Lemma 2.38 and Prop. 2.49 [10]).** — *We have the equalities*

(1)

$$\overline{c_1} = h\colon LB \to U(1)^\delta\ .$$

(2)
$$\overline{\int_\pi d}_{|LB_1} = g\colon LB_1 \to U(1)^\delta \ .$$

5.3.2. — We now have functions $h, \hat{h}\colon LB \to U(1)^\delta$ associated to the $U(1)$-principal bundles $\pi\colon E \to B$ and $\hat{\pi}\colon \hat{E} \to B$. We define

$$LB_{(1,*)} := h^{-1}(1) \ , \quad LB_{(*,1)} := \hat{h}^{-1}(1) \ .$$

We furthermore have functions (see 5.2.1)

$$g\colon LB_{(1,*)} \to U(1)^\delta \ , \quad \hat{g}\colon LB_{(*,1)} \to U(1)^\delta$$

measuring the holonomy of $\tilde{G}^\delta \to LE$ and $\tilde{\hat{G}}^\delta \to L\hat{E}$ along the fibers.

Proposition 5.3.2. — *We have the equalities*

$$\hat{g} = h^{-1}_{|LB_{(*,1)}} \ , \quad g = \hat{h}^{-1}_{|LB_{(1,*)}} \ .$$

Proof. — Let

$$d \in H^3(E;\mathbb{Z}) \ , \quad \hat{d} \in H^3(\hat{E};\mathbb{Z})$$

be the Dixmier-Douady classes of the gerbes $G_L \to E$ and $\hat{G}_L \to \hat{E}$. Furthermore let

$$c_1, \hat{c}_1 \in H^2(B;\mathbb{Z})$$

denote the first Chern classes of the $U(1)$-principal bundles $\pi\colon E \to B$ and $\hat{\pi}\colon \hat{E} \to B$. The theory of T-duality for orbispaces [**8**] gives the equalities

$$c_1 = -\hat{\pi}_!(\hat{d}) \ , \quad \hat{c}_1 = -\pi_!(d) \ .$$

Hence the assertion follows from Proposition 5.3.1. □

5.4. The T-duality transformation in twisted periodic delocalized cohomology is an isomorphism

5.4.1. — Let us consider a $U(1)$-principal bundle $\pi\colon E \to B$ in locally compact orbispaces with first Chern class $c_1 \in H^2(B;\mathbb{Z})$ and a topological $U(1)$-banded gerbe $f\colon G \to E$ with Dixmier-Douady class $d \in H^3(E;\mathbb{Z})$. In Definition 5.1.2 we have introduced the object $\mathcal{L}_G \in D(\mathrm{Sh_{Ab}}\mathbf{LE})$. Furthermore we have $U(1)^\delta$-valued functions $h = \overline{c_1}$ and $g = \overline{\pi_!(d)}$ on LB. Let $LB_1 := h^{-1}(1)$ and note that $L\pi\colon LE \to LB$ factors over the $U(1)$-principal bundle $L\pi\colon LE \to LB_1$. We fix $u \in U(1)^\delta \setminus \{1\}$ and consider the component $LB_{(1,u)} := h^{-1}(1) \cap g^{-1}(u)$.

Lemma 5.4.1. — *We have* $R\pi_*(\mathcal{L}_G)_{|LB_{(1,u)}} \cong 0$.

Proof. — Let $(T \to LB_{(1,u)}) \in \mathbf{LB}_{(1,u)}$. After refining T by a covering we can assume that there is a diagram

$$\begin{array}{ccccccc}
\mathcal{B}U(1) & \xleftarrow{z} & U(1) \times \mathcal{B}U(1) & \longleftarrow & s^*G_L & \longrightarrow & G_{L(1,u)} \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x} & & \downarrow & & \downarrow \\
* & \xleftarrow{q} & U(1) & \xleftarrow{v} & T \times U(1) & \xrightarrow{s} & LE_{(1,u)} \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
 & & * & \xleftarrow{w} & T & \xrightarrow{t} & LB_{(1,u)}
\end{array}$$

of Cartesian squares. We get

$$\begin{aligned}
t^*R\pi_*(\mathcal{L}_G) &\cong Rp_*s^*(\mathcal{L}_G) \\
&= Rp_*s^*(P_{G_L}(\mathcal{L})) \\
&\cong Rp_*P_{s^*G_L}(s^*\mathcal{L}) \ .
\end{aligned}$$

Let $\mathcal{H} \in \mathrm{Sh}_{\mathrm{Ab}}(\mathrm{Site}(U(1)))$ be the locally constant sheaf over $U(1)$ with fiber $\mathbb{C}$ and holonomy $u \in U(1) \setminus \{1\}$. Then we have $s^*\mathcal{L} \cong v^*\mathcal{H}$. We calculate further

$$\begin{aligned}
Rp_*P_{s^*G_L}(s^*\mathcal{L}) &\cong Rp_*P_{s^*G_L}(v^*\mathcal{H}) \\
&\cong Rp_*v^*P_{U(1)\times\mathcal{B}U(1)}(\mathcal{H}) \\
&\cong w^*Rq_*P_{U(1)\times\mathcal{B}U(1)}(\mathcal{H}) \ .
\end{aligned}$$

It remains to show that

$$Rq_*P_{U(1)\times\mathcal{B}U(1)}(\mathcal{H}) \cong 0 \ .$$

Recall from 3.4.9 that the object $P_{U(1)\times\mathcal{B}U(1)}(\mathcal{H}) \in D(\mathrm{Sh}_{\mathrm{Ab}}\mathrm{Site}(U(1)))$ is given (up to non-canonical isomorphism) by the holim of a diagram

$$0 \leftarrow Rx_*x^*(\mathcal{H}) \stackrel{D}{\leftarrow} Rx_*x^*(\mathcal{H})[2] \stackrel{D}{\leftarrow} Rx_*x^*(\mathcal{H})[4] \stackrel{D}{\leftarrow} Rx_*x^*(\mathcal{H})[6] \cdots \ .$$

The functor Rq_* commutes with this holim [1]. Therefore $Rq_*P_{U(1)\times\mathcal{B}U(1)}(\mathcal{H})$ is given by the holim of the diagram

$$\begin{aligned}
0 \leftarrow Rq_*Rx_*x^*(\mathcal{H}) &\stackrel{Rq_*(D)}{\leftarrow} Rq_*Rx_*x^*(\mathcal{H})[2] \\
&\stackrel{Rq_*(D)}{\leftarrow} Rq_*Rx_*x^*(\mathcal{H})[4] \stackrel{Rq_*(D)}{\leftarrow} Rq_*Rx_*x^*(\mathcal{H})[6] \cdots \ .
\end{aligned}$$

The following calculation uses the projection formula twice, first by Lemma 6.2.10 for the non-representable map x and a tensor product with a one-dimensional local system of complex vector spaces $\mathcal{H}$, secondly using Lemma 6.2.13 for the proper representable

[1] Rq_* is a right-adjoint and commutes with products and mapping cones

map q and the tensor product with the bounded below object $Ry_*(i^\sharp\mathbb{Z}_{\mathrm{Site}([*/U(1)])}) \in D^+(\mathrm{Sh_{Ab}Site}(U(1)))$

$$\begin{aligned} Rq_*Rx_*x^*(\mathcal{H}) &\cong Rq_*Rx_*(\underline{\mathbb{Z}}_{\mathrm{Site}(U(1)\times\mathcal{B}U(1)} \otimes x^*(\mathcal{H})) \\ &\cong Rq_*(Rx_*(\underline{\mathbb{Z}}_{\mathrm{Site}(U(1)\times\mathcal{B}U(1))}) \otimes \mathcal{H}) \\ &\cong Rq_*(Rx_*(z^*\underline{\mathbb{Z}}_{\mathrm{Site}(\mathcal{B}U(1))}) \otimes \mathcal{H}) \\ &\cong Rq_*(q^*(Ry_*\underline{\mathbb{Z}}_{\mathrm{Site}(\mathcal{B}U(1))}) \otimes \mathcal{H}) \\ &\cong Ry_*\underline{\mathbb{Z}}_{\mathrm{Site}(\mathcal{B}U(1))} \otimes Rq_*(\mathcal{H}) \ . \end{aligned}$$

Since the holonomy of $\mathcal{H}$ along $U(1)$ is non-trivial, and the cohomology of S^1 with coefficients in a non-trivial flat line bundle is trivial, we have

$$Rq_*(\mathcal{H}) \cong 0 \ . \qquad \square$$

5.4.2. — We now consider a T-duality diagram (5.2.1) where B is a locally compact orbispace.

Theorem 5.4.2. — *The local T-duality transformation (Definition 5.2.5)*

$$T_{\mathrm{loc}} \colon RL\pi_*(\mathcal{L}_G) \to RL\hat{\pi}_*(\hat{\mathcal{L}}_{\hat{G}})[-2]$$

is an isomorphism in $D(\mathrm{Sh_{Ab}}\mathbf{LB})$. In particular, the T-duality transformation

$$T\colon H^*_{\mathrm{deloc,per}}(E,G) \to H^*_{\mathrm{deloc,per}}(\hat{E},\hat{G})$$

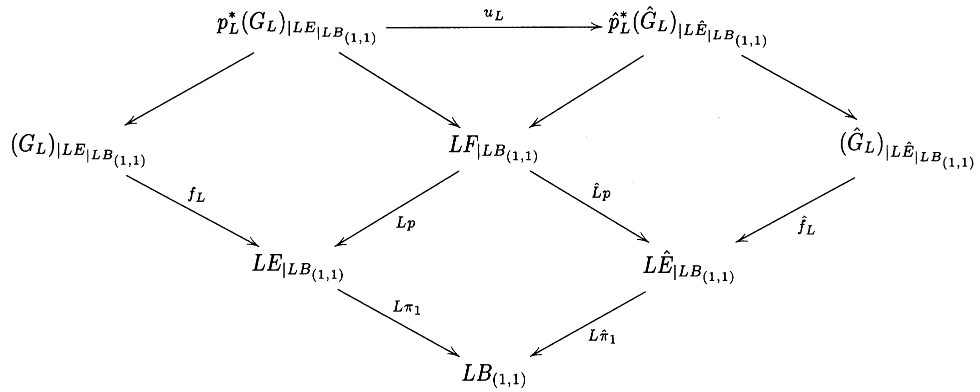
is an isomorphism.

Proof. — We have functions $h,\hat{h}\colon LB \to U(1)$ which define substacks $LB_{(1,*)} := h^{-1}(1)$ and $LB_{(*,1)} := \hat{h}^{-1}(1)$. By Proposition 5.3.2 we have $g = \hat{h}^{-1}_{|LB_{(1,*)}} \colon LB_{(1,*)} \to U(1)^\delta$. By Lemma 5.4.1 the object $RL\pi_*(\mathcal{L}_G) \in D(\mathrm{Sh_{Ab}}\mathbf{LB})$ is supported on

$$g^{-1}(1) = LB_{(1,*)} \cap LB_{(*,1)} =: LB_{(1,1)} \ .$$

Note that $\hat{g} = h^{-1}_{|LB_{(*,1)}}$, so that $RL\hat{\pi}_*\hat{\mathcal{L}}_{\hat{G}}$ is supported on $LB_{(1,1)}$, too. Let $i\colon LB_{(1,1)} \to LB$ denote the inclusion. The following diagram is the pull-back

of (5.2.1) via the map $LB_{(1,1)} \to LB \to B$
(5.4.3)



We consider

$$\mathcal{L}_1 := \mathcal{L}_{|LE_{|LB_{(1,1)}}} \ , \quad \hat{\mathcal{L}}_1 := \hat{\mathcal{L}}_{|L\hat{E}_{|LB_{(1,1)}}} \ .$$

Because we restrict to the subset $LB_{(1,1)}$ of trivial holonomy we have isomorphisms

$$\mathcal{L}_1 \cong L\pi_1^* \underline{\mathbb{C}}_{\mathbf{LB}_{(1,1)}} \quad \hat{\mathcal{L}}_1 \cong L\hat{\pi}_1^* \underline{\mathbb{C}}_{\mathbf{LB}_{(1,1)}} \ .$$

The local T-duality transformation T_{loc} is now locally equal to the transformation J defined in 4.3.5 applied to the T-duality diagram (5.4.3) and the sheaf $\underline{\mathbb{C}}_{\mathbf{LB}_{(1,1)}}$. As in the proof of Theorem 4.3.7 one shows, using the commutative double T-duality diagram, that T_{loc} is an isomorphism.

The global second assertion can be deduced directly from Theorem 4.3.7. By the observation on the support of $RL\pi_*(\mathcal{L}_G) \in D(\text{Sh}_{\text{Ab}}\mathbf{LB})$ made above we get

$$H^*_{\text{deloc,per}}(E, G) \cong H^*_{\text{per}}(LB_{(1,1)}; RL(\pi_1)_* P_{(G_L)_{|LE_{|LB_{(1,1)}}}}(L\pi_1^* \underline{\mathbb{C}}_{\mathbf{LB}_{(1,1)}})) \ ,$$

and similarly

$$H^*_{\text{deloc,per}}(\hat{E}, \hat{G}) \cong H^*_{\text{per}}(LB_{(1,1)}; RL(\hat{\pi}_1)_* P_{(\hat{G}_L)_{|L\hat{E}_{|LB_{(1,1)}}}}(L\pi_1^* \underline{\mathbb{C}}_{\mathbf{LB}_{(1,1)}})) \ .$$

With these identifications the T-duality transformation in twisted periodic delocalized cohomology is then equal to the T-duality transformation in twisted periodic cohomology for the diagram (5.4.3) and the sheaf $\underline{\mathbb{C}}_{\mathbf{LB}_{(1,1)}} \in D^+(\text{Sh}_{\text{Ab}}\mathbf{LB}_{1,1})$. $\square$

CHAPTER 6

VERDIER DUALITY FOR LOCALLY COMPACT STACKS

6.1. Elements of the theory of stacks on Top and sheaf theory

6.1.1. — In the present paper we consider stacks on the site Top. A prestack is a lax presheaf X of groupoids on Top. The prefix "lax" indicates that for a pair of composable morphisms $u\colon U \to V$, $v\colon V \to W$ we have a natural transformation of functors $\phi_{u,v}\colon X(u) \circ X(v) \to X(v \circ u)$ which is not necessarily the identity, and which satisfies a compatibility condition for triples. A prestack is a stack if it satisfies the standard descent conditions on the level of objects and morphisms. A sheaf of sets can be considered as a stack in the canonical way. Via the Yoneda embedding Top $\to$ ShTop (note that the topology of Top is sub-canonical, i.e. representable presheaves are sheaves) we consider topological spaces as stacks in the natural way.

6.1.2. — In the following we collect some definitions and facts of the theory of stacks in topological spaces. Stacks are objects of a two-category, and fiber products and more general limits in stacks are understood in the two-categorial sense. Note that two-categorial limits in stacks exists (see [**10**] for more information), and that the inclusion of spaces into stacks preserves those limits. A useful reference for stacks in topological spaces and manifolds is the survey [**14**].

(1) A morphism of stacks $G \to H$ is called representable, if for each space U and map $U \to H$ the fiber product $U \times_H G$ is equivalent to a space.
(2) A representable map $G \to H$ between stacks is called proper if for every map $K \to H$ from a compact space the fiber product $K \times_H G$ is a compact space.
(3) A map $f : A \to B$ of topological spaces has local sections if for each point $b \in B$ in the image of f there exists a neighbourhood $b \in U \subseteq B$ and a map $s : U \to A$ such that $f \circ u = \mathrm{id}_U$.
(4) A representable morphism $G \to H$ has local sections if for every map $U \to H$ from a space the induced map $U \times_H G \to U$ of spaces has local sections.

(5) A representable map $G \to H$ is surjective if for every map $U \to H$ from a space the induced map $U \times_H G \to U$ is a surjective map of spaces.
(6) A map $A \to X$ from a space A to a stack X is called an atlas of X, if it is surjective, representable and admits local sections. A stack which admits an atlas is called a topological stack.
(7) A morphism (not necessarily representable) between topological stacks $G \to H$ is surjective (or has local sections, respectively) if for an atlas $A \to G$ the composition $A \to G \to H$ is surjective (or has local sections, respectively) (note that this composition is representable by Proposition 6.1.1 below).
(8) A composition of maps with local sections has local sections. The corresponding assertion is true for the following properties of maps:
   (a) representable
   (b) representable and proper
   (c) surjective.
(9) Consider a two-cartesian diagram of stacks

$$\begin{array}{ccc} H & \xrightarrow[v]{} & G \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[u]{} & X \end{array}$$

If u has local sections, then so has v. If f is representable, then so is g.

6.1.3. — The inclusion of spaces into sheaves and of sheaves into stacks preserves small limits, where limits in stacks are understood in the 2-categorical sense. This implies that a map of spaces $X \to Y$ is representable. In fact we have the following more general result.

Proposition 6.1.1. — *Let G be a topological stack and X a space. Then every morphism $f: X \to G$ is representable.*

The proof will be given in 6.1.5 and needs some preparations.

6.1.4. — We will need the notion of an open substack.

Definition 6.1.2. — *Let G be a stack in topological spaces. A morphism $H \to G$ of stacks is an embedding of an open substack, if it is representable and for each map $T \to G$ from a space T the induced map of spaces $T \times_G H \to T$ is an open embedding of topological spaces.*

Note that, via Yoneda, an open embedding of spaces is an open embedding of stacks.

Definition 6.1.3. — *A morphism $U \to G$ of topological stacks is locally an open embedding if $U \cong \bigsqcup_{i\in I} U_i$ for a collection $(U_i)_{i\in I}$ of topological stacks and $U_i \to G$ is an embedding of an open substack for every $i \in I$.*

Let us first characterize spaces as stacks which can be covered by a collection of spaces.

Lemma 6.1.4. — *Let X be a stack in topological spaces for which there exists a morphism $U \to X$ from a space which is surjective and locally an open embedding. Then X is equivalent to a space.*

Proof. — Let $U \cong \sqcup_i U_i$ be such that $U_i \to X$ is an open embedding for all i. Then we define the space B as the coequalizer in spaces

$$B := \operatorname{coeq}(\bigsqcup_{i,j} U_i \times_X U_j \rightrightarrows \bigsqcup_i U_i) \ . \tag{6.1.5}$$

Since $U_i \to X$ is an open embedding we see that $\mathrm{pr}_{U_i} : U_i \times_X U_j \to U_i$ is an open embedding. We can now refer to [**22**, Prop. 16.1] and deduce that the equalizer in spaces B is also the two-categorical equalizer in stacks of the diagram (6.1.5), which is of course equivalent to X. Note that the difficulty at this point is that the embedding of the category of spaces (viewed as a two-category) into the two-category of stacks does not preserve general small colimits, as opposed to the case of limits.

For completeness we will give an argument. First note that $\mathrm{pr}_{U_i} : U_i \times_X U_i \xrightarrow{\sim} U_i$ is a homeomorphism. It thus follows from the groupoid structure of the coequalizer diagram that $U_i \to B$ is injective for all i. Since $\bigsqcup_i U_i \to B$ is a topological quotient map it is open. Therefore $\bigsqcup_i U_i \to B$ is a open covering. We further conclude that the natural map $U_i \times_X U_j \to U_i \times_B U_j$ is in fact a homeomorphism.

The claim is that X is equivalent to B. We first construct a morphism $X \to B$. Let $(T \to X) \in X(T)$. Then $(T_i := T \times_X U_i)_i$ is an open covering of T. Using the identification $T_i \times_T T_j \cong T \times_X (U_i \times_X U_j)$ we get a diagram

$$\begin{array}{ccc} \bigsqcup_{i,j} T_i \times_T T_j & \longrightarrow & U_i \times_X U_j \\ \downarrow\downarrow & & \downarrow\downarrow \\ \bigsqcup_i T_i & \longrightarrow & \bigsqcup_i U_i \\ \downarrow & & \downarrow \\ T & \dashrightarrow & B \end{array} ,$$

where the horizontal maps are induced by the projections $T \times_X U_i \to U_i$, and the left vertical is the representation of T as a coequalizer. Therefore we obtain a unique factorization $(T \to B) \in B(T)$. The construction is functorial in T and therefore induces a morphism $X \to B$.

In order to see that it has an inverse let $(T \to B) \in B(T)$ be given. Then we define the open covering $(T_i := T \times_B U_i)_i$ of T. The compositions

$$\phi_i : T_i \cong T \times_B U_i \stackrel{\mathrm{pr}_{U_i}}{\to} U_i \to X$$

can be considered as a collection of objects $(\phi_i \in X(T_i))_i$. The induced map

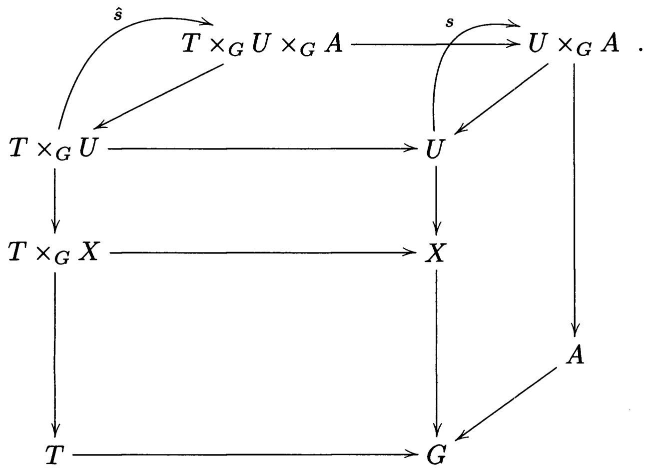
$$T_i \cap T_j \cong T_i \times_T T_j \cong (T \times_B U_i) \times_T (T \times_B U_j) \cong T \times_B (U_i \times_B U_j)$$
$$\stackrel{\mathrm{pr}_{U_i \times_B U_j}}{\to} U_i \times_B U_j \cong U_i \times_X U_j \to X \times_X X$$

can be considered as a collection of isomorphisms $\phi_{ij} : (\phi_i)_{|T_i \cap T_j} \xrightarrow{\sim} (\phi_j)_{|T_i \cap T_j}$ which satisfy the cocycle condition on triple intersections. Since X is a stack we can therefore glue the local maps and get a map $(T \to X) \in X(T)$ which is unique up to unique isomorphism. This construction is again functorial in T and provides the map $B \to X$.

It is easy to see that both maps $X \to B$ and $B \to X$ constructed above are mutually inverse. □

6.1.5. — We now show Proposition 6.1.1

Proof. — Consider a map $T \to G$ from a space T. We have to prove that the fiber product $T \times_G X$ is equivalent to a space. Using the assumption that G is topological we choose an atlas $A \to G$ of G. Because $A \to G$ has local sections, we can find an open covering $\bigsqcup_{i \in I} U_i =: U \to X$ such that $U \times_G A \to U$ has a section $s \colon U \to U \times_G A$. We first want to show that $T \times_G U$ is a space. Since the structure map $A \to G$ of an atlas is representable we know that $U \times_G A$ and $T \times_G A$ are spaces. Therefore, $T \times_G U \times_G A \cong (T \times_G A) \times_A (U \times_G A)$ is a space, too. The section s pulls back to a section $\hat{s} : T \times_G U \to T \times_G U \times_G A$ which implements $T \times_G U$ as a subspace of the space $T \times_G U \times_G A$.



Since the map $U \to X$ is surjective and locally an open embedding its pull-back $T \times_G U \to T \times_G X$ is surjective and locally an open embedding, too. Therefore by Lemma 6.1.4 the stack $T \times_G X$ is equivalent to a space. □

6.1.6. — Recall that a topological stack is called locally compact if it admits a locally compact atlas $A \to G$ such that $A \times_G A$ is a locally compact space. Furthermore recall that the site $\mathbf{X} = \mathrm{Site}(X)$ associated to a locally compact stack X is the full subcategory of locally compact spaces $U \to X$ over X such that the structure map has local sections. A morphism in this site $\mathbf{X}$ is a diagram

$$\begin{array}{ccc} U & \xrightarrow{\qquad} & V \\ & \searrow \;\Downarrow\; \swarrow & \\ & X & \end{array} \qquad (6.1.6)$$

consisting of a morphism of spaces over X and a two-morphism. The topology on $\mathbf{X}$ is given by the covering families of the objects $(U \to X)$ induced by open covering of U.

Much of the general theory would work without the assumption of local compactness. But local compactness is important in connection with the projection formula Lemma 6.2.11 which is a crucial ingredient of the theory of integration. Since the latter is our main goal of the present section we generally adopt the restriction to locally compact stacks.

6.1.7. — The sheaf theory for topological stacks can be built in a parallel manner to the sheaf theory for smooth stacks developed in [**9**]. The transition goes via the following replacements of words:

(1) For the definition of stacks the site of smooth manifolds Mf^∞ is replaced by the site of topological spaces Top. In the definition of the site of a locally compact stack manifolds are replaced by locally compact spaces.
(2) The concept of a *smooth stack* is replaced by the concept of a *locally compact stack.*
(3) The notion of a *smooth map* is replaced by the notion of a *map which admits local sections.*

In the present paper we freely use results in the general sheaf theory for topological stacks from [**9**, Sec. 2] in the case of stacks in topological spaces which are proved there for manifolds. It should be noted that with the conventions just made, all statements and proofs carry over verbatim

6.1.8. — Let X be a locally compact stack. By $\mathrm{Pr}\mathbf{X}$ and $\mathrm{Sh}\mathbf{X}$ we denote the categories of presheaves and sheaves on $\mathbf{X}$. They are related by a pair of adjoint functors

$$i^\sharp : \mathrm{Pr}\mathbf{X} \leftrightarrows \mathrm{Sh}\mathbf{X} : i\ .$$

The sheafification functor $i^\sharp$ is exact.

6.1.9. — Let $f : X \to Y$ be a morphism of locally compact stacks. In induces a functor ${}^pf_* : \mathrm{Pr}\mathbf{X} \to \mathrm{Pr}\mathbf{Y}$ by

$$ {}^pf_*F(V \to Y) := \lim F(U \to X) \ , $$

where the limit is taken over the category of diagrams

$$ \begin{array}{ccc} U & \longrightarrow & X \\ \Big\downarrow & \nearrow & \Big\downarrow f \\ V & \longrightarrow & Y \end{array} \tag{6.1.7} $$

with $(U \to X) \in \mathbf{X}$. For details we refer to [**9**, Sections 2.1, 2.2]. This functor fits into an adjoint pair

$$ {}^pf^* : \mathrm{Pr}\mathbf{Y} \leftrightarrows \mathrm{Pr}\mathbf{X} : {}^pf_* \ . $$

The functor ${}^pf^*$ is given by

$$ {}^pf^*G(U \to X) = \operatorname{colim} G(V \to Y) \ , $$

where the colimit is again taken over the category of diagrams with $(V \to Y) \in \mathbf{Y}$.

We extend these functors to sheaves by

$$ f_* := i^\sharp \circ {}^pf_* \circ i \ , \quad f^* := i^\sharp \circ {}^pf^* \circ i $$

and obtain an adjoint pair

$$ f^* : \mathrm{Sh}\mathbf{Y} \leftrightarrows \mathrm{Sh}\mathbf{X} : f_* \ . $$

Note that ${}^pf_*$ preserves sheaves (see [**9**, Lemma 2.13]). The right-adjoint functor $f_* : \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$ is left exact and therefore admits a right-derived functor

$$ Rf_* : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) $$

between the bounded below derived categories.

6.1.10. — If $g : Y \to Z$ is a second morphism of locally compact stacks, then we have natural isomorphisms of functors

$$ (g \circ f)_* \cong g_* \circ f_* \ , \quad f^* \circ g^* \cong (g \circ f)^* $$

(see 6.6.9). Furthermore, we have

$$ Rg_* \circ Rf_* \cong R(g \circ f)_* $$

on the level of bounded below derived categories by Lemma 6.6.13. The relation $f^* \circ g^* \cong (g \circ f)^*$ descends to the derived categories if the pull-back functors are exact, e.g. if f and g have local sections (see 6.1.11). These facts generalize corresponding results shown in [**9**].

6.1.11. — Let $f : G \to H$ be a morphism between topological stacks which has local sections. It induces a morphism between sites $f_\sharp : \mathbf{G} \to \mathbf{H}$ by composition. On objects it is given by $f_\sharp(U \to G) := (U \to G \to H)$ (we will often use the short hand U for $(U \to G)$ and write $f_\sharp U$). In fact, since $U \to \mathbf{G}$ and f have local sections, the composition $U \to H$ has local sections. Furthermore, the map $U \to H$ from a space to a topological stack is representable by Lemma 6.1.1. One checks that $f_\sharp$ maps covering families to covering families and preserves the fiber products as in [**25**, 1.2.2].

If $f : G \to H$ has local sections, then the functor $f^* : \mathrm{Sh}\mathbf{H} \to \mathrm{Sh}\mathbf{G}$ is the pull-back $f^* = (f_\sharp)^*$ associated to a morphism of sites. Explicitly it is given by $f^*F(U) := F(f_\sharp U)$, compare Lemma [**9**, 2.7]. In addition, the functor $f^* : \mathrm{Sh}\mathbf{H} \to \mathrm{Sh}\mathbf{G}$ is exact (see [**9**, 2.5.9]) and preserves flat sheaves of abelian groups.

Lemma 6.1.8. — *If $f : X \to Y$ is a morphism between locally compact stacks which has local sections, then we have the derived adjunction*

$$f^* \colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \leftrightarrows D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \colon Rf_* \ .$$

Proof. — Since f^* is exact its right adjoint f_* preserves injectives. If $G \in C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ is a complex of injectives and $F \in C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$, then we have

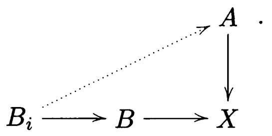
$$\begin{aligned} R\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(F, Rf_*(G)) &\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(F, f_*(G)) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(f^*(F), G) \cong R\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(f^*(F), G) \ . \end{aligned}$$

This implies the assertion. □

6.1.12.

Lemma 6.1.9. — *Let X be a locally compact stack. If $C, B \to X$ are maps from locally compact spaces, then $C \times_X B$ is locally compact.*

Proof. — By assumption X is locally compact so that we can chose an atlas $A \to X$ such that A and $A \times_X A$ are locally compact. Since $A \to X$ is surjective and has local sections, there exists an open covering (B_i) of B such that we have lifts



Then $(A \times_X B_i)$ is an open covering of $A \times_X B$. In order to show that $A \times_X B$ is locally compact it suffices to show that the space $A \times_X B_i$ is locally compact. By $A \times_X B_i \cong (A \times_X A) \times_A B_i \subseteq A \times_X A \times B_i$, this space is a closed (note that A is Hausdorff) subspace of a locally compact space and hence itself locally compact.

The same argument shows that $C \times_X A$ is locally compact. We now write $C \times_X B_i \cong (C \times_X A) \times_A B_i \subseteq (C \times_X A) \times B_i$ in order to see that $C \times_X B_i$ is locally compact.

Since $(C \times_X B_i)$ is an open covering of $C \times_X B$ we conclude that $C \times_X B$ is locally compact. □

6.1.13. — Let $f : X \to Y$ be a morphism between locally compact stacks.

Lemma 6.1.10. — *If f is representable, then it induces a morphism of sites $f^\sharp : \mathbf{Y} \to \mathbf{X}$ given by $f^\sharp(V \to Y) := (X \times_Y V \to X)$.*

Proof. — Let $B \to X$ be a locally compact atlas. We consider $(V \to Y) \in \mathbf{Y}$ and form the diagram of Cartesian squares

$$\begin{array}{ccc} V \times_Y B & \longrightarrow & B \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \\ \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array} .$$

In order to check that $(U \to X) \in \mathbf{X}$ we must show that U is locally compact. Since $B \to X$ is surjective and has local sections we see that $V \times_Y B \to U$ is surjective and has local sections, too. Since Y is locally compact we see by Lemma 6.1.9 that $V \times_Y B$ is locally compact. Let $u \in U$ and $W \subseteq U$ be a neighborhood of u such that there exists a section

$$\begin{array}{ccc} & & V \times_Y B \\ & \overset{s}{\nearrow} & \downarrow \pi \\ W & \longrightarrow & U \end{array} .$$

Let $K \subseteq \pi^{-1}(W)$ be a compact neighborhood of $s(u)$. Then $s^{-1}(K)$ is a compact neighborhood of u. Indeed, $s^{-1}(K)$ is a closed subset of the compact set $\pi(K)$.

It is easy to see that $f^\sharp$ maps covering families to covering families and preserves the fiber products required for a morphism of sites, see [**25**, 1.2.2]. □

If $f : X \to Y$ is a representable morphism between locally compact stacks, then we have the relations $f^* = (f^\sharp)_* : \mathrm{Sh}\mathbf{Y} \to \mathrm{Sh}\mathbf{X}$ and $f_* = (f^\sharp)^* : \mathrm{Sh}\mathbf{X} \to \mathrm{Sh}\mathbf{Y}$, see [**9**, Lemma 2.9].

6.1.14. — Let X be a topological stack and $(U \to X) \in \mathbf{X}$. Let (U) denote the site whose objects and morphisms are the open subsets of U and inclusions, and whose coverings are coverings by families of open subsets. We have restriction functors $\nu_U : \mathrm{Sh}\mathbf{X} \to \mathrm{Sh}(U)$ and ${}^p\nu_U : \mathrm{Pr}\mathbf{X} \to \mathrm{Pr}(U)$. For $F \in \mathrm{Sh}\mathbf{X}$ we also write $\nu_U(F) =: F_U$. We have the following assertions, most of which are straightforward to prove.

(1) Let $i^\sharp$ and $i^\sharp_U$ denote the sheafification functors on the sites $\mathbf{X}$ and (U). Then we have a natural isomorphism

$$i^\sharp_U \circ {}^p\nu_U \cong \nu_U \circ i^\sharp \ ,$$

see [**9**, Lemma 2.4.7]

(2) Let $F \in \mathrm{Sh}\mathbf{X}$. If $f : U \to V$ is a morphism (6.1.6) in $\mathbf{X}$, then we have a natural map $f^*F_V \to F_U$.

(3) There is a one-to one correspondence of sheaves $F \in \mathrm{Sh}\mathbf{X}$ on the one hand, and of collections $(F_U)_{(U\to X)\in\mathbf{X}}$ of sheaves $F_U \in \mathrm{Sh}(U)$ together with functorial maps $f^*F_V \to F_U$ for all morphisms $f : U \to V$ in $\mathbf{X}$ on the other hand.

(4) Let $F, G \in \mathrm{Sh}\mathbf{X}$. There is a one-to-one correspondence between compatible collections of morphisms $g_U : F_U \to G_U$ for all $(U \to X) \in \mathbf{X}$ and maps $g : F \to G$.

(5) If $F, G \in \mathrm{Sh}\mathbf{X}$ or $F, G \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$, then a map $F \to G$ is an isomorphism if and only if the induced map $F_U \to G_U$ is an isomorphism for all $(U \to X) \in \mathbf{X}$.

(6) Let $f : X \to Y$ be a representable map of locally compact stacks, $(A \to Y) \in \mathbf{Y}$ and $(B := A \times_Y X \to X) \in \mathbf{X}$. Let $g : B \to A$ be the projection onto the first factor and $g_* : \mathrm{Sh}(B) \to \mathrm{Sh}(A)$. Then we have for $F \in \mathrm{Sh}\mathbf{X}$ or $G \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$

$$(f_*F)_A \cong g_*(F_B) \ , \quad (Rf_*G)_A \cong Rg_*(G_B) \ .$$

The second isomorphism follows from the first using the fact that the restriction ν_B preserves flabby or even injective sheaves (see Lemma 6.1.11).

(7) If $f : X \to Y$ is a map of topological stacks which has local sections, $(B \to X) \in \mathbf{X}$, then we have $(B \to X \to Y) \in \mathbf{Y}$ and for $F \in \mathrm{Sh}\mathbf{Y}$

$$(f^*F)_B \cong F_B \ .$$

(8) The collection of restriction functors $(\nu_U)_{(U\to X)\in\mathbf{X}}$ detects flabby (flasque, flat) sheaves (see Definition 3.1.1), i.e. a sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is flabby (flasque, flat) if and only if $F_U \in \mathrm{Sh}_{\mathrm{Ab}}(U)$ is flabby for all $(U \to X) \in \mathbf{X}$ (compare 6.2.6 for the flat case).

(9) The collection of restriction functors $(\nu_U)_{(U\to X)\in\mathbf{X}}$ detects exact sequences, i.e. a sequence $F \to G \to H$ of sheaves of abelian groups on $\mathbf{X}$ is exact if and only if $F_U \to G_U \to H_U$ is exact for all $(U \to X) \in \mathbf{X}$.

Lemma 6.1.11. — *Let $(U \to X) \in \mathbf{X}$. The functor $\nu_U : \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}(U)$ preserves injective sheaves.*

Proof. — We show that ν_U has an exact left adjoint $\nu^U_{\mathbb{Z}} : \mathrm{Sh}_{\mathrm{Ab}}(U) \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$. We first show that the restriction functor ${}^p\nu_U : \mathrm{Pr}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Pr}_{\mathrm{Ab}}(U)$ fits into an adjoint pair

$${}^p\nu^U_{\mathbb{Z}} : \mathrm{Pr}_{\mathrm{Ab}}(U) \leftrightarrows \mathrm{Pr}_{\mathrm{Ab}}\mathbf{X} : {}^p\nu_U \ .$$

The left-adjoint is given by

$$ {}^p\nu_{\mathbb{Z}}^U(F)(A \to X) := \mathrm{colim} F(V) \ , $$

where the colimit is taken over the category of diagrams

$$ \begin{array}{ccc} V & \longleftarrow & A \\ \downarrow & \swarrow^{\phi} & \downarrow \\ U & \longrightarrow & X \end{array} \ , $$

where $V \to U$ is the embedding of an open subset. As explained in [**20**, II.3.18] we have a decomposition of this category into a union of categories $S(\phi)$ with $\phi \in \mathrm{Hom}_{\mathbf{X}}((A \to X), (U \to X))$. The category $S(\phi)$ is the category of open neighborhoods of $\phi(A)$ and their inclusions. It is cofiltered. Therefore $F \mapsto \mathrm{colim}_{S(\phi)} F(V)$ preserves finite limits and is in particular left exact. This implies that ${}^p\nu_{\mathbb{Z}}^U$ given by

$$ {}^p\nu_{\mathbb{Z}}^U(F)(A \to X) \cong \bigoplus_{\phi} \mathrm{colim}_{S(\phi)} F(V) $$

is left-exact, too. We now get $\nu_{\mathbb{Z}}^U := i^\sharp \circ {}^p\nu_{\mathbb{Z}}^U \circ i_U$. As a left-adjoint it is right-exact. Since i_U is left exact and $i^\sharp$ is exact, this composition is also left-exact. □

6.1.15.

Lemma 6.1.12. — *Consider the following Cartesian diagram in locally compact topological stacks*

$$ \begin{array}{ccc} H & \xrightarrow{v} & G \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & X \end{array} $$

In this situation the two canonical ways to define a natural transformation

$$ u^* f_* \to g_* v^* \colon \mathrm{Sh}_{\mathrm{Ab}}(\mathbf{G}) \to \mathrm{Sh}_{\mathrm{Ab}}(\mathbf{Y}) $$

give the same result, i.e. the diagram

$$ \begin{array}{ccccccc} u^* f_* & \xrightarrow{\text{unit}} & g_* g^* u^* f_* & \xrightarrow{ug=fv} & g_* v^* f^* f_* & \xrightarrow{\text{counit}} & g_* v^* \\ \| & & & & & & \| \\ u^* f_* & \xrightarrow{\text{unit}} & u^* f_* v_* v^* & \xrightarrow{ug=fv} & u^* u_* g_* v^* & \xrightarrow{\text{counit}} & g_* v^* \end{array} \tag{6.1.13} $$

commutes. This transformation is functorial with respect to composition of Cartesian diagrams.

Moreover, if u has local sections, then this transformation induces isomorphisms

$$ u^* f_* \cong g_* v^* \colon \mathrm{Sh}_{\mathrm{Ab}}(\mathbf{G}) \to \mathrm{Sh}_{\mathrm{Ab}}(\mathbf{Y}), \tag{6.1.14} $$

$$ u^* R f_* \cong R g_* v^* \colon D^+\mathrm{Sh}_{\mathrm{Ab}}(\mathbf{G}) \to D^+\mathrm{Sh}_{\mathrm{Ab}}(\mathbf{Y}). \tag{6.1.15} $$

If u and f have local sections, then we get commutative diagrams

$$\begin{array}{ccccc} & & u_* & & \\ & \overset{\text{unit}}{\swarrow} & & \overset{\text{unit}}{\searrow} & \\ u_*g_*g^* & \xrightarrow{\sim} & f_*v_*g^* & \xleftarrow{\sim} & f_*f^*u_* \end{array} \quad , \quad \begin{array}{ccccc} & & v_* & & \\ & \overset{\text{counit}}{\nearrow} & & \overset{\text{counit}}{\nwarrow} & \\ f^*f_*v_* & \xrightarrow{\sim} & f^*u_*g_* & \xrightarrow{\sim} & v_*g^*g_* \end{array}$$

$$\begin{array}{ccccc} & & u^* & & \\ & \overset{\text{unit}}{\swarrow} & & \overset{\text{unit}}{\searrow} & \\ u^*f_*f^* & \xrightarrow{\sim} & g_*v^*f^* & \xleftarrow{\sim} & g_*g^*u^* \end{array} \quad , \quad \begin{array}{ccccc} & & v^* & & \\ & \overset{\text{counit}}{\nearrow} & & \overset{\text{counit}}{\nwarrow} & \\ v^*f^*f_* & \xrightarrow{\sim} & g^*u^*f_* & \xrightarrow{\sim} & g^*g_*v^* \end{array}$$

and their derived versions, e.g.

(6.1.16)
$$\begin{array}{ccccc} & & u^* & & \\ & \overset{\text{unit}}{\swarrow} & & \overset{\text{unit}}{\searrow} & \\ u^*Rf_*f^* & \xrightarrow{\sim} & Rg_*v^*f^* & \xleftarrow{\sim} & Rg_*g^*u^* \end{array} \quad ,$$

and also
(6.1.17)
$$\begin{array}{ccccccccc} & & & & Ru_*u^* & & & & \\ & \overset{\text{unit}}{\swarrow} & & & & & & \overset{\text{unit}}{\searrow} & \\ Ru_*u^*Rf_*f^* & \xrightarrow{\sim} & Ru_*Rg_*v^*f^* & \xrightarrow{\sim} & Rf_*Rv_*v^*f^* & \xrightarrow{\sim} & Rf_*Rv_*g^*u^* & \xleftarrow[\sim]{} & Rf_*f^*Ru_*u^* \end{array}$$

Proof. — *Most of the following arguments and the large diagrams were supplied by Ansgar Schneider. We thank Ansgar Schneider for the premission to use these ideas in the present article.* For convenience we present a proof of (6.1.13), see also [**13**, Expose XVII, Proposition2.1.3]. We first observe that

(6.1.18)
$$\begin{array}{ccc} v^*f^*f_*v_* & \xrightarrow{\text{counit}} & v^*v_* \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\text{counit}} \\ (fv)^*(fv)_* & \xrightarrow{\text{counit}} & \mathrm{id} \end{array}$$

commutes. Using this in addition to standard functorial properties we check that all squares in the following diagram commute:

$$
\begin{array}{ccccccccccccc}
u^*f_* & \xrightarrow{\text{unit}} & g_*g^*u^*f_* & \xrightarrow{\sim} & g_*(ug)^*f_* & \xrightarrow{=} & g_*(fv)^*f_* & \xrightarrow{\sim} & g_*v^*f^*f_* & \xrightarrow{\text{counit}} & g_*v^* & & \\
\| & & \Big\downarrow{\scriptstyle\text{unit}} & & \Big\downarrow{\scriptstyle\text{unit}} & & \Big\downarrow{\scriptstyle\text{unit}} & & \Big\downarrow{\scriptstyle\text{unit}} & & \Big\downarrow{\scriptstyle\text{unit}} & & \\
\| & & g_*g^*u^*f_*v_*v^* & \xrightarrow{\sim} & g_*(ug)^*f_*v_*v^* & \xrightarrow{=} & g_*(fv)^*f_*v_*v^* & \xrightarrow{\sim} & g_*v^*f^*f_*v_*v^* & \xrightarrow{\text{counit}} & g_*v^*v_*v^* & & \text{id} \\
\| & & \| & & \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\text{counit}} & & \\
\| & & g_*g^*u^*f_*v_*v^* & \xrightarrow{\sim} & g_*(ug)^*(fv)_*v^* & \xrightarrow{=} & g_*(fv)^*(fv)_*v^* & = & g_*(fv)^*(fv)_*v^* & \xrightarrow{\text{counit}} & g_*v^* & & \\
\| & & \| & & \| & & \Big\uparrow{\scriptstyle=} & & \Big\uparrow{\scriptstyle=} & & \| & & \\
\| & & g_*g^*u^*f_*v_*v^* & \xrightarrow{\sim} & g_*(ug)^*(fv)_*v^* & \xrightarrow{=} & g_*(ug)^*(ug)_*v^* & = & g_*(ug)^*(ug)_*v^* & \xrightarrow{\text{counit}} & g_*v^* & & \\
\| & & \| & & \Big\uparrow{\scriptstyle\sim} & & \Big\uparrow{\scriptstyle\sim} & & \Big\uparrow{\scriptstyle\sim} & & \Big\uparrow{\scriptstyle\text{counit}} & & \\
\| & & g_*g^*u^*f_*v_*v^* & \xrightarrow{\sim} & g_*g^*u^*(fv)_*v^* & \xrightarrow{=} & g_*g^*u^*(ug)_*v^* & \xrightarrow{\sim} & g_*g^*u^*u_*g^*v^* & \xrightarrow{\text{counit}} & g_*g^*g_*v^* & & \text{id} \\
\| & & \Big\uparrow{\scriptstyle\text{unit}} & & \Big\uparrow{\scriptstyle\text{unit}} & & \Big\uparrow{\scriptstyle\text{unit}} & & \Big\uparrow{\scriptstyle\text{unit}} & & \Big\uparrow{\scriptstyle\text{unit}} & & \\
u^*f_* & \xrightarrow{\text{unit}} & u^*f_*v_*v^* & \xrightarrow{\sim} & u^*(fv)_*v^* & \xrightarrow{=} & u^*(ug)_*v^* & \xrightarrow{\sim} & u^*u_*g^*v^* & \xrightarrow{\text{counit}} & g_*v^*. & &
\end{array}
$$

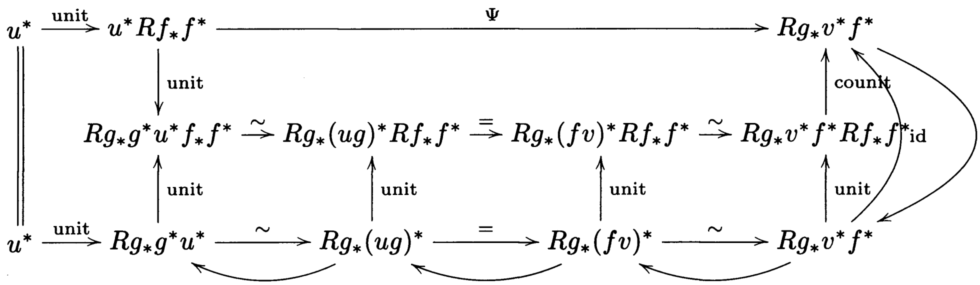
The two ways to go along the boundary from the upper left to lower right corner give the two maps $u^*f_* \to g_*v^*$ in question.

The isomorphism (6.1.14) can be shown as in [**9**, Lemma 2.16], where the assumption of smoothness of u in [**9**] corresponds to the assumption of local sections in the present setting. The derived version (6.1.15) can be shown using the simplicial models as in [**9**, Lemma 2.43]. Alternatively one can use the commutativity of the diagram asserted in Lemma 3.2.6 and the isomorphism (3.2.5).

We now show the compatibility of the units and counits with Cartesian diagrams. The arguments are purely formal and only use that the functors involved occur as parts of adjoint pairs. We will only give the details for the two triangles involving derived functors. If in addition to u also f has local sections, then so has g. In this case we have the adjoint pairs (f^*, Rf_*) and (g^*, Rg_*). In order to see (6.1.16) we must show that

$$
u^* \xrightarrow{\text{unit}} u^*Rf_*f^* \xrightarrow{\Psi} Rg_*v^*f^* \xrightarrow{\sim} Rg_*(fv)^* \xrightarrow{=} Rg_*(ug)^* \xrightarrow{\sim} Rg_*g^*u^*,
$$
$$
u^* \xrightarrow{\text{unit}} Rg_*g^*u^*
$$

commutes, where $\Psi : u^*Rf_*f^* \to Rg_*v^*f^*$ is induced by (6.1.15). This is a consequence of the commutativity of



which follows from standard functorial properties of units and counits.

The same properties are used in the proof of (6.1.17) which is represented by the boundary of the following big array of small commutative squares and triangles (see Fig. 1 on next page). □

6.2. Tensor products and the projection formula

6.2.1. — We consider a Grothendieck site $\mathbf{X}$ and a commutative ring R. The goal of the present Subsection is to discuss aspects of the closed monoidal structures on the categories of presheaves $\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$ and sheaves $\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ of R-modules on $\mathbf{X}$. The material is standard, but we need to understand in detail the relation between the sheaf and presheaf versions in order to show the compatibility with the operations induced by a morphism of stacks.

6.2.2. — Let $F, G \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$ be presheaves of R-modules. The tensor product $F \otimes^p G \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$ is defined as the presheaf which associates to $(U \to X)$ the R-module $F(U) \otimes^p_R G(U)$. In this way $\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$ becomes a symmetric monoidal category.

Since colimits of presheaves are defined objectwise we have for a diagram of presheaves of R-modules $(F_i)_{i\in I}$ that

$$\mathrm{colim}_{i\in I}(F_i \otimes^p_R G) \cong (\mathrm{colim}_{i\in I} F_i) \otimes^p_R G \ .$$

6.2.3. — For $U \in \mathbf{X}$ let $h_U \in \mathrm{Pr}\mathbf{X}$ denote the presheaf represented by U and $h^R_U \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$ be the presheaf of R-modules generated by h_U. Let $F, G \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$. We define the presheaf

$$\underline{\mathrm{Hom}}^p(F, G) \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$$

by

$$\underline{\mathrm{Hom}}^p(F, G)(U) := \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(h^R_U \otimes^p F, G) \ .$$

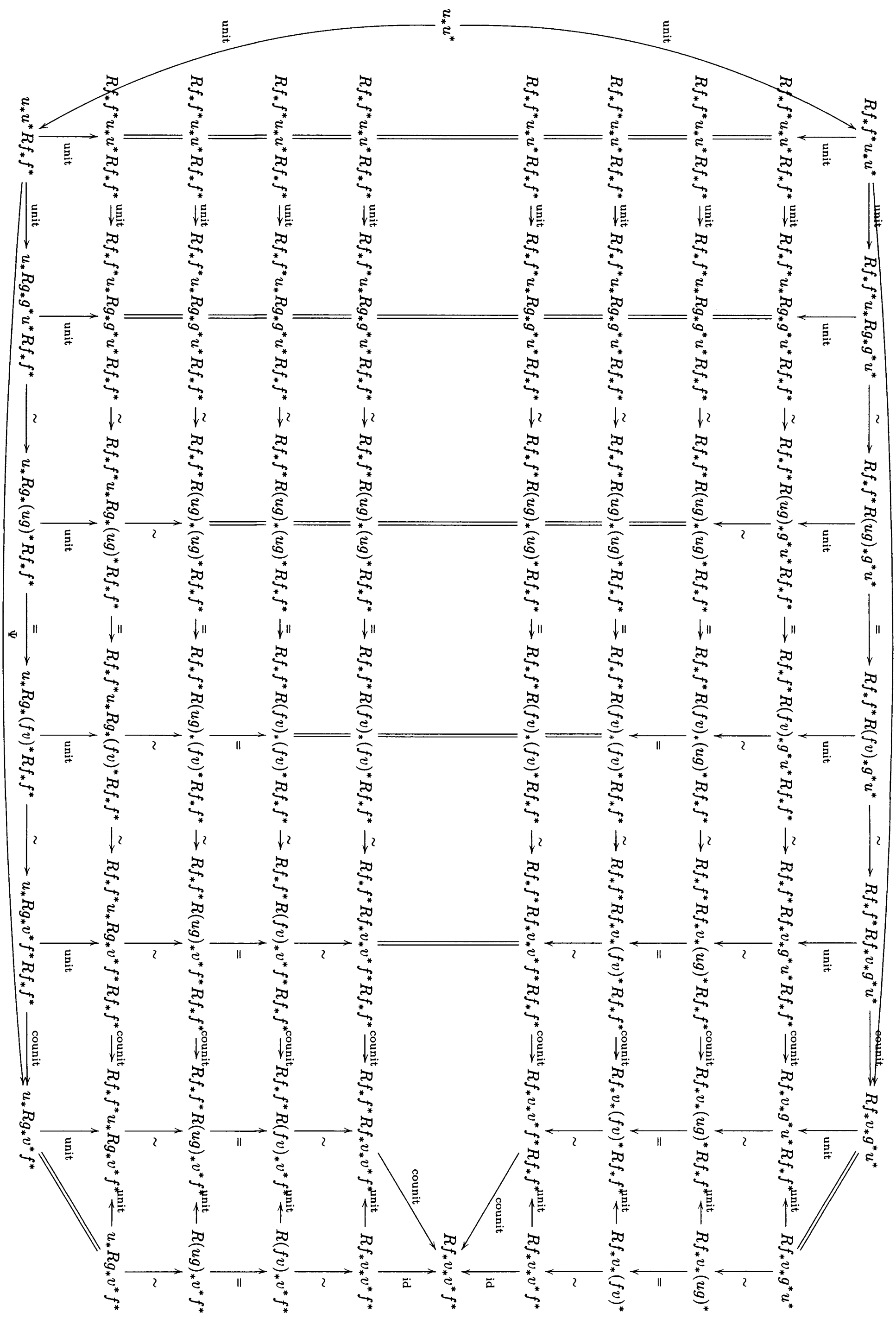


FIGURE 1.

The topology of the site of a locally compact stack is sub-canonical. Hence, in this case h_U is actually a sheaf. But even in the case of a sub-canonical topology h_U^R is only a presheaf, in general.

If $U \to V$ is a morphism in $\mathbf{X}$, then $\underline{\mathrm{Hom}}^p(F,G)(V) \to \underline{\mathrm{Hom}}^p(F,G)(U)$ is induced by the morphism $h_U \to h_V$. If $H \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$, then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(H, \underline{\mathrm{Hom}}^p(F,G)) &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(\mathrm{colim}_{h_V^R \to H} h_V^R, \underline{\mathrm{Hom}}^p(F,G)) \\
&\cong \lim_{h_V^R \to H} \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(h_V^R, \underline{\mathrm{Hom}}^p(F,G)) \\
&\cong \lim_{h_V^R \to H} \underline{\mathrm{Hom}}^p(F,G)(V) \\
&= \lim_{h_V^R \to H} \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(h_V^R \otimes^p F, G) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(\mathrm{colim}_{h_V^R \to H}(h_V^R \otimes^p F), G) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}((\mathrm{colim}_{h_V^R \to H} h_V^R) \otimes^p F, G) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(H \otimes^p F, G)
\end{aligned}$$

In other words, the pair $(\otimes^p, \underline{\mathrm{Hom}}^p)$ together with this natural isomorphism defines a closed symmetric monoidal structure on $\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$. In particular, if $(F_i)_{i\in I}$ is a diagram of presheaves, then we have

(6.2.1) $$\underline{\mathrm{Hom}}^p(\mathrm{colim}_{i\in I} F_i, G) \cong \lim_{i\in I} \underline{\mathrm{Hom}}^p(F_i, G)\ .$$

6.2.4. — An element of

$$\underline{\mathrm{Hom}}(F,G)(U) = \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(h_U^R \otimes^p F, G)$$

is given by a collection of R-linear maps $(\phi_{V\to U} : F(V) \to G(V))_{(V\to U)\in \mathbf{X}/U}$ such that for a morphism $(W \to U) \mapsto (V \to U)$ in $\mathbf{X}/U$ the diagram

$$\begin{array}{ccc}
F(V) & \longrightarrow & F(W) \\
\Big\downarrow {\scriptstyle \phi_{V\to U}} & & \Big\downarrow {\scriptstyle \phi_{W\to U}} \\
G(V) & \longrightarrow & G(W)
\end{array}$$

commutes. Therefore

$$\underline{\mathrm{Hom}}(F,G)(U) \cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}/U}(F_{|U}, G_{|U})\ .$$

Lemma 6.2.2. — *If G is a sheaf, then $\underline{\mathrm{Hom}}(F,G)$ is a sheaf.*

Proof. — Let $U \in \mathbf{X}$ and $(U_i \to U)_{i\in I}$ be a covering. In order to simplify the notation we consider $V := \sqcup_{i\in I} U_i$. We must show that the sequence

$$0 \to \underline{\mathrm{Hom}}(F,G)(U) \to \underline{\mathrm{Hom}}(F,G)(V) \to \underline{\mathrm{Hom}}(F,G)(V \times_U V)$$

is exact.

Let $\psi \in \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}/U}(F_{|U}, G_{|U})$ be such that its restriction to V vanishes. If $(W \to U) \in \mathbf{X}/U$, then $W \times_U V \to W$ is a covering of W, and $\mathrm{pr}_W^* : G(W) \to G(W \times_U V)$ is injective since G is a sheaf. In view of the commutative diagram

$$\begin{array}{ccc} F(W) & \xrightarrow{\mathrm{pr}_W^*} & F(W \times_U V) \\ \downarrow{\scriptstyle \psi_W} & & \downarrow{\scriptstyle (\psi_{|V})_{W \times_U V}} \\ G(W) & \xrightarrow{\mathrm{pr}_W^*} & G(W \times_U V) \end{array}$$

we see that $\psi_W = 0$.

Let now $\phi \in \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}/V}(F_{|V}, G_{|V})$ be such that the induced map

$$\Phi \in \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}/(V \times_U V)}(F_{|V \times_U V}, G_{|V \times_U V})$$

vanishes. We will construct $\psi \in \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}/U}(F_{|U}, G_{|U})$ such that $\psi_{|V} = \phi$. Let $(W \to U) \in \mathbf{X}/U$ and $f \in F_{|U}(W \to U) = F(W)$. Then $W \times_U V \to W$ is a covering of W and $\mathrm{pr}_W^* f \in F_{|V}(W \times_U V \to V) = F(W \times_U V)$. We get an element

$$\phi_{W \times_U V \to V}(\mathrm{pr}_W^*(f)) \in G(W \times_U V) = G_{|V}(W \times_U V \to V) \ .$$

Note that $(W \times_U V) \times_W (W \times_U V) \cong W \times_U (V \times_U V)$. The difference of the pull-backs of $\phi_{W \times_U V \to V}(\mathrm{pr}_W^*(f))$ with respect to the two projections to $W \times_U V$ induces

$$\Phi_{W \times_U (V \times_U V)}(\mathrm{pr}_W^*(f)) = 0 \in G((W \times_U V) \times_W (W \times_U V)) \ .$$

Again, since G is a sheaf there is a unique element $\psi_W(f) \in G(W)$ such that

$$\psi_W(f)_{|W \times_U V} = \phi_{W \times_U V \to V}(\mathrm{pr}_W^*(f)) \ .$$

The morphism ψ is now given by the collection $(\psi_W)_{(W \to U) \in \mathbf{X}/U}$. □

6.2.5. — If $F, G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$, then we define $F \otimes G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ to be

$$F \otimes G := i^\sharp(i(F) \otimes^p i(G)) \ .$$

We furthermore define

$$\underline{\mathrm{Hom}}(F, G) := i^\sharp \underline{\mathrm{Hom}}^p(i(F), i(G)) \ .$$

Using the fact 6.2.2 that $\underline{\mathrm{Hom}}^p(i(F), i(G))$ is a sheaf at the isomorphism marked by ! we get for every $H \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(H \otimes F, G) &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(i^\sharp(i(H) \otimes^p i(F), G) \\ &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i(H) \otimes^p i(F), i(G)) \\ &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i(H), \underline{\mathrm{Hom}}^p(i(F), i(G)) \\ &\stackrel{!}{\cong} \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i(H), i \circ i^\sharp(\underline{\mathrm{Hom}}^p(i(F), i(G)))) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(i^\sharp \circ i(H), \underline{\mathrm{Hom}}(F, G)) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(H, \underline{\mathrm{Hom}}(F, G)) \ . \end{aligned}$$

In other words, the pair $(\otimes, \underline{\mathrm{Hom}})$ together with this natural isomorphism make $\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ into a closed symmetric monoidal category.

6.2.6. — Let $F, G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ and $(U \to X) \in \mathbf{X}$. Then we have

$$(F \otimes G)_U \cong F_U \otimes G_U \ .$$

Indeed, this follows from the fact that sheafification commutes with the restriction from the site $\mathbf{X}$ to the site (U), see 6.1.14. Since the collection of functors $(\nu_U)_{(U \to X) \in \mathbf{X}}$ detects exact sequences it now follows that a sheaf $F \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ is flat if and only if $F_U \in \mathrm{Sh}_{R\text{-Mod}}(U)$ is flat for all $(U \to X) \in \mathbf{X}$. This fact was claimed in 6.1.14.

6.2.7.

Lemma 6.2.3. — *For $F, G \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{X}$ we have $i^\sharp(F \otimes^p G) \cong i^\sharp(F) \otimes i^\sharp(G)$.*

Proof. — This follows from (we omit the functor i at various places in order to simplify the formula)

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(i^\sharp(F \otimes^p G), H) &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(F \otimes^p G, H) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(F, \underline{\mathrm{Hom}}^p(G, H)) \\
&\overset{!}{\cong} \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i^\sharp(F), \underline{\mathrm{Hom}}^p(G, H)) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i^\sharp(F) \otimes^p G, H) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(G, \underline{\mathrm{Hom}}^p(i^\sharp F, H)) \\
&\overset{!}{\cong} \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i^\sharp G, \underline{\mathrm{Hom}}^p(i^\sharp F, H)) \\
&\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}(i^\sharp G \otimes^p i^\sharp F, H) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(i^\sharp G \otimes i^\sharp F, H)
\end{aligned}$$

for arbitrary $H \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$, where we use Lemma 6.2.2 at the isomorphisms marked by !. □

6.2.8. — Let $f\colon X \to Y$ be a morphism of locally compact stacks. Let $\mathbf{X}$ and $\mathbf{Y}$ be the sites associated to X and Y. Consider the adjoint pair of functors

$$^pf^*\colon \mathrm{Pr}_{R\text{-Mod}}\mathbf{Y} \leftrightarrows \mathrm{Pr}_{R\text{-Mod}}\mathbf{X} \colon {}^pf_* \ .$$

The proof of the following Lemma uses the product in $\mathbf{Y}$ described in [**9**, Lemma 3.1] in a specific way.

Lemma 6.2.4. — *For $F, G \in \mathrm{Pr}_{R\text{-Mod}}\mathbf{Y}$ we have a natural isomorphism*

$$^pf^*(F \otimes^p G) \cong {}^pf^*F \otimes^p {}^pf^*G \ .$$

Proof. — We use the notation introduced in [**9**, 2.1.4]. For $(U \to X) \in \mathbf{X}$ we consider the category $U/\mathbf{Y}$ of diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} .$$

The functor ${}^p f^*$ is defined in [**9**, Definition 2.3] as a colimit over this category.

We consider the diagonal functor $U/\mathbf{Y} \to U/\mathbf{Y} \times U/\mathbf{Y}$ given on objects by

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} \quad \mapsto \quad \left(\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} , \begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} \right) .$$

In view of the definition of ${}^p f^*$ by colimits it induces a map

$${}^p f^*(F \otimes^p G) \to {}^p f^* F \otimes^p {}^p f^* G .$$

In the other direction we have the functor $U/\mathbf{Y} \times U/\mathbf{Y} \to U/\mathbf{Y}$ given by

$$\left(\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array} , \begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V' & \longrightarrow & Y \end{array} \right) \quad \mapsto \quad \begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V \times_Y V' & \longrightarrow & Y \end{array} .$$

This together with the projections $V \times_Y V' \to V$ and $V \times_Y V' \to V'$ induces the inverse map

$${}^p f^* F \otimes^p {}^p f^* G \to {}^p f^*(F \otimes^p G) . \qquad \square$$

6.2.9. — Let $f\colon X \to Y$ be a morphism of locally compact stacks.

Lemma 6.2.5. — *For $F, G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$ we have a natural isomorphism*

$$f^*(F \otimes G) \cong f^* F \otimes f^* G .$$

Proof. — For $H \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$, using the fact that ${}^p f_*$ preserves sheaves (see 6.1.9) and Lemma 6.2.3, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*(F \otimes G), H) &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(F \otimes G, f_*(H)) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}}(i^\sharp(i(F) \otimes^p i(G)), i^\sharp \circ f_*^p \circ i(H)) \\ &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{Y}}((i(F) \otimes^p i(G)), f_*^p \circ i(H)) \\ &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}({}^p f^*(i(F) \otimes^p i(G)), i(H)) \\ &\cong \mathrm{Hom}_{\mathrm{Pr}_{R\text{-Mod}}\mathbf{X}}({}^p f^* \circ i(F) \otimes^p {}^p f^* \circ i(G), i(H)) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(i^\sharp({}^p f^* \circ i(F) \otimes^p {}^p f^* \circ i(G)), H) \\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*(F) \otimes f^*(G), H) \end{aligned} \qquad \square$$

6.2.10. — For a derived version of Lemma 6.2.5 we assume that the morphism $f : X \to Y$ of locally compact stacks has local sections. For simplicity we only consider the case $R = \mathbb{Z}$, i.e. sheaves of abelian groups (finite cohomological dimension of R would suffice). Then the exact functor $f^* = (f_\sharp)^*$ preserves torsion-free sheaves of abelian groups. Since the derived tensor product can be calculated using torsion-free resolutions we get the following corollary.

Corollary 6.2.6. — *If $f : X \to Y$ has local sections, then for $F, G \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ we have a natural isomorphism*

$$f^*(F \otimes^L G) \cong f^*F \otimes^L f^*G \ .$$

6.2.11. — Let $f\colon X \to Y$ be a morphism of locally compact stacks.

Lemma 6.2.7. — *For $F \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$ and $G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ we have a natural isomorphism*

$$\underline{\mathrm{Hom}}(F, f_*G) \cong f_*\underline{\mathrm{Hom}}(f^*F, G)$$

in $\mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$

Proof. — For any $T \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$ we calculate

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}}(T, f_*\underline{\mathrm{Hom}}(f^*F, G)) &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*T, \underline{\mathrm{Hom}}(f^*F, G))\\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*T \otimes f^*F, G)\\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*(T \otimes F), G)\\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}}(T \otimes F, f_*G)\\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}}(T, \underline{\mathrm{Hom}}(F, f_*G)) \qquad \square\end{aligned}$$

6.2.12. — Let $f\colon X \to Y$ be a morphism of locally compact stacks.

Lemma 6.2.8. — *For $F \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$ and $G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ we have a natural morphism*

$$f_*G \otimes F \to f_*(G \otimes f^*F) \ .$$

Proof. — The transformation is the image of the identity under the following chain of maps, where the first is induced by the counit $f^* \circ f_* \to \mathrm{id}$ of the adjoint pair (f^*, f_*), and the second isomorphism is given by Lemma 6.2.5.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(G \otimes f^*F, G \otimes f^*F) &\to \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*f_*G \otimes f^*F, G \otimes f^*F)\\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{X}}(f^*(f_*G \otimes F), G \otimes f^*F)\\ &\cong \mathrm{Hom}_{\mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}}(f_*G \otimes F, f_*(G \otimes f^*F)) \ . \qquad \square\end{aligned}$$

Lemma 6.2.9. — *If f has local sections, then for $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$ and $G \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ we have a natural morphism*

$$f_*G \otimes^L F \to f_*(G \otimes^L f^*F) \ .$$

Proof. — We use the same argument as for Lemma 6.2.8 based on the adjoint pair (f^*, Rf_*) and Lemma 6.2.6. □

6.2.13. — Let $f\colon X \to Y$ be a morphism of locally compact stacks.

Lemma 6.2.10. — *Let $F \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$ be a sheaf which is locally isomorphic to $\underline{R}_{\mathbf{Y}}$, i.e. there exist an atlas $a\colon U \to Y$ such that $a^*F \cong \underline{R}_{\mathbf{U}}$. In this case we have the projection formula: For all $G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$ or $H \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ the natural morphisms*

$$f_*G \otimes F \to f_*(G \otimes f^*F)\ , \quad Rf_*H \otimes^L F \to Rf_*(H \otimes^L f^*F)$$

are isomorphisms.

Proof. — This can be checked locally on the atlas $U \to Y$. We consider the pull-back

$$\begin{array}{ccc} V & \xrightarrow{b} & X \\ \big\downarrow{\scriptstyle g} & \nearrow & \big\downarrow{\scriptstyle f} \\ U & \xrightarrow{a} & Y \end{array} .$$

We must check that

$$a^* \circ (f_*G \otimes F) \to a^* \circ f_*(G \otimes f^*F)$$

is an isomorphism. This map is equivalent to

$$\begin{aligned} a^*(f_*G \otimes F) &\cong a^*f_*G \otimes a^*F \\ &\cong a^*f_*G \otimes \underline{R}_U \\ &\cong a^*f_*G \\ &\cong g_*b^*G \\ &\cong g_*b^*(G \otimes \underline{R}_X) \\ &\cong g_*(b^*G \otimes b^*f^*\underline{R}_Y) \\ &\cong g_*(b^*G \otimes g^*a^*\underline{R}_Y) \\ &\cong g_*(b^*G \otimes g^*a^*F) \\ &\cong g_*b^*(G \otimes f^*F) \\ &\cong a^*f_*(G \otimes f^*F)\ . \end{aligned}$$

The derived version is shown in similar manner. □

6.2.14. — We will also need the projection formula with different assumptions. Let $f\colon X \to Y$ be a map of locally compact stacks. We consider $F \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{Y}$ and $G \in \mathrm{Sh}_{R\text{-Mod}}\mathbf{X}$.

Lemma 6.2.11. — *Assume that f is proper and representable, and that F is flat. Then the natural transformation*

$$f_*G \otimes F \to f_*(G \otimes f^*F)$$

of 6.2.8 is an isomorphism.

Proof. — Using the observations 6.1.14 we see that it suffices to show that for all $(U \to Y) \in \mathbf{Y}$ the induced morphism

$$g_*G_V \otimes F_U \to g_*(G_V \otimes g^*F_U) \tag{6.2.12}$$

is an isomorphism. Here $g\colon V \to U$ is the proper map of locally compact spaces defined by the Cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & Y \end{array} .$$

The map (6.2.12) is an isomorphism by [**17**, Prop. 2.5.13]. □

6.2.15. — We also have a derived version of the projection formula in the case of sheaves of abelian groups. The main point is that the ring $\mathbb{Z}$ has finite cohomological dimension (in fact equal to 1). Let $f\colon X \to Y$ be a morphism of locally compact stacks.

Lemma 6.2.13. — *Assume that f is proper and representable. If $G \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ and $F \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$, then we have*

$$Rf_*G \otimes^L F \xrightarrow{\sim} Rf_*(G \otimes^L f^*F)$$

in $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$.

Proof. — As in the proof of Lemma 6.2.11 we can reduce to the small sites (U) for all objects $(U \to Y) \in \mathbf{Y}$. After this reduction we apply [**17**, Prop. 2.6.6] and the fact that the cohomological dimension of $\mathbb{Z}$ is 1, hence finite. □

6.2.16. — The following derived adjunction again uses the finiteness of the cohomological dimension of $\mathbb{Z}$.

Lemma 6.2.14. — *For $F, G, H \in D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ we have a natural isomorphism*

$$R\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(F \otimes^L G, H) \cong R\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(F, R\underline{\mathrm{Hom}}(G,H)) .$$

Proof. — If $G \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is flat and $H \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is injective, then the functor

$$\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \ni F \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(F, \underline{\mathrm{Hom}}(G,H)) \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(F \otimes G, H) \in \mathrm{Ab}$$

is, as a composition of exact functors, exact. It follows that $\underline{\mathrm{Hom}}(G,H)$ is again injective. We now show the Lemma. We can assume that H is a complex of injectives.

Furthermore, since the cohomological dimension of $\mathbb{Z}$ is one, hence in particular finite, we can assume that G is a complex of flat sheaves. Then we have

$$\begin{aligned} R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(F \otimes^L G, H) &\cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(F \otimes G, H) \\ &\cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(F, \underline{\mathrm{Hom}}(G, H)) \\ &\cong R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(F, \underline{\mathrm{Hom}}(G, H)) \ . \end{aligned}$$ □

6.3. Verdier duality for locally compact stacks in detail

6.3.1. — Let $f\colon X \to Y$ be a map of locally compact stacks.

Definition 6.3.1. — *We say that the cohomological dimension of f_* is not greater than $n \in \mathbb{N}$ if the derived functor $R^i f_* : \mathrm{Sh_{Ab}}\mathbf{X} \to \mathrm{Sh_{Ab}}\mathbf{Y}$ vanishes for all $i > n$.*

The main theorem of the present subsection is

Theorem 6.3.2. — *Assume that $f : X \to Y$ is a representable and proper map between locally compact stacks such that f_* has finite cohomological dimension. Then the functor $Rf_*\colon D^+(\mathrm{Sh_{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh_{Ab}}\mathbf{Y})$ admits a right adjoint $f^!\colon D^+(\mathrm{Sh_{Ab}}\mathbf{Y}) \to D^+(\mathrm{Sh_{Ab}}\mathbf{X})$.*

The proof of Theorem 6.3.2 will be finished in 6.3.6. The main idea is to transfer the construction of $f^!$ from [**17**, Section 3.1] to the present situation.

6.3.2. — We consider the functorial flabby resolution (see 3.1.10) of the constant sheaf $\underline{\mathbb{Z}}_{\mathbf{X}} \to \mathcal{Fl}(\underline{\mathbb{Z}}_{\mathbf{X}})$ and form the truncated complex $K := \tau^{\le n}\mathcal{Fl}(\underline{\mathbb{Z}}_{\mathbf{X}})$ so that in particular $K^n = \ker(\mathcal{Fl}^n(\underline{\mathbb{Z}}_{\mathbf{X}}) \to \mathcal{Fl}^{n+1}(\underline{\mathbb{Z}}_{\mathbf{X}}))$.

Lemma 6.3.3. — *Assume that f is representable and that f_* has cohomological dimension not greater than n. Then the complex*

$$0 \to \underline{\mathbb{Z}}_{\mathbf{X}} \to K^0 \to K^1 \to \cdots \to K^n \to 0 \qquad (6.3.4)$$

is a flat and f_*-acyclic resolution of $\underline{\mathbb{Z}}_{\mathbf{X}}$.

Proof. — The sheaf $\ker(K^n \to K^{n+1})$ is a torsion-free subsheaf of a torsion-free sheaf and therefore flat (compare [**17**, Lemma 3.1.4]). By Lemma 3.1.4 the flabby sheaves K^i for $i = 0, \dots, n-1$ are f_*-acyclic. In order to see that K^n is f_*-acyclic, it suffices to show that $R^i f_*(\ker(K^n \to K^{n+1})) \cong 0$ for $i \geq 1$. In fact, we have $R^i f_*(\ker(K^n \to K^{n+1})) \cong R^{i+n} f_* \underline{\mathbb{Z}}_{\mathbf{X}} \cong 0$. □

6.3.3. — The fibers of a representable and proper morphism of topological stacks are compact. This is explicitly used in the proof of the following Lemma.

Lemma 6.3.5. — *If $f : X \to Y$ is a representable and proper morphism of locally compact stacks, then the functor $f_*\colon \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$ preserves direct sums.*

Proof. — Let $(G_i)_{i\in I}$ be a family of sheaves in $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$. Then we have a canonical map

$$\bigoplus_{i\in I} \circ f_*(G_i) \to f_* \circ \bigoplus_{i\in I}(G_i) \ .$$

In order to show that this map is an isomorphism we show that the induced map

$$(\bigoplus_{i\in I} \circ f_*(G_i))_U \to (f_* \circ \bigoplus_{i\in I}(G_i))_U$$

is an isomorphism for all $(U \to Y) \in \mathbf{Y}$. Choose such $(U \to Y)$ and consider the Cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & Y \end{array} \ .$$

It suffices to show that the induced map

$$\bigoplus_{i\in I} \circ g_*(G_i)_U \to g_* \circ \bigoplus_{i\in I}(G_i)_U$$

is an isomorphism. We consider the induced map on the stalk at $x \in U$. Since the restriction to $g^{-1}(x)$ commutes with the sum and $g^{-1}(x)$ is compact it is given by

$$\bigoplus_{i\in I} \circ\Gamma(g^{-1}(x), [(G_i)_U]_{|g^{-1}(x)}) \to \Gamma(g^{-1}(x), \bigoplus_{i\in I}[(G_i)_U]_{|g^{-1}(x)})$$

(see [**17**, Proposition 2.5.2]). But this last map is an isomorphism since the global section functor on sheaves on a compact space commutes with sums. □

6.3.4. — Fix $j \in \{0, 1, 2, \ldots, n\}$ and set $K := K^j$, see 6.3.2

Lemma 6.3.6. — *Let $f : X \to Y$ be a representable, proper morphism of locally compact stacks such that f_* has cohomological dimension not greater than n. Then the functor $G \mapsto f_*(G \otimes K)$ is an exact functor* $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$. *Furthermore, $G \otimes K$ is f_*-acyclic.*

Proof. — In the following proof we freely use the facts listed in 6.1.14. Let $G^{\cdot}$ be an exact complex in $\mathrm{Sh_{Ab}}\mathbf{X}$. For $(U \to Y) \in \mathbf{Y}$ consider the Cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ U & \longrightarrow & Y \end{array} .$$

Note that $(V \to X) \in \mathbf{X}$. By construction (see [**17**, Lemma 3.1.4]) K_V is flat and g-soft. The complex $G^{\cdot}_V$ is exact. By [**17**, Lemma 3.1.2 (ii)] the complex $g_*(G^{\cdot}_V \otimes K_V) = (f_*(G^{\cdot} \otimes K))_U$ is exact. Since this is true for all $(U \to Y) \in \mathbf{Y}$ we conclude that $f_*(G^{\cdot} \otimes K)$ is exact.

We now show that $G \otimes K$ is f_*-acyclic. We must show that $R^i f_*(G \otimes K) \cong 0$ for all $i \geq 1$. For $(U \to Y) \in \mathbf{Y}$ as above we have $(R^i f_*(G \otimes K))_U \cong R^i g_*(G_U \otimes K_U) \cong 0$, since $G_U \otimes K_U$ is g-soft by [**17**, Lemma 3.1.2 (i)] (note that K_U is flasque and flat). Since $(U \to Y)$ was arbitrary this implies that $R^i f_*(G \otimes K) \cong 0$ □

6.3.5. — For $(V \to X) \in \mathbf{X}$ let $\hat{h}^{\mathbb{Z}}_V$ denote the sheafification of the presheaf $h^{\mathbb{Z}}_V$, the presheaf of free abelian groups generated by the sheaf h_V represented by V. We consider the functor $f^!_K \colon \mathrm{Sh_{Ab}}\mathbf{Y} \to \mathrm{Pr_{Ab}}\mathbf{X}$ which associates to a sheaf $F \in \mathrm{Sh_{Ab}}\mathbf{Y}$ the presheaf $f^!_K(F) \in \mathrm{Pr_{Ab}}\mathbf{X}$ given by

$$\mathbf{X} \ni (V \to X) \mapsto f^!_K F(V) := \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(f_*(\hat{h}^{\mathbb{Z}}_V \otimes K), F) \in \mathrm{Ab} .$$

Note that $K \to f^!_K(F)$ is also a functor in K (for fixed F).

Lemma 6.3.7. — *Let K be as in 6.3.4 and $f : X \to Y$ be a representable, proper morphism of locally compact stacks such that f_* has cohomological dimension not greater than n. Assume that $F \in \mathrm{Sh_{Ab}}\mathbf{Y}$ is an injective sheaf. Then $f^!_K(F)$ is an injective sheaf. Furthermore, for $G \in \mathrm{Sh_{Ab}}\mathbf{X}$ there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(f_*(G \otimes K), F) \cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(G, f^!_K(F)) . \tag{6.3.8}$$

Proof. — We show that $f^!_K F$ is a sheaf by copying the corresponding argument in the proof of [**17**, Lemma 3.1.3]. The functor $G \mapsto \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(f_*(G \otimes K), F)$ is exact by Lemma 6.3.6 and injectivity of F. If we establish the isomorphism (6.3.8), then we also have shown that $f^!_K(F)$ is injective.

For $(W \to X) \in \mathbf{X}$ we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(f_*(\hat{h}^{\mathbb{Z}}_W \otimes K), F) = f^!_K(F)(W) \cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(\hat{h}^{\mathbb{Z}}_W, f^!_K(F)) . \tag{6.3.9}$$

For a system $(G_i)_{i\in I}$ of sheaves we have a natural map $\operatorname{colim}_{i\in I}\circ f_*(G_i)\to f_*\circ\operatorname{colim}_{i\in I}(G_i)$. For $G\in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ we get

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(G\otimes K),F) &\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*((\operatorname{colim}_{\hat h^{\mathbb{Z}}_W\to G}\hat h^{\mathbb{Z}}_W)\otimes K),F)\\
&\overset{!}{\cong} \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*\circ\operatorname{colim}_{\hat h^{\mathbb{Z}}_W\to G}(\hat h^{\mathbb{Z}}_W\otimes K),F)\\
&\to \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(\operatorname{colim}_{\hat h^{\mathbb{Z}}_W\to G}\circ f_*(\hat h^{\mathbb{Z}}_W\otimes K),F)\\
&\cong \lim_{\hat h^{\mathbb{Z}}_W\to G}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(\hat h^{\mathbb{Z}}_W\otimes K),F)\\
&\cong \lim_{\hat h^{\mathbb{Z}}_W\to G}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(\hat h^{\mathbb{Z}}_W,f^!_K(F))\\
&\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(\operatorname{colim}_{\hat h^{\mathbb{Z}}_W\to G}\hat h^{\mathbb{Z}}_W,f^!_K(F))\\
&\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(G,f^!_K(F))\ .
\end{aligned}$$

The marked isomorphism uses that the tensor product of sheaves commutes with colimits, a consequence of the fact 6.2.5 that it is part of a closed monoidal structure. It remains to show that this composition is an isomorphism. If we write out the definition of the colimit in $G\cong\operatorname{colim}_{\hat h^{\mathbb{Z}}_W\to G}\hat h^{\mathbb{Z}}_W$ we obtain an exact sequence of the form

$$\bigoplus_{j\in J}\hat h^{\mathbb{Z}}_{W_j}\to\bigoplus_{i\in I}\hat h^{\mathbb{Z}}_{V_i}\to G\to 0\ . \tag{6.3.10}$$

Now observe that for any collection $(G_i)_{i\in I}$ of sheaves in $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ we have

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*((\bigoplus_i G_i)\otimes K),F)\cong\prod_{i\in I}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(G_i\otimes K),F)$$

since f_* (Lemma 6.3.5) and $\cdots\otimes K$ commute with sums. Clearly we also have

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(\bigoplus_i G_i,f^!_K(F))\cong\prod_{i\in I}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(G_i,f^!_K(F))\ .$$

From (6.3.10) we thus get the diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(G\otimes K),F) & \longrightarrow & \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(G,f^!_K(F))\\
\downarrow & & \downarrow\\
\prod_{i\in I}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(\hat h^{\mathbb{Z}}_{V_i}\otimes K),F) & \xrightarrow{\alpha} & \prod_{i\in I}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(\hat h^{\mathbb{Z}}_{V_i},f^!_K(F))\\
\downarrow & & \downarrow\\
\prod_{j\in J}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(\hat h^{\mathbb{Z}}_{W_j}\otimes K),F) & \xrightarrow{\beta} & \prod_{j\in J}\mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(\hat h^{\mathbb{Z}}_{W_j},f^!_K(F)).
\end{array}$$

Because of the isomorphism (6.3.9) the maps α and β are isomorphisms. The left vertical sequence is exact by Lemma 6.3.6. The right vertical sequence is exact by the left-exactness of the Hom-functor. It follows from the 5-lemma that (6.3.8) is an isomorphism. □

6.3.6. — Let $I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \subset \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ denote the full subcategory of injective objects and $K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ be the category of complexes in $I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ which are bounded below, and whose morphisms are homotopy classes of chain maps. Then we have an equivalence of triangulated categories

$$K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \cong D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) .$$

Let $f : X \to Y$ be a representable, proper morphism of locally compact stacks such that f_* has cohomological dimension not greater than n, and let $K^\cdot$ be as in 6.3.2. We then define the functor $f^!: K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \to K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ by

$$f^!(F^\cdot) = (f^!_{K^\cdot}(F^\cdot))_{tot} ,$$

where $E^{\cdot,\cdot}_{tot}$ denotes the total complex of the double complex $E^{\cdot,\cdot}$. Since $f^!_K$ preserves injective sheaves by Lemma 6.3.7 this functor is well-defined. Furthermore, for $F \in K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ and $G \in K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ we have by Lemma 6.3.7 a natural isomorphism between spaces of chain maps

$$\mathrm{Hom}_{C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})}(f_*(G^\cdot \otimes K^\cdot)_{tot}, F^\cdot) \cong \mathrm{Hom}_{C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})}(G^\cdot, f^!(F^\cdot))$$

which descends to an isomorphism on the level of homotopy classes

$$\mathrm{Hom}_{K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})}(f_*(G^\cdot \otimes K^\cdot)_{tot}, F^\cdot) \cong \mathrm{Hom}_{K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})}(G^\cdot, f^!(F^\cdot)) .$$

Since $f^!(F^\cdot)$ is a complex of injective sheaves we have

$$\mathrm{Hom}_{K^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})}(G^\cdot, f^!(F^\cdot)) \cong \mathrm{Hom}_{D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})}(G^\cdot, f^!(F^\cdot)) .$$

Note that $G^\cdot \cong G^\cdot \otimes \underline{\mathbb{Z}}_{\mathbf{X}} \to (G^\cdot \otimes K^\cdot)_{tot}$ is a quasi-isomorphism, and the complex $G^\cdot \otimes K^\cdot$ consists of f_*-acyclic sheaves by Lemma 6.3.6. Therefore $f_*(G^\cdot \otimes K^\cdot)_{tot} \cong Rf_*(G^\cdot)$. Since $F^\cdot$ is injective we have

$$\mathrm{Hom}_{K^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})}(f_*(G^\cdot \otimes K^\cdot)_{tot}, F^\cdot) \cong \mathrm{Hom}_{D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})}(Rf_*(G^\cdot), F^\cdot) .$$

We conclude that

$$\mathrm{Hom}_{D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})}(Rf_*(G^\cdot), F^\cdot) \cong \mathrm{Hom}_{D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})}(G^\cdot, f^!(F^\cdot)) .$$

This finishes the proof of Theorem 6.3.2. □

6.3.7. — We consider morphisms $f\colon X \to Y$ and $u : U \to Y$ of locally compact stacks and form the Cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{u} & Y \end{array} .$$

Lemma 6.3.11. — *Assume the f is representable, proper and that f_* has finite cohomological dimension. Assume furthermore that u has local sections. Then we have a natural transformation $v^* \circ f^! \to g^! \circ u^*$.*

Proof. — First note that g is representable, proper and that g_* has finite cohomological dimension. Furthermore, v has local sections. We apply $f^!$ to the unit $\mathrm{id} \to Ru_* \circ u^*$ and obtain a morphism

$$f^! \to f^! \circ Ru_* \circ u^* . \tag{6.3.12}$$

Since f is representable and u has local sections we have the isomorphism (see Lemma 6.1.12 or [**9**, Lemma 2.43])

$$u^* \circ Rf_* \cong Rg_* \circ v^* .$$

Taking its right adjoint yields the isomorphism

$$f^! \circ Ru_* \cong Rv_* \circ g^! .$$

We plug this into (6.3.12) and obtain a transformation

$$f^! \to Rv_* \circ g^! \circ u^* .$$

Its adjoint is the desired transformation □

6.3.8. — The following adjunction is a consequence of the derived projection formula Lemma 6.2.13 and the derived adjunction Lemma 6.2.14

Lemma 6.3.13. — *If $f : X \to Y$ is a representable proper morphism of locally compact stacks which has local sections and is such that f_* has finite cohomological dimension, then for $G \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ and $F \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ we have a natural isomorphism*

$$Rf_* R\underline{\mathrm{Hom}}(G, f^! F) \cong R\underline{\mathrm{Hom}}(Rf_* G, F) .$$

Proof. — Let $H \in D^+(\mathrm{Sh_{Ab}}\mathbf{X})$ be arbitrary. Then we calculate using Lemma 6.1.8 and Lemma 6.2.13 that

$$\begin{aligned} R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(H, Rf_*R\underline{\mathrm{Hom}}(G, f^!F)) &\cong R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(f^*H, R\underline{\mathrm{Hom}}(G, f^!F)) \\ &\cong R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{X}}(f^*H \otimes^L G, f^!F) \\ &\cong R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(Rf_*(f^*H \otimes^L G), F) \\ &\cong R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(H \otimes^L Rf_*G, F) \\ &\cong R\mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{Y}}(H, R\underline{\mathrm{Hom}}(Rf_*G, F)) . \qquad \square \end{aligned}$$

6.3.9.

Definition 6.3.14. — *If $f : X \to Y$ is a proper morphism of locally compact stacks such that f_* has finite cohomological dimension, then we define the relative dualizing complex by*

$$\omega_{X/Y} := f^!(\mathbb{Z}_{\mathbf{Y}}) .$$

It would be interesting to know the structure of $\omega_{X/Y}$ for a topological submersion f in the sense of [**17**, Def. 3.3.1].

6.3.10. — In a different setup of Artin stacks and the lisse-étale site in [**18**] a six functor calculus was constructed. Starting with the observation that dualizing sheaves on the small sites are sufficiently functorial the functors $Rf_!$ and $f^!$ are constructed on constructible sheaves by duality. In this approach one can relate the global $f^!$ with the local versions without any difficulty.

A similar approach may work in the present topological context as well, but it is not clear how the resulting $f^!$ will relate to the construction in the present paper.

6.4. The integration map

6.4.1. — Let M be a closed connected orientable n-dimensional topological manifold.

Definition 6.4.1. — *A map between locally compact stacks $f : X \to Y$ is a locally trivial fiber bundle with fiber M if for every space $U \to X$ the pull-back $U \times_Y X \to U$ is a locally trivial fiber bundle of spaces with fiber M.*

Note that a locally trivial fiber bundle f with fiber M is representable, proper and has local sections, and f_* has finite cohomological dimension. In order to see the last fact and to calculate $R^nf_*(\mathbb{Z}_{\mathbf{X}})$ we consider $(U \to Y) \in \mathbf{Y}$ and let $V \to U$ be surjective and locally an open embedding such that we have a diagram with Cartesian squares

(6.4.2)
$$\begin{array}{ccccccc} M & \longleftarrow & V \times_Y X & \longrightarrow & U \times_Y X & \longrightarrow & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ * & \underset{p}{\longleftarrow} & V & \longrightarrow & U & \overset{u}{\longrightarrow} & Y \end{array} .$$

The map g is a topological submersion in the sense of [**17**, Def. 3.3.1]. As remarked in [**17**, Sec. 3.3] the cohomological dimension of g_* is not greater than n. This implies $(R^i f_* F)_U \cong R^i g_*(F_{U \times_Y X}) = 0$ for all $i > n$. Since this holds true for all $(U \to Y) \in \mathbf{Y}$ we conclude that $R^i f_* F = 0$ for all $i > n$.

We use the left part of the diagram (6.4.2) in order to see that $R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}})$ is locally isomorphic to $\underline{\mathbb{Z}}_{\mathbf{Y}}$. In fact, we have

$$Rf_*(\underline{\mathbb{Z}}_{\mathbf{X}})_V \cong Rh_* \underline{\mathbb{Z}}_{(V \times_Y X)} \cong p^* Rq_* \underline{\mathbb{Z}}_{(M)} .$$

A choice of an orientation of M gives an isomorphism $R^n q_* \underline{\mathbb{Z}}_{(M)} \cong \underline{\mathbb{Z}}_{(*)}$ and therefore $R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}})_V \cong p^* \underline{\mathbb{Z}}_{(*)} \cong \underline{\mathbb{Z}}_{(V)}$.

Definition 6.4.3. — *A locally trivial fiber bundle $f \colon X \to Y$ with fiber M is called orientable if there exists an isomorphism $R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}) \cong \underline{\mathbb{Z}}_{\mathbf{Y}}$. An orientation of f is a choice of such an isomorphism.*

6.4.2. — Let $f : X \to Y$ be a locally trivial fiber bundle with fiber M, where M is a compact closed n-dimensional topological manifold. We consider the f_*-acyclic and flat resolution K defined in (6.3.4).

Corollary 6.4.4. — *The functor $Rf_* \colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ is represented by $f_* \circ T_K$, where T_K is tensor product with the complex K.*

We now define a natural transformation

$$R\underline{\mathrm{Hom}}(R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}), F) \to Rf_* \circ f^!(F) .$$

Let $F \in C^+(I\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ be a complex of injectives. We start from the observation that

$$R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}) \cong f_*(K^n)/\mathrm{im}(f_*(K^{n-1}) \to f_*(K^n)) .$$

For $(U \to Y) \in \mathbf{Y}$ we thus obtain a chain of maps of complexes

$$\begin{aligned}
\underline{\mathrm{Hom}}(R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}}, F)(U) &\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(\hat{h}_U^{\mathbb{Z}}, \underline{\mathrm{Hom}}(R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}}, F)) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(\hat{h}_U^{\mathbb{Z}} \otimes R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}), F) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(\hat{h}_U^{\mathbb{Z}} \otimes f_*(K^n)/\mathrm{im}(f_*(K^{n-1}) \to f_*(K^n)), F) \\
&\stackrel{!}{\to} \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(\hat{h}_U^{\mathbb{Z}} \otimes f_*(K), F) \\
&\stackrel{6.2.11}{\cong} \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}}(f_*(f^*\hat{h}_U^{\mathbb{Z}} \otimes K), F) \\
&\stackrel{6.3.7}{\cong} \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(f^*\hat{h}_U^{\mathbb{Z}}, f_K^!(F)) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}}(\hat{h}_U^{\mathbb{Z}}, f_* \circ f_K^!(F)) \\
&\cong f_* \circ f_K^!(F)(U)\ ,
\end{aligned}$$

where the map marked by ! has degree n. The projection formula Lemma 6.2.11 can be applied since $f^*\hat{h}_U^{\mathbb{Z}}$ is flat. This transformation preserves homotopy classes of morphisms $F \to F'$. Since F is injective we have

$$\underline{\mathrm{Hom}}(R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}}, F) \cong R\underline{\mathrm{Hom}}(R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}}, F)\ .$$

Further note that $f_K^!(F)$ is still a complex of injectives by Lemma 6.3.7. Therefore $f_* \circ f_K^!(F) \cong Rf_* \circ f^!(F)$. Hence this chain of maps of complexes induces a transformation

$$R\underline{\mathrm{Hom}}(R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}}, F) \to Rf_* \circ f^!(F)\ . \tag{6.4.5}$$

6.4.3. — Its adjoint is a natural transformation

$$Rf_* f^* R\underline{\mathrm{Hom}}(R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}}, F) \to F\ .$$

Let us now assume that $f : X \to Y$ is in addition oriented by an isomorphism $R^n f_* \underline{\mathbb{Z}}_{\mathbf{X}} \cong \underline{\mathbb{Z}}_{\mathbf{Y}}$. We precompose with this isomorphism and get the integration map.

Definition 6.4.6. — *The integration map*

$$\int_f : Rf_* \circ f^* \to \mathrm{id}$$

is the natural transformation of functors $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ *of degree* $-n$ *defined as the composition*

$$Rf_* f^*(F) \cong Rf_* f^*(\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}_{\mathbf{Y}}, F)) \cong Rf_* f^*(\underline{\mathrm{Hom}}(R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}), F)) \to F\ .$$

In Lemmas 6.5.20 and 6.5.31 we will verify in the more general case of unbounded derived categories that the integration map is functorial with respect to compositions and compatible with pull-back diagrams.

6.5. Operations with unbounded derived categories

6.5.1. — The category of sheaves $\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ on a locally compact stack is a Grothendieck abelian category (see 3.3.1). The category of complexes in a Grothendieck abelian category carries a model category structure (see 3.3.2). The unbounded derived category is the associated homotopy category. The goal of the present subsection is to extend the sheaf theory operations (f^*, f_*) and the integration map $\int_f$ to the unbounded derived category.

Many results of the present subsection would continue to hold if one drops the assumption of local compactness in the definition of the site associated to stacks as well as for the stacks themselves. But the assumption of local compactness is important for the integration map since it uses versions of the projection formula.

6.5.2. — Let $f\colon X \to Y$ be a morphism between locally compact stacks. Then we have an adjoint pair of functors

$$f^*\colon C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \leftrightarrows C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) : f_* \ .$$

In order to descend the functor f_* to the bounded below derived category it was sufficient to know that f_* is left exact. In this case the idea is to apply f_* to injective resolutions. The descent of the other functor f^* is usually only considered if it is exact, but see e.g. [**23**] for more general constructions. We know by 6.1.11 that the functor f^* is exact if f has local sections.

It is not possible to show using the left exactness that f_* preserves quasi-isomorphisms between unbounded complexes of injectives. Even worse, it is not clear how to resolve an unbounded complex by an injective complex. The method to descend f_* to the derived category uses abstract homotopy theory and works under the additional assumption that f has local sections.

Recall that we use a model structure on the category $C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ of unbounded complexes of sheaves for which the equivalences are the quasi-isomorphisms, and the cofibrations are the level-wise injections (see 3.3.2). The inclusion $C^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \hookrightarrow C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ of the full subcategory of bounded below complexes induces an identification $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \cong hC^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \hookrightarrow hC(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) =: D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ of the bounded below derived category as a full subcategory of the unbounded derived category.

The functor $Rf_* : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ is the adjoint of the restriction of f^* to the bounded below derived categories, and it is therefore the restriction of $Rf_* : D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ to be defined below.

Lemma 6.5.1. — *If the morphism $f\colon X \to Y$ of locally compact stacks has local sections, then (f^*, f_*) is a Quillen adjoint pair.*

Proof. — We use the criterion [**15**, Def. 1.3.1 (2)] in order to show that f^* is a left Quillen functor. We must show that it preserves cofibrations and trivial cofibrations. In other words, we must show that f^* preserves injections and injections which induce isomorphisms on cohomology. Both properties follow from the exactness of $f^*\colon \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$. □

6.5.3. — Let $f\colon X \to Y$ be a map between locally compact stacks which has local sections. Since (f^*, f_*) is a Quillen adjoint pair it induces a derived adjoint pair

$$Lf^*\colon hC(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \leftrightarrows hC(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) : Rf_*$$

(see Lemma [**15**, Lemma 1.3.10]). Since f^* is exact it directly descends to the homotopy category.

6.5.4. — Let $g\colon Y \to Z$ be a second map of locally compact stacks which admits local sections. Then we have adjoint canonical isomorphisms

$$(g \circ f)^* \cong f^* \circ g^* \ , \quad (g \circ f)_* \cong g_* \circ f_* \ . \tag{6.5.2}$$

Lemma 6.5.3. — *We have a canonical isomorphism*

$$R(g \circ f)_* \cong Rg_* \circ Rf_* \ .$$

Proof. — Using [**15**, Thm. 1.3.7] we have a natural transformation

$$R(g \circ f)_* \cong R(g_* \circ f_*) \to Rg_* \circ Rf_* \tag{6.5.4}$$

which is adjoint to

$$Lf^* \circ Lg^* \to L(f^* \circ g^*) \cong L(g \circ f)^* \ . \tag{6.5.5}$$

Since Lf^*, Lg^*, and $L(g\circ f)^*$ are plain descents of f^*, g^*, and $(g\circ f)^*$ to the homotopy category it follows from (6.5.2) that (6.5.5) is an isomorphism. Therefore its adjoint (6.5.4) is also an isomorphism. □

6.5.5. — Consider a Cartesian diagram of locally compact stacks

$$\begin{array}{ccc} U & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{u} & Y \end{array} \ ,$$

where all maps have local sections. Using the unit $\mathrm{id} \to v_* \circ v^*$, the counit $u^* \circ u_* \to \mathrm{id}$, and (6.5.2) we define (see Lemma 6.1.12) a transformation

$$u^* \circ f_* \to u^* \circ f_* \circ v_* \circ v^* \cong u^* \circ u_* \circ g_* \circ v^* \to g_* \circ v^* \ .$$

It is functorial with respect to compositions of such Cartesian diagrams. By the same method we obtain a transformation

$$Lu^* \circ Rf_* \to Rg_* \circ Lv^* \ . \tag{6.5.6}$$

6.5.6. — By Lemma 6.1.12 we know that the transformation

$$u^* \circ f_* \to g_* \circ v^*$$

is in fact an isomorphism. The derived version is more complicated and needs an additional assumption.

Lemma 6.5.7. — *Assume that g is representable and $g_*\colon \mathrm{Sh_{Ab}}\mathbf{U} \to \mathrm{Sh_{Ab}}\mathbf{V}$ has finite cohomological dimension. Then the transformation* (6.5.6) *is an isomorphism.*

Proof. — We choose fibrant replacement functors

$$I_X\colon C(\mathrm{Sh_{Ab}}\mathbf{X}) \to C(\mathrm{Sh_{Ab}}\mathbf{X}) \ , \quad I_U : C(\mathrm{Sh_{Ab}}\mathbf{U}) \to C(\mathrm{Sh_{Ab}}\mathbf{U}) \ .$$

In terms of these replacement functors we can write the compositions of derived functors as descents of quasi-isomorphism preserving functors on the level of chain complexes:

$$Lu^* \circ Rf_* \cong u^* \circ f_* \circ I_X \ , \quad Rg_* \circ Lv^* \cong g_* \circ I_U \circ v^* \ .$$

Let $F \in C(\mathrm{Sh_{Ab}}\mathbf{X})$. We must show that the marked arrows (induced by $\mathrm{id} \to I_U$ and $\mathrm{id} \to I_X$) in the following sequence are quasi-isomorphisms

$$u^* f_* I_X(F) \cong g_* v^* I_X(F) \stackrel{(*)}{\to} g_* I_U v^* I_X(F) \stackrel{(**)}{\leftarrow} g_* I_U v^*(F) \ .$$

The arrow marked by $(**)$ is a quasi-isomorphism since the functors $g_* I_U$ and v^* preserve quasi-isomorphisms, and $F \to I_X(F)$ is a quasi-isomorphism.

The morphism $(*)$ is more complicated, and it is here where we need the assumption. It is a property of the injective model structure on the chain complexes of a Grothendieck abelian category that a fibrant complex consists of injective objects. An injective sheaf is in particular flabby. Since v has local sections v^* preserves flabby sheaves (Lemma 3.1.5). We conclude that $v^* I_X(F)$ is a complex of flabby sheaves.

Let $G \in C(\mathrm{Sh_{Ab}}\mathbf{U})$ be a complex of flabby sheaves. We must show that $g_*(G) \to g_* I_U(G)$ is a quasi-isomorphism. Since g_* is an additive functor this assertion is equivalent to the assertion that $g_*(C)$ is exact, where C is the mapping cone of $G \to I_U(G)$. Note that C is an exact complex of flabby sheaves. It decomposes into short exact sequences

$$0 \to Z^n \to C^n \to Z^{n+1} \to 0 \ ,$$

where $Z^n := \ker(C^n \to C^{n+1})$. Since g is representable we know by Lemma 3.1.4 that flabby sheaves are g_*-acyclic. Therefore we obtain the exact sequence

$$0 \to g_*(Z^n) \to g_*(C^n) \to g_*(Z^{n+1}) \to R^1 g_*(Z^n) \to 0$$

and the isomorphisms

$$R^k g_*(Z^{n+1}) \cong R^{k+1} g_*(Z^n)$$

for all $k \geq 1$. By induction we show that for $k \geq 1$ and all $l \in \mathbb{N}$ we have

$$R^k g_*(Z^n) \cong R^{k+l} g_*(Z^{n-l}) .$$

Since we assume that g_* has bounded cohomological dimension we conclude that $R^k(Z^n) \cong 0$ for all $n \in \mathbb{Z}$ and $k \geq 1$. In particular the sequences

$$0 \to g_*(Z^n) \to g_*(C^n) \to g_*(Z^{n+1}) \to 0$$

are exact for all $n \in \mathbb{Z}$. This shows the exactness of $g_*(C)$. □

6.5.7. — Let now $f\colon X \to Y$ be a representable map between locally compact stacks which is an oriented locally trivial fiber bundle of closed oriented manifolds of dimension n. In particular, f has local sections and is proper, and f_* has cohomological dimension $\leq n$. We consider the canonical flabby resolution (see 3.1.10)

$$0 \to \underline{\mathbb{Z}}_{\mathbf{X}} \to \mathscr{Fl}^0(\underline{\mathbb{Z}}_{\mathbf{X}}) \to \mathscr{Fl}^1(\underline{\mathbb{Z}}_{\mathbf{X}}) \to \cdots .$$

Then we know that $f_*\mathscr{Fl}(\underline{\mathbb{Z}}_{\mathbf{X}})$ is exact above degree n. We let K denote the truncation (6.3.4) of this resolution at n. Then the orientation of the bundle (see 6.4.3) gives the last isomorphism in the following composition

$$f_*K^n \to f_*K^n/\mathrm{im}(f_*K^{n-1} \to f_*K^n) \cong R^n f_*\underline{\mathbb{Z}}_{\mathbf{X}} \cong \underline{\mathbb{Z}}_{\mathbf{Y}} .$$

We let $T_K\colon C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ denote the functor which associates to the complex F the total complex $T_K(F)$ of $F \otimes K$. The projection formula Lemma 6.2.11 for the proper representable map f gives an isomorphism

$$f_* \circ T_K \circ f^*(F) \cong T_{f_*K}(F)$$

for complexes of flat sheaves $F \in C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$. The inclusion $\underline{\mathbb{Z}}_{\mathbf{X}} \to K$ and the projection $f_*K \to \underline{\mathbb{Z}}_{\mathbf{Y}}[-n]$ induce transformations

$$\mathrm{id} \to T_K \ , \quad T_{f_*K} \to \mathrm{id}[-n] . \tag{6.5.8}$$

6.5.8. — We know by Lemma 6.3.6 that the functor

$$f_* \circ T_K\colon \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$$

is exact. We choose a functorial fibrant replacement functor $\mathrm{id} \to I$ on $C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$. Let $R : C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}) \to C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y})$ be the functorial flat resolution functor of 3.4.1, extended to unbounded complexes. Then we consider the sequence

(6.5.9)

$$f_*\circ I\circ f^* \to f_*\circ T_K\circ I\circ f^* \overset{!}{\leftarrow} f_*\circ T_K\circ f^* \overset{!}{\leftarrow} f_*\circ T_K\circ f^*\circ R \cong T_{f_*K}\circ R \to R[-n] \to \mathrm{id}[-n] .$$

All functors in this sequence preserve quasi-isomorphisms and therefore descend plainly to the homotopy category $hC(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$. Since $f_* \circ T_K$ is exact the arrows marked by ! induce isomorphisms of functors on the homotopy category. Now observe that the descent of $f_* \circ I \circ f^*$ to the homotopy category is isomorphic to $Rf_* \circ Lf^*$. Therefore (6.5.9) induces a transformation

$$\int_f : Rf_* \circ Lf^* \to \mathrm{id}[-n] \ . \tag{6.5.10}$$

Definition 6.5.11. — *The transformation* (6.5.10) *is called the integration map.*

It generalizes Definition 6.4.6 from the bounded below to the unbounded derived category.

6.5.9. — In order to have a simple definition we have defined the integration map using a canonical resolution of $\underline{\mathbb{Z}}_{\mathbf{X}}$ of length n. In fact, we can use more general resolutions. This will turn out to be useful for the verification of functorial properties of the integration map.

6.5.10. — Let us first recall some notation. An object $(U \to X) \in \mathbf{X}$ represents the presheaf $h_U \in \mathrm{Pr}\mathbf{X}$ (see also 6.2.3). We let $h_U^{\mathbb{Z}} \in \mathrm{Pr}_{\mathrm{Ab}}\mathbf{X}$ be the free abelian presheaf generated by h_U and form $\hat{h}_U^{\mathbb{Z}} := i^\sharp h_U^{\mathbb{Z}} \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$.

Definition 6.5.12. — *Let $f : X \to Y$ be a map of locally compact stacks. A sheaf $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is called locally f_*-acyclic, if for every $(U \to X) \in \mathbf{X}$ and $k \geq 1$ we have $R^k f_*(\hat{h}_U^{\mathbb{Z}} \otimes F) \cong 0$.*

6.5.11. — Let $f : X \to Y$ be a map of locally compact stacks.

Lemma 6.5.13. — *Assume that the cohomological dimension of f_* is bounded by n. If*

$$L^0 \to L^1 \to \cdots \to L^{n-1} \to L^n \to 0$$

is an exact complex such that the L^i are f_*-acyclic (or locally f_*-acyclic) for $i = 0, \ldots, n-1$, then L^n is f_*-acyclic (or locally f_*-acyclic, respectively).

This can be shown by a similar induction argument as used in the proof of Lemma 6.5.7. □

6.5.12. — Let $f : X \to Y$ be a map of locally compact stacks.

Lemma 6.5.14. — *Let $(V \to X) \in \mathbf{X}$ and F be locally f_*-acyclic. Then $\hat{h}_V^{\mathbb{Z}} \otimes F$ is locally f_*-acyclic.*

Proof. — Indeed, let $(U \to X) \in \mathbf{X}$. Then we have

$$\hat{h}_U^{\mathbb{Z}} \otimes (\hat{h}_V^{\mathbb{Z}} \otimes F) \cong (\hat{h}_U^{\mathbb{Z}} \otimes \hat{h}_V^{\mathbb{Z}}) \otimes F \ .$$

Furthermore we have

$$\hat{h}_U^{\mathbb{Z}} \otimes \hat{h}_V^{\mathbb{Z}} \overset{\text{Lemma 6.2.3}}{\cong} i^\sharp (h_U^{\mathbb{Z}} \otimes^p h_V^{\mathbb{Z}}) \cong i^\sharp (h_U \times h_V)^{\mathbb{Z}} \cong i^\sharp h_{U \times_{\mathbf{X}} V}^{\mathbb{Z}} \cong \hat{h}_{U \times_X V}^{\mathbb{Z}} \ ,$$

where we use the fact, that the absolute product in $\mathbf{X}$ is given by the fiber product spaces over X ([**9**, Lemma 2.3.3]). It follows that

$$R^k f_*(\hat{h}^{\mathbb{Z}}_U \otimes (\hat{h}^{\mathbb{Z}}_V \otimes F)) \cong R^k f_*(\hat{h}^{\mathbb{Z}}_{U \times_X V} \otimes F) \cong 0$$

for all $k \geq 1$. □

6.5.13. — Let $f : X \to Y$ be a map of locally compact stacks.

Lemma 6.5.15. — *Assume that f is proper, representable, and that the cohomological dimension of f_* is finite. If $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ is flat and locally f_*-acyclic, then for any sheaf $G \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ the tensor product $G \otimes F$ is f_*-acyclic and locally f_*-acyclic).*

Proof. — We construct a resolution $\dots \to G_j \to G_{j-1} \to \dots \to G_0 \to G$, where all G_j are coproducts of sheaves of the form $\hat{h}^{\mathbb{Z}}_U$. In fact, we have a surjection

$$\bigoplus_{\hat{h}^{\mathbb{Z}}_U \to G} \hat{h}^{\mathbb{Z}}_U \to G\ .$$

If we have already constructed $G_j \to \dots \to G_0 \to G$, then we extend this complex by

$$\bigoplus_{\hat{h}^{\mathbb{Z}}_U \to \ker(G_j \to G_{j-1})} \hat{h}^U_{\mathbb{Z}} \to G_j\ .$$

Since F is flat, the complex

$$\dots \to G_j \otimes F \to \dots \to G_0 \otimes F \to G \otimes F$$

is exact. The tensor product commutes with direct sums. Therefore $G_j \otimes F$ is a sum of f_*-acyclic sheaves, and by Lemma 6.5.14 also of locally f_*-acyclic sheaves. Since f_* commutes with direct sums (Lemma 6.3.5) the sheaves $G_j \otimes F$ are themselves f_*-acyclic and locally f_*-acyclic. With Lemma 6.5.13 we conclude that $G \otimes F$ is f_*-acyclic and locally f_*-acyclic. □

6.5.14. — Let $f : X \to Y$ be a map of locally compact stack.

Lemma 6.5.16. — *If f is representable, then a flasque sheaf is locally f_*-acyclic.*

Proof. — Let $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ be flasque. We consider $(U \to Y) \in \mathbf{Y}$ and form the Cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & Y \end{array}\ .$$

Then $(V \to X) \in \mathbf{X}$ and we have $Rf_*(F)_U \cong Rg_*(F_V)$. The restriction $F_V \in \mathrm{Sh}_{\mathrm{Ab}}(V)$ is still flasque. A flasque sheaf on (V) is g-soft (see [**17**, Definition 3.1.1]). But this implies that $R^k g_*(F_V) = 0$ for $k \geq 1$. Since $U \to Y$ was arbitrary we see that $R^k f_*(F) = 0$ for $k \geq 1$. □

6.5.15. — Let us from now on until the end of this subsection assume that $f : X \to Y$ is a proper representable map of locally compact stacks which is an oriented locally trivial fiber bundle with fiber a closed connected topological manifold of dimension n.

Since a flat and flasque sheaf is locally f_*-acyclic and K is a truncation of a flat and flasque resolution of $\mathbb{Z}_{\mathbf{X}}$ we see by Lemma 6.5.13 that K is a complex of flat and locally f_*-acyclic sheaves. These are the two properties which make the theory work.

Let $L \to M$ be a quasi-isomorphism between upper bounded complexes of locally f_*-acyclic and flat sheaves.

Lemma 6.5.17. — *For every complex $F \in C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ the induced map*

$$f_*(F \otimes L) \to f_*(F \otimes M)$$

is a quasi-isomorphism.

Proof. — We form the mapping cone C of $L \to M$. It is an exact complex of locally f_*-acyclic and flat sheaves. Since the tensor product and g_* commute with the formation of a mapping cone it suffices to show that $f_*(F \otimes C)$ is exact.

We know by Lemma 6.5.15 that $F \otimes C$ is a complex of f_*-acyclic sheaves. We claim that $F \otimes C$ is exact.

To this end we first show that $H \otimes C$ is exact for an arbitrary sheaf $H \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$. We decompose the exact complex C into short exact sequences

$$S(k) \colon 0 \to Z^k \to C^k \to Z^{k+1} \to 0$$

where $Z^k := \ker(C^k \to C^{k+1})$. Using the fact that the sheaves C^k are flat we obtain the exact sequence

$$0 \to \mathrm{Tor}_1(H, Z^{k+1}) \to H \otimes Z^k \to H \otimes C^k \to H \otimes Z^{k+1} \to 0$$

and the isomorphisms $\mathrm{Tor}_{m+1}(H, Z^{k+1}) \cong \mathrm{Tor}_m(H, Z^k)$ for all $m \geq 1$. Since $\mathbb{Z}$ is one-dimensional we know that $\mathrm{Tor}_m \cong 0$ for $m \geq 2$. Inductively we conclude that $\mathrm{Tor}_1(H, Z^k) \cong 0$ for all $k \in \mathbb{Z}$. It follows that $H \otimes S(k)$ is exact for all $k \in \mathbb{Z}$. This implies that $H \otimes C$ is exact.

Let now F be a complex. Using the previous result and a spectral sequence argument we conclude that the total complex associated to the double complex $F \otimes C$ is exact.

This finishes the proof of the claim.

Let now $C \in C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ be an exact complex of f_*-acyclic sheaves. We show that this implies that $f_*(C)$ is exact. The complex C decomposes into short exact sequences

$$0 \to Z^n \to C^n \to Z^{n+1} \to 0 \ ,$$

where $Z^n := \ker(C^n \to C^{n+1})$. Using the fact that C^n is f_*-acyclic we obtain the exact sequence

$$0 \to f_*(Z^n) \to f_*(C^n) \to f_*(Z^{n+1}) \to R^1 f_*(Z^n) \to 0$$

and the isomorphisms

$$R^k f_*(Z^{n+1}) \cong R^{k+1} f_*(Z^n)$$

for all $k \geq 1$. By induction we show that for $k \geq 1$ and all $l \in \mathbb{N}$ we have

$$R^k f_*(Z^n) \cong R^{k+l} f_*(Z^{n-l}) \ .$$

Since f_* has bounded cohomological dimension we conclude that $R^k f_*(Z^n) \cong 0$ for all $n \in \mathbb{Z}$ and $k \geq 1$. In particular the sequences

$$0 \to f_*(Z^n) \to f_*(C^n) \to f_*(Z^{n+1}) \to 0$$

are exact for all $n \in \mathbb{Z}$. This shows the exactness of $f_*(C)$. □

6.5.16.

Lemma 6.5.18. — *The integration map is independent of the choice of a flat locally f_*-acyclic resolution K of $\underline{\mathbb{Z}}_{\mathbf{X}}$ of length n.*

Proof. — Let K, L are two such resolutions. Assume that there exists a quasi-isomorphism $K \to L$. The identification

$$\operatorname{coker}(f_*L^{n-1} \to f_*L^n) \cong \operatorname{coker}(f_*K^{n-1} \to f_*K^n) \cong R^n f_*(\underline{\mathbb{Z}}_{\mathbf{X}}) \cong \underline{\mathbb{Z}}_{\mathbf{Y}}$$

gives a map $f_*L \to \underline{\mathbb{Z}}_{\mathbf{Y}}[-n]$ which induces the transformation $T_{f_*L} \to \mathrm{id}$ of degree $-n$.

It induces a commutative diagram

$$\begin{array}{ccccccccccccc}
f_*If^* & \longrightarrow & f_*T_KIf^* & \xleftarrow{\sim} & f_*T_Kf^* & \xleftarrow{\sim} & f_*T_Kf^*R & \xrightarrow{\cong} & T_{f_*K}R & \longrightarrow & R & \longrightarrow & \mathrm{id} \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \| \\
f_*If^* & \longrightarrow & f_*T_LIf^* & \xleftarrow{\sim} & f_*T_Lf^* & \xleftarrow{\sim} & f_*T_Lf^*R & \xrightarrow{\cong} & T_{f_*L}R & \longrightarrow & R & \longrightarrow & \mathrm{id}
\end{array}$$

The upper horizontal composition is the integration map defined using K (see 6.5.9), and the lower horizontal composition is the integration map defined using L. We see that both maps are equal.

Let now K, L again be flat and locally f_*-acyclic resolutions of $\underline{\mathbb{Z}}_{\mathbf{X}}$ of length n. We complete the proof of the Lemma by showing that there exists a third such resolution M together with quasi-isomorphisms $K \xrightarrow{\sim} M \xleftarrow{\sim} L$.

The maps $\underline{\mathbb{Z}}_{\mathbf{X}} \to K$ and $\underline{\mathbb{Z}}_{\mathbf{X}} \to L$, respectively, induce maps $K \to K \otimes L$ and $L \to K \otimes L$ which are quasi-isomorphisms. We further get induced quasi-isomorphisms

$$K \to \mathcal{H}(K \otimes L) \ , \quad L \to \mathcal{H}(K \otimes L) \ . \qquad (6.5.19)$$

We let $M := \tau^{\leq n}\mathscr{H}(K \otimes L)$. The maps (6.5.19) factorize over M. Note that $K \otimes L$ is flat. Since $\mathscr{H}$ and truncation preserve flatness (see Lemma 3.1.12), we see that M is flat. Since $\mathscr{H}$ in fact produces flasque and hence locally f_*-acyclic resolutions, and the cohomological dimension of f_* is bounded by n we conclude by Lemma 6.5.13 that M is locally f_*-acyclic. □

6.5.17. — In this paragraph we show that the integration map is functorial. Let $g\colon Y \to Z$ be a second proper and representable map of locally compact stacks which is an oriented locally trivial fiber bundle of closed m-dimensional manifolds.

Lemma 6.5.20. — *We have a commutative diagram*

$$\begin{array}{ccc} Rg_* \circ Rf_* \circ Lf^* \circ Lg^* & \xrightarrow{\cong} & R(g\circ f)_* \circ L(g \circ f)^* \\ \Big\downarrow {\scriptstyle Rg_*(\int_f)} & & \Big\downarrow {\scriptstyle \int_{g\circ f}} \\ Rg_* \circ Lg^*[-n] & \xrightarrow{\quad \int_g \quad} & \mathrm{id}[-n-m] \end{array}.$$

Proof. — The following sequence of modifications transforms the down-right composition into the right-down composition.

$$\begin{aligned} (6.5.21)\quad & g_*If_*If^*g^* \to g_*If_*T_KIf^*g^* \xleftarrow{\sim} g_*If_*T_Kf^*g^*R \to g_*Ig^*R \\ & \qquad \to g_*T_LIg^*R \xleftarrow{\sim} g_*T_Lg^*R \to \mathrm{id} \end{aligned}$$

$$\begin{aligned} (6.5.22)\quad & g_*If_*If^*g^* \to g_*T_LIf_*If^*g^* \to g_*T_LIf_*T_KIf^*g^* \xleftarrow{\sim} g_*T_Lf_*T_KIf^*g^* \\ & \qquad \xleftarrow{\sim} g_*T_Lf_*T_Kf^*g^*R \to g_*T_Lg^*R \to \mathrm{id} \end{aligned}$$

$$\begin{aligned} (6.5.23)\quad & g_*If_*If^*g^* \to g_*T_LIf_*If^*g^* \xleftarrow{\sim} g_*T_Lf_*If^*g^* \to g_*T_Lf_*T_KIf^*g^* \\ & \qquad \xleftarrow{\sim} g_*T_Lf_*T_Kf^*g^*R \to g_*T_Lg^*R \to \mathrm{id} \end{aligned}$$

$$\begin{aligned} (6.5.24)\quad & g_*If_*If^*g^* \xleftarrow{\sim} g_*f_*If^*g^* \to g_*T_Lf_*If^*g^* \to g_*T_Lf_*T_KIf^*g^* \\ & \qquad \xleftarrow{\sim} g_*T_Lf_*T_Kf^*g^*R \to g_*T_Lg^*R \to \mathrm{id} \end{aligned}$$

$$\begin{aligned} (6.5.25)\quad & g_*f_*If^*g^* \to g_*T_Lf_*If^*g^* \to g_*T_Lf_*T_KIf^*g^* \xleftarrow{\sim} g_*T_Lf_*T_KRIf^*g^* \\ & \qquad \xleftarrow{\sim} g_*T_Lf_*T_KRf^*g^*R \to g_*T_Lg^*R \to \mathrm{id} \end{aligned}$$

$$\begin{aligned} (6.5.26)\quad & g_*f_*If^*g^* \to g_*T_Lf_*T_KIf^*g^* \xleftarrow{\sim} g_*f_*T_{f^*L\otimes K}RIf^*g^* \\ & \qquad \xleftarrow{\sim} g_*f_*T_{f^*L\otimes K}Rf^*g^*R \to g_*T_Lg^*R \to \mathrm{id} \end{aligned}$$

$$(6.5.27)\quad (g\circ f)_*I(g\circ f)^* \to (g\circ f)_*T_MI(g\circ f)^* \xleftarrow{\sim} (g\circ f)_*T_M(g\circ f)^*R \to \mathrm{id}$$

The transition from (6.5.21) to (6.5.22) uses the fact that tensoring with L and the map $\mathrm{id} \to T_L$ can be commuted with the intermediate operations. In order to go from (6.5.22) to (6.5.23) we use the fact that g_*T_L preserves quasi-isomorphisms. The same reason and the fact that f_* preserves fibrant objects is behind the transition from (6.5.23) to (6.5.24). We use e.g. the isomorphism $g_*f_*If^*g^* \xrightarrow{\sim} g_*If_*If^*g^*$. There is a vertical quasi-isomorphism from (6.5.25) to (6.5.24). The step from (6.5.25) to (6.5.26) uses the isomorphism $T_Lf_*T_KR \xrightarrow{\sim} f_*T_{f^*L\otimes K}R$ given by the projection formula. The weak equivalence in (6.5.26) is not obvious (since $f^*L \otimes K$ is not obviously g_*f_*-acyclic), but follows from the fact, that this line is isomorphic to the previous (6.5.25). In the last step from (6.5.26) to (6.5.27) we use the map $f^*L \otimes K \to M$ given by a truncated flabby resolution of $f^*L \otimes K$ and the fact that the integration map is independent of the choice of the resolution. □

6.5.18. — Consider a cartesian diagram of locally compact stacks

(6.5.28)
$$\begin{CD} V @>v>> X \\ @VgVV @VVfV \\ U @>u>> Y \end{CD} .$$

We assume that f and u, and hence also g and v have local sections. Furthermore we assume that f is representable and a locally trivial oriented fiber bundle with a closed manifold as fiber. Then g has these properties, too. The orientation of g is induced by

$$R^n g_*\underline{\mathbb{Z}}_{\mathbf{V}} \cong R^n g_* v^*\underline{\mathbb{Z}}_{\mathbf{X}} \cong u^* R^n f_*\underline{\mathbb{Z}}_{\mathbf{X}} \cong u^*\underline{\mathbb{Z}}_{\mathbf{Y}} \cong \underline{\mathbb{Z}}_{\mathbf{U}}$$

We get diagrams

(6.5.29)
$$\begin{CD} u^*Rf_*f^* @>(6.5.6)>> Rg_*v^*f^* \\ @Vu^*\int_fVV @VV(6.5.5)V \\ u^* @<<\int_g< Rg_*g^*u^* \end{CD}$$

(6.5.30)
$$\begin{CD} Ru_*Rg_*g^* @>>> Rf_*Rv_*g^* \\ @VRu_*\int_gVV @AA\sim A \\ Ru_* @<<\int_f Ru_*< Rf_*f^*Ru_* \end{CD}$$

For the upper horizontal transformation in (6.5.29) we use 6.5.3, and for the right vertical one (6.1.15) or 6.5.7. Note that only in the bounded below derived category the right vertical morphism is an equivalence for general u (which is anyway the situation in which we will apply the assertion).

Lemma 6.5.31. — *The diagrams* (6.5.29) *and* (6.5.30) *commutes.*

To prove Lemma 6.5.31, we start with the following two technical lemmas.

Lemma 6.5.32. — *Given a Cartesian diagram* (6.5.28) *of locally compact stacks such that f and u have local sections, then for sheaves $K \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ and $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{U}$ the following diagram commutes:*

$$\begin{array}{ccc}
f_*K \otimes u_*F & \xrightarrow{=} & f_*K \otimes u_*F \\
\downarrow{\scriptstyle 6.2.8} & & \downarrow{\scriptstyle 6.2.8} \\
f_*(K \otimes f^*u_*F) & & u_*(u^*f_*K \otimes F) \\
\sim\downarrow{\scriptstyle 6.1.12} & & \sim\downarrow{\scriptstyle 6.1.12} \\
f_*(K \otimes v_*g^*F) & & u_*(g_*v^*K \otimes F) \\
\downarrow{\scriptstyle 6.2.8} & & \downarrow{\scriptstyle 6.2.8} \\
f_*v_*(v^*K \otimes g^*F) & & u_*g_*(v^*K \otimes g^*F) \\
\sim\downarrow{\scriptstyle 6.6.8} & & \sim\downarrow{\scriptstyle 6.6.8} \\
h_*(v^*K \otimes g^*F) & \xrightarrow{=} & h_*(v^*K \otimes g^*F)
\end{array},$$

where $h := f \circ v = u \circ g$.

Proof. — By Definition 6.2.8, the left vertical morphism is the image of the identity under the following sequence of maps

$$\begin{aligned}
&\mathrm{Hom}(v^*K \otimes g^*K, v^*K \otimes g^*K) \to \mathrm{Hom}(v^*f^*f_*K \otimes v^*v_*g^*K, v^*K \otimes g^*K) \\
&\to \mathrm{Hom}(v^*(f^*f_*K \otimes f^*u_*K), v^*K \otimes g^*K) \to \mathrm{Hom}(f^*(f_*K \otimes u_*K), v_*(v^*K \otimes g^*K)) \\
&\quad \to \mathrm{Hom}(f_*K \otimes u_*K, f_*v_*(v^*K \otimes g^*K)) \to \mathrm{Hom}(f_*K \otimes u_*K, h_*(v^*K \otimes g^*F)).
\end{aligned}$$

The right vertical morphism, on the other hand, is given by

$$\begin{aligned}
&\mathrm{Hom}(v^*K \otimes g^*K, v^*K \otimes g^*K) \to \mathrm{Hom}(g^*g_*v^*K \otimes g^*u^*u_*K, v^*K \otimes g^*K) \\
&\to \mathrm{Hom}(g^*(u^*f_*K \otimes u^*u_*K), v^*K \otimes g^*K) \to \mathrm{Hom}(u^*(f_*K \otimes u_*K), g_*(v^*K \otimes g^*K)) \\
&\quad \to \mathrm{Hom}(f_*K \otimes u_*K, u_*g_*(v^*K \otimes g^*K)) \to \mathrm{Hom}(f_*K \otimes u_*K, h_*(v^*K \otimes g^*F)).
\end{aligned}$$

In both cases, we first use the counit, then "commute" pushdown and pullback using Lemma 6.1.12 and finally use adjunctions. By Lemma 6.1.12, the two ways to apply the counit and the push-pull isomorphism commute. This implies commutativity of the diagram of homomorphism sets, and therefore the commutativity of the original diagram. □

Lemma 6.5.33. — *In the situation of Lemma 6.5.32 for* $K \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ *and* $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$ *the following diagram commutes:*

$$\begin{array}{ccc}
u^*(f_*K \otimes F) & \xrightarrow{6.2.8} & u^*f_*(K \otimes f^*F) \\
\downarrow{\scriptstyle 6.2.5} & & \downarrow{\scriptstyle 6.1.12} \\
u^*f_*K \otimes u^*F & & g_*v^*(K \otimes f^*F) \\
\downarrow{\scriptstyle 6.1.12} & & \downarrow{\scriptstyle 6.2.5} \\
g_*v^*K \otimes u^*F & & g_*(v^*K \otimes v^*f^*F) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle 6.6.9} \\
g_*v^*K \otimes u^*F & \xrightarrow[6.2.8]{} & g_*(v^*K \otimes g^*u^*F)
\end{array} .$$

Proof. — The left vertical and lower composition is by definition the image of the identity under the sequence of maps

$$\begin{aligned}
\mathrm{Hom}(K \otimes f^*F, K \otimes f^*F) &\xrightarrow{\text{unit}} \mathrm{Hom}(K \otimes f^*F, v_*v^*(K \otimes f^*F)) \\
&\xrightarrow{\text{adj}} \mathrm{Hom}(v^*(K \otimes f^*F), v^*(K \otimes f^*F)) \\
&\to \mathrm{Hom}(v^*K \otimes g^*u^*F, v^*K \otimes g^*u^*F) \\
&\xrightarrow{\text{counit}} \mathrm{Hom}(g^*g_*v^*K \otimes g^*u^*F, v^*K \otimes g^*u^*F) \\
&\xrightarrow{\text{adj}} \mathrm{Hom}(g_*v^*K \otimes u^*F, g_*(v^*K \otimes g^*u^*F)) \\
&\to \mathrm{Hom}(u^*(f_*K \otimes F), g_*(v^*K \otimes g^*u^*F)).
\end{aligned}$$

The upper and right vertical composition is the image of the identity under the sequence of maps

$$\begin{aligned}
\mathrm{Hom}(K \otimes f^*F, K \otimes f^*F) &\xrightarrow{\text{counit}} \mathrm{Hom}(f^*f_*K \otimes f^*F, K \otimes f^*F) \\
&\xrightarrow{\text{adj}} \mathrm{Hom}(f_*K \otimes F, f_*(K \otimes f^*F)) \\
&\xrightarrow{\text{unit}} \mathrm{Hom}(f_*K \otimes F, u_*u^*f_*(K \otimes f^*F)) \\
&\xrightarrow{\text{adj}} \mathrm{Hom}(u^*(f_*K \otimes F), u^*f_*(K \otimes f^*F)) \\
&\to \mathrm{Hom}(u^*(f_*K \otimes F), g_*v^*(K \otimes f^*F)) \\
&\to \mathrm{Hom}(u^*(f_*K \otimes F), g_*(v^*K \otimes v^*f^*F)) \\
&\to \mathrm{Hom}(u^*(f_*K \otimes F), g_*(v^*K \otimes g^*u^*F)).
\end{aligned}$$

These two maps coincide, as follows from the fact that units and counits commute (in the appropriate sense) with α_* and β^*. □

6.5.19. — We now show that (6.5.29) commutes. We simplify the definition of the integration map which is represented by all horizontal compositions in the following diagram.

$$\begin{array}{ccccccc}
f_*If^* & \longrightarrow & f_*T_KIf^* & \xleftarrow{\sim} & f_*T_Kf^*R & \longrightarrow & \mathrm{id} \\
\downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} \\
f_*If^*I & \xrightarrow{\sim} & f_*T_KIf^*I & \xleftarrow{\sim} & f_*T_Kf^*RI & \longrightarrow & I \\
{\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & \| & & \| \\
f_*f^*I & \longrightarrow & f_*T_Kf^*I & \xleftarrow[\sim]{} & f_*T_Kf^*RI & \longrightarrow & I \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
f_*f^*I\mathcal{F}l & \longrightarrow & f_*T_Kf^*I\mathcal{F}l & \xleftarrow[\sim]{} & f_*T_Kf^*RI\mathcal{F}l & \longrightarrow & I\mathcal{F}l \\
{\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow \\
f_*f^*\mathcal{F}l & \longrightarrow & f_*T_Kf^*\mathcal{F}l & \xleftarrow[\sim]{} & f_*T_Kf^*R\mathcal{F}l & \longrightarrow & \mathcal{F}l
\end{array}$$

Let us comment about the isomorphisms in the first column. Let $F \in C(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$. Then $f_*If^*(F) \to f_*If^*I(F)$ is a quasi-isomorphism since f_*If^* preserves quasi-isomorphisms and $F \to I(F)$ is a quasi-isomorphism. The map $f_*f^*I(F) \to f_*If^*I(F)$ is a quasi-isomorphism since $I(F)$ is a complex of injective, hence flabby sheaves, the functor f^* preserves flabby sheaves, and therefore the acyclic mapping cone of $C := C(f^*I(F) \to If^*I(F))$ is an exact complex of flabby sheaves. In particular it is an exact complex of f_*-acyclic sheaves. Since f_* has bounded cohomological dimension this implies that $f_*(C)$ is exact (see the argument in the proof of Lemma 6.5.17), and therefore $f_*f^*I(F) \to f_*If^*I(F)$ is a quasi-isomorphism. The map $f_*f^*I(F) \to f_*f^*I\mathcal{F}l(F)$ is a quasi-isomorphism by a similar argument. In fact, $f^*\mathcal{F}l(F) \to f^*I\mathcal{F}l(F)$ is a quasi-isomorphism of f_*-acyclic sheaves. This implies again by the mapping cone argument, that $f_*f^*\mathcal{F}l(F) \to f_*f^*I\mathcal{F}l(F)$ is a quasi-isomorphism.

The lower line of the diagram (6.5.29) expresses the integration map in terms of the flabby resolution functor $\mathcal{F}l$. Since we know that $\mathcal{F}l$ preserves flat sheaves (we do not know this for I) we can drop the flat resolution functor R from the construction of the integration by adopting the convention that the functors are applied to complexes of flat sheaves.

We get the following commutative diagram

$$
\begin{array}{ccccc}
u^*Rf_*f^* & \xrightarrow{\sim} & u^*Rf_*f^* & \xrightarrow{u^*\int_f} & u^* \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
u^*f_*T_K f^*\mathcal{H} & \xleftarrow{\sim} & u^*T_{f_*K}\mathcal{H} & \longrightarrow & u^*\mathcal{H} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
g_*v^*T_K f^*\mathcal{H} & & T_{u^*f_*K}u^*\mathcal{H} & \longrightarrow & T_{u^*\mathbb{Z}}u^*\mathcal{H} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
g_*T_{v^*K}v^*f^*\mathcal{H} & & T_{g^*v_*K}u^*\mathcal{H} & \longrightarrow & u^*\mathcal{H} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle=} \\
g_*T_{v^*K}g^*u^*\mathcal{H} & \xleftarrow{\sim} & T_{g_*v^*K}u^*\mathcal{H} & \longrightarrow & u^*\mathcal{H} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle=} \\
g_*T_{v^*K}g^*\mathcal{H}u^* & \xleftarrow{\sim} & T_{g_*v^*K}\mathcal{H}u^* & \longrightarrow & \mathcal{H}u^* \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
Rg_*g^*u^* & \xrightarrow{=} & Rg_*g^*u^* & \xrightarrow{\int_g u^*} & u^*
\end{array}
\tag{6.5.34}
$$

The commutativity of all the small squares is evident. The commutativity of the large rectangle relies on the fact that the projection formula is compatible with pullbacks, this is the statement of Lemma 6.5.33. The commutativity of the boundary of this diagram gives (6.5.29).

6.5.20. — In order to show that (6.5.30) commutes we start with the following observation.

Lemma 6.5.35. — *Assume, in the situation of Lemma 6.5.32, that K is a flat locally f_*-acyclic resolution of $\mathbb{Z}_X$ of length n, and that f is a projection of a locally trivial orientable fiber bundle of n-dimensional closed manifolds. Assume that $f_*K \to \mathbb{Z}_Y$ is an orientation. Let $g_*v^*K \to \mathbb{Z}_U$ be the induced orientation of the pullback bundle g. Then the following diagram commutes, where all the horizontal maps are given by the* *orientations.*

$$\begin{array}{ccc} f_*K\otimes u_*F & \longrightarrow & \underline{\mathbb{Z}}_Y\otimes u_*F \\ \downarrow & & \downarrow \\ u_*(u^*f_*K\otimes F) & \longrightarrow & u_*(u^*\underline{\mathbb{Z}}_Y\otimes F) \\ \downarrow\sim & & \downarrow\sim \\ u_*(g_*v^*K\otimes F) & \longrightarrow & u_*(\underline{\mathbb{Z}}_U\otimes F) \end{array}$$

Proof. — The upper diagram commutes because of the naturality of the homomorphism of the projection formula, the lower diagram commutes by the definition of the induced orientation of g. □

To understand the relation between derived pushdown along a non-representable map and integration we need to use an explicit model of the derived pushdown. If $u\colon U\to Y$ is a morphism between locally compact stacks which has local sections, then Ru_* is given by $C_A\circ\mathcal{Fl}$, where $\mathcal{Fl}$ is the functorial flabby resolution functor, and C_A is defined in Section 3.2, using an atlas $A\to U$. Note that C_A indeed can be decomposed as the composition of a functor L_A on sheaves on U and u_*. Here L_A is the sheafification of the functor on presheaves given by

$$^{p}L^k_AF(W\to U):=F(\underbrace{A\times_U\cdots\times_U A}_{k+1\text{ factors}}\times_U W\to U)\ .$$

i.e. $^{p}L^k_A=p_{k*}p_k^*$, with $p_k\colon \underbrace{A\times_U\cdots\times_U A}_{k+1\text{ factors}}\to U$.

Lemma 6.5.36. — *In the situation of Lemma 6.5.35, we obtain a commutative diagram*

$$\begin{array}{ccccccc} f_*T_Kf^*u_*L_A\mathcal{Fl} & \xrightarrow{=} & f_*T_Kf^*u_*L_A\mathcal{Fl} & \xleftarrow{\sim} & T_{f_*K}u_*L_A\mathcal{Fl} & \longrightarrow & u_*L_A\mathcal{Fl} \\ \downarrow\sim & & \downarrow\sim & & \downarrow & & \downarrow= \\ f_*T_Kv_*L_{g^*A}g^*\mathcal{Fl} & \xrightarrow[\sim]{3.2.4} & f_*T_Kv_*g^*L_A\mathcal{Fl} & & u_*T_{u^*f_*K}L_A\mathcal{Fl} & \longrightarrow & u_*L_A\mathcal{Fl} \\ & & \downarrow & & \downarrow\sim & & \downarrow= \\ & & f_*v_*T_{v^*K}g^*L_A\mathcal{Fl} & & u_*T_{g_*v^*K}L_A\mathcal{Fl} & \longrightarrow & u_*L_A\mathcal{Fl} \\ & & \downarrow\sim & & \downarrow= & & \downarrow= \\ & & u_*g_*T_{v^*K}g^*L_A\mathcal{Fl} & \longleftarrow & u_*T_{g_*v^*K}L_A\mathcal{Fl} & \longrightarrow & u_*L_A\mathcal{Fl}. \end{array}$$

Here, the right horizontal maps are given by the orientations $f_*K\to\underline{\mathbb{Z}}_Y$ and $g_*v^*K\to\underline{\mathbb{Z}}_U$.

Proof. — This is the direct translation of Lemma 6.5.32 and Lemma 6.5.35. □

Note that the upper composition is a representation (when applied to flat sheaves) of

$$Rf_*f^*Ru_* \xrightarrow{\int_f} Ru_*.$$

The leftmost vertical arrow represents the morphism

$$Rf_*f^*Ru_* \to Rf_*Rv_*g^*, \tag{6.5.37}$$

since g^* preserves flabby sheaves, and $v_*L_{g^*A}$ indeed is a model for C_{g^*A}, which can be used to calculate Rv_*.

Therefore the diagram in Lemma 6.5.36 contains one part (lower right-up) of the diagram (6.5.30).

6.5.21. — To represent the other composition of the diagram (6.5.30), we have to commute not only u_* but also L_A with the other operations. Recall that L_A provides some kind of a resolution, i.e. we have a canonical map id $\to L_A$, which is used in the Lemma below.

Lemma 6.5.38. — *In the situation of Lemma 6.5.35, the following diagram commutes, where the horizontal maps are induced by the orientation of g.*

$$\begin{array}{ccc}
u_*T_{g_*v^*K}L_A\mathcal{H} & \longrightarrow & u_*T_{\mathbb{Z}}L_A\mathcal{H} \\
\downarrow & & \downarrow \\
u_*T_{L_Ag_*v^*K}L_A\mathcal{H} & \longrightarrow & u_*T_{L_A\mathbb{Z}}L_A\mathcal{H} \\
\downarrow & & \downarrow \\
u_*L_AT_{g_*v^*K}\mathcal{H} & \longrightarrow & u_*L_AT_{\mathbb{Z}}\mathcal{H}
\end{array}$$

The second vertical map in each column follows from a variant of the projection formula, using that L_A is given by application of $(p_k)_*p_k^*$ (or by directly inspecting the definitions).

Proof. — If $G \to H$ is a morphism of sheaves, then we get a natural transformation of functors $T_G \to T_H$. This naturality implies the commutativity of the first square. The second square is commutative by the naturality of the morphism in the projection formula. □

Observe that we have a natural isomorphism $g^*L_A \cong L_{g^*A}g^*$.

Lemma 6.5.39. — *In the situation of Lemma 6.5.35, we obtain the following commutative diagram*

$$\begin{array}{ccc}
u_*g_*T_{v^*K}g^*L_A\mathscr{H}l & \longleftarrow & u_*T_{g_*v^*K}L_A\mathscr{H}l \\
\downarrow & & \downarrow \\
u_*g_*T_{L_{g^*A}v^*K}g^*L_A\mathscr{H}l & \longleftarrow & u_*T_{g_*L_{g^*A}v^*K}L_A\mathscr{H}l \\
{\scriptstyle 3.2.4}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
u_*g_*T_{L_{g^*A}v^*K}L_{g^*A}g^*\mathscr{H}l & & u_*T_{L_Ag_*v^*K}L_A\mathscr{H}l \\
\downarrow & & \downarrow \\
u_*g_*L_{g^*A}T_{v^*K}g^*\mathscr{H}l & & u_*L_AT_{g_*v^*K}\mathscr{H}l \\
{\scriptstyle 3.2.4}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle =} \\
u_*L_Ag_*T_{g_*v^*K}g^*\mathscr{H}l & \longleftarrow & u_*L_AT_{g_*v^*K}\mathscr{H}l
\end{array}$$

Proof. — The upper square is commutative because of the naturality of the morphism in the projection formula. The commutativity of the lower rectangle follows from Lemma 6.5.32, as we basically have to commute two different applications of the projection formula. □

We now prove the commutativity of (6.5.30). Using explicit representatives of the maps in question, we obtain (applied to flat sheaves)

$$\begin{array}{ccccc}
Rf_*f^*Ru_* & \xrightarrow[=]{} & Rf_*f^*Ru_* & \xrightarrow{\int_f Ru_*} & Ru_* \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
f_*T_Kf^*u_*L_A\mathscr{H}l & \xleftarrow{\sim} & T_{f_*K}u_*L_A\mathscr{H}l & \longrightarrow & u_*L_A\mathscr{H}l \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\
u_*g_*T_{v^*K}g^*L_A\mathscr{H}l & \longleftarrow & u_*T_{g_*v^*K}L_A\mathscr{H}l & \longrightarrow & u_*L_A\mathscr{H}l \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\
u_*L_Ag_*T_{g_*v^*K}g^*\mathscr{H}l & \longleftarrow & u_*L_AT_{g_*v^*K}\mathscr{H}l & \longrightarrow & u_*L_AT_{\mathbb{Z}}\mathscr{H}l \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\
u_*L_A\mathscr{H}lg_*T_{g_*v^*K}g^*\mathscr{H}l & \longleftarrow & u_*L_A\mathscr{H}lT_{g_*v^*K}\mathscr{H}l & \longrightarrow & u_*L_A\mathscr{H}l \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
Ru_*Rg_*g^* & \xrightarrow[=]{} & Ru_*Rg_*g^* & \xrightarrow{Ru_*\int_g} & Ru_*
\end{array}$$

Here, the first and the last rows are just added as illustration what the next or preceding line, respectively, computes in the derived category. The map from the third-last to the second-last row is induced by the inclusion into the flabby resolution. This step is necessary because we don't know that the functors in question are u_*-acyclic, and explains why one can directly define only the map $f^*Ru_* \to Rv_*g^*$, and why it is hard to show that this is an equivalence. The other vertical maps, and the commutativity of the remaining four squares, are given by Lemmas 6.5.36, 6.5.38, 6.5.39.

Note that the left vertical composition is the composition

$$Rf_*f^*Ru_* \to Rf_*Rv_*g^* \to Ru_*Rg_*g^*,$$

as shown in the reasoning for (6.5.37). The assertion follows. □

6.5.22. — Compared with the simplicity of its statement the proof of Lemma 6.5.31 seems to be too long. But let us mention that the proof of a similar result in the algebraic context is quite involved, too. The book [**12**] is devoted to this problem.

6.6. Extended sites

6.6.1. — We consider the lower right Cartesian square of the diagram

$$\begin{array}{ccccc} & & U \times_Y B & \dashrightarrow & B \\ & & \vdots & & \vdots \\ A \times_Y X & \dashrightarrow & U \times_Y X & \longrightarrow & X \\ \vdots & & \downarrow & & \downarrow f \\ A & \dashrightarrow & U & \longrightarrow & Y \end{array}$$

in stacks where U, X, Y are locally compact.

Lemma 6.6.1. — *If U is a space or f is representable, then $U \times_Y X$ is a locally compact stack.*

Proof. — We first assume that U is a locally compact space. Let $B \to X$ be a locally compact atlas. Then $U \times_Y B \to U \times_Y X$ is an atlas. Indeed, surjectivity, representability, and local sections for this map are implied by the corresponding properties of the map $B \to X$. The stack $U \times_Y B$ is a space since $U \to Y$ is representable by Proposition 6.1.1. By Lemma 6.1.9 the space $U \times_Y B$ is locally compact. Furthermore, again by Lemma 6.1.9,

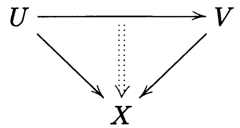
$$(U \times_Y B) \times_{(U \times_Y X)} (U \times_Y B) \cong U \times_Y (B \times_X B)$$

is locally compact since $B \times_X B$ is locally compact. Hence the atlas $U \times_Y B \to U \times_Y X$ has the properties required in Definition 2.1.2 so that $U \times_Y X$ is a locally compact stack.

We now assume that f is representable. Let $A \to U$ be a locally compact atlas such that $A \times_U A$ is locally compact. Then $A \times_Y X \cong A \times_U (U \times_Y X) \to U \times_Y X$ is an atlas of $U \times_Y X$. We again verify the properties required in Definition 2.1.2. By the special case of the Lemma already shown this atlas is locally compact. Moreover $[A \times_U (U \times_Y X)] \times_{U \times_Y X} [A \times_U (U \times_Y X)] \cong (A \times_U A) \times_Y X$ is locally compact. □

6.6.2. — If $f \colon X \to Y$ is a representable map with local sections between locally compact stacks, then for $(U \to Y) \in \mathbf{Y}$ we have ${}^p f^* h_U \cong h_{U \times_X Y}$ (see the proof of Lemma 6.6.6 below). If we drop the assumption that f is representable, then in general ${}^p f^* h_U$ is not representable. In order to overcome this defect we enlarge the site $\mathbf{X}$ to $\tilde{\mathbf{X}}$ so that it contains the stacks $U \times_X Y \to X$ over X.

We consider the 2-category $\mathrm{Stacks}^{\mathrm{top,lc}}/_{\mathrm{ls,rep}}X$ of locally compact stacks $U \to X$ over X, where the structure map is representable and has local sections. A morphism in this category is a diagram



consisting of a one-morphism and a two-morphism. The composition is defined in the obvious way. If there is a two-morphism between two such one-morphisms, then it is unique by the representability of the structure maps. Therefore $\mathrm{Stacks}^{\mathrm{top,lc}}/_{\mathrm{ls,rep}}X$ is equivalent in two-categories to the one-category obtained by identifying all isomorphic one-morphisms.

6.6.3. — Let $f : X \to Y$ be a map between locally compact stacks.

Definition 6.6.2. — *We let $\tilde{\mathbf{X}}$ be the category obtained from* $\mathrm{Stacks}^{\mathrm{top,lc}}/_{\mathrm{ls,rep}}X$ *by identifying all isomorphic one-morphisms.*

We now define the topology on $\tilde{\mathbf{X}}$. A covering family $(U_i \to U)$ of $(U \to X) \in \tilde{\mathbf{X}}$ is a family of locally compact stacks over U such that $U_i \to U$ is representable, has local sections and $\sqcup_{i \in I} U_i \to U$ is surjective [(1)]. Using Lemma 6.6.1 one easily checks the axioms listed in [**25**, 1.2.1].

Let $\hat{\mathbf{X}}$ be the site with the same underlying category as $\tilde{\mathbf{X}}$, but with the topology generated by the covering families of $(U \to X)$ given by families $(U_i \to U) \in$

(1) These maps are actually equivalence classes, but in order to simplify the language we will not mention this explicitly in the following

$\mathrm{Stacks}^{\mathrm{top,lc}}/X$ such that $U_i \to U$ is a map from a locally compact space with local sections and $\sqcup_i U_i \to U$ is surjective.

Lemma 6.6.3. — *We have a canonical isomorphism*

$$\mathrm{Sh}\tilde{\mathbf{X}} \cong \mathrm{Sh}\hat{\mathbf{X}} .$$

Proof. — The covering families of $\hat{\mathbf{X}}$ are covering families in $\tilde{\mathbf{X}}$. Here we use Proposition 6.1.1 in order to see that the maps $U_i \to U$ from spaces U_i are representable. On the other hand, every covering family $(U_i \to U)$ of $(U \to X)$ in $\tilde{\mathbf{X}}$ can be refined to a covering family in $\hat{\mathbf{X}}$ by choosing a locally compact atlas $A_i \to U_i$ for each U_i. This implies the lemma. □

6.6.4. — The natural functor $\mathrm{Top}^{\mathrm{lc}}/X \to \mathrm{Stacks}^{\mathrm{top,lc}}/X$ from locally compact spaces over X to locally compact stacks over X induces a map of sites $j\colon \mathbf{X} \to \tilde{\mathbf{X}}$.

Lemma 6.6.4. — *The restriction functor*

$$j^*\colon \mathrm{Sh}\tilde{\mathbf{X}} \to \mathrm{Sh}\mathbf{X}$$

is an equivalence of categories.

Proof. — The inverse of j^* is the functor j_* given by

$$j_*F(U) := \lim_{(V \to U) \in \mathbf{X}//U} F(V)$$

for all $(U \to X) \in \tilde{\mathbf{X}}$, where $\mathbf{X}//U$ is the category of all pairs $(V \in \mathbf{X}, j(V) \to U \in \mathrm{Mor}(\tilde{\mathbf{X}}))$ such that the map $j(V) \to U$ has local sections.

If $U \in j(\mathbf{X})$, then $(U, \mathrm{id}_{j(U)}\colon j(U) \to j(U))$ is the final object of $\mathbf{X}//U$. This gives a natural isomorphism $j^*j_*(F)(U) \cong F(U)$.

We now define a natural isomorphism $j_*j^*(F) \to F$ for all $F \in \mathrm{Sh}\tilde{\mathbf{X}}$. Let $(U \to X) \in \tilde{\mathbf{X}}$. The family $(V \to U)_{\mathbf{X}//U}$ is a covering family of $U \to X$ in $\hat{\mathbf{X}}$. Since F is also a sheaf on $\hat{\mathbf{X}}$ by Lemma 6.6.3 we get an isomorphism

$$j_*j^*(F)(U) \cong \lim_{(V \to U) \in \mathbf{X}//U} j^*(F)(V) \cong F(U) .$$ □

6.6.5.

Lemma 6.6.5. — *A map $f : X \to Y$ between locally compact stacks induces a map of sites*

$$\tilde{f}^\sharp\colon \mathbf{Y} \to \tilde{\mathbf{X}}$$

by

$$\tilde{f}^\sharp(U \to Y) := U \times_Y X \to X .$$

Proof. — Indeed, if $U \to Y$ is a map from a locally compact space, then the stack $U \times_Y X$ is locally compact by Lemma 6.6.1. If $(U_i \to U)$ is a covering family of $(U \to Y) \in \mathbf{Y}$ by open subspaces, then $(U_i \times_Y X \to U \times_Y X)$ is a covering family in $\tilde{\mathbf{X}}$ by open substacks.

Furthermore it is easy to see that $\tilde{f}^\sharp$ preserves fiber products, i.e. if $(U_i \to U)$ is a covering family and $V \to U$ is a morphism in $\mathbf{Y}$, then $\tilde{f}^\sharp(U_i \times_U V) \cong \tilde{f}^\sharp(U_i) \times_{\tilde{f}^\sharp(U)} \tilde{f}^\sharp(V)$. □

6.6.6. — We consider a map $f : X \to Y$ between locally compact stacks. Then we have an adjoint pair of functors

$$\tilde{f}^\sharp_* : \mathrm{Sh}\mathbf{Y} \leftrightarrows \mathrm{Sh}\tilde{\mathbf{X}} : (\tilde{f}^\sharp)^* .$$

Lemma 6.6.6. — *We have an isomorphism of functors* $j^* \circ \tilde{f}^\sharp_* \cong f^* : \mathrm{Sh}\mathbf{Y} \to \mathrm{Sh}\mathbf{X}$

Proof. — The map $j : \mathbf{X} \to \tilde{\mathbf{X}}$ induces a map ${}^p j^* : \mathrm{Pr}\tilde{\mathbf{X}} \to \mathrm{Pr}\mathbf{X}$. We show the relation first on representable presheaves. Let $(U \to Y) \in \mathbf{Y}$ and observe that $(U \times_Y X \to X) \in \tilde{\mathbf{X}}$ by Lemma 6.6.1. The following chain of natural isomorphisms (for arbitrary $F \in \mathrm{Pr}\tilde{\mathbf{X}}$) shows that $\tilde{f}^\sharp_* h_U \cong h_{U \times_Y X}$:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Pr}\tilde{\mathbf{X}}}(\tilde{f}^\sharp_* h_U, F) &\cong \mathrm{Hom}_{\mathrm{Pr}\mathbf{Y}}(h_U, (\tilde{f}^\sharp)^* F) \\ &\cong (\tilde{f}^\sharp)^* F(U) \\ &\cong F(\tilde{f}^\sharp(U)) \\ &\cong F(U \times_Y X) \\ &\cong \mathrm{Hom}_{\mathrm{Pr}\tilde{\mathbf{X}}}(h_{U \times_Y X}, F) . \end{aligned}$$

For $(U \to Y) \in \mathbf{Y}$ we have ${}^p f^* h_U \cong {}^p j^* h_{U \times_Y X}$. Indeed, for $(V \to X) \in \mathbf{X}$ we have

$${}^p j^* h_{U \times_Y X}(V) \cong \mathrm{Hom}_{\tilde{\mathbf{X}}}(j(V), U \times_Y X) \overset{!}{\cong} {}^p f^* h_U(V) ,$$

where the marked isomorphism can be seen by making the definition of ${}^p f^*$ explicit. Since ${}^p j^* \circ {}^p \tilde{f}^\sharp_*$ and ${}^p f^*$ commute with colimits the equation ${}^p j^* \circ {}^p \tilde{f}^\sharp_* \cong {}^p f^*$ holds on all presheaves. The restriction to sheaves (note that all functors preserve sheaves) gives $j^* \circ \tilde{f}^\sharp_* \cong f^*$. □

By adjointness we get

$$(\tilde{f}^\sharp)^* \circ j_* \cong f_* . \tag{6.6.7}$$

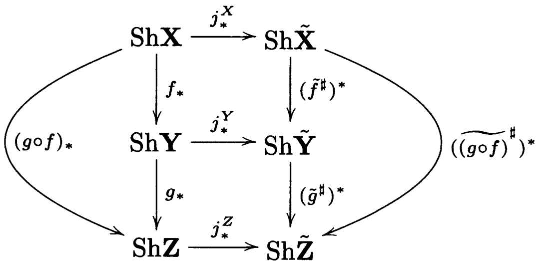
6.6.7. — Consider two composeable maps between locally compact stacks.

$$X \xrightarrow{f} Y \xrightarrow{g} Z .$$

The following lemma generalizes [**9**, Lemma 2.23] by dropping the unnecessary additional assumptions that f has local sections or g is representable.

Lemma 6.6.8. — *We have an isomorphism of functors* $g_* \circ f_* \cong (g \circ f)_* \colon \mathrm{Sh}\mathbf{X} \to \mathrm{Sh}\mathbf{Z}$.

Proof. — We consider the following diagram:



We know that the squares commute (Equation (6.6.7)), and that the horizontal arrows are isomorphisms (Lemma 6.6.4). It follows from the constructions that

$$\tilde{f}^\sharp \circ \tilde{g}^\sharp = \widetilde{(g \circ f)}^\sharp$$

on the level of sites. Hence the right triangle commutes, too. This implies commutativity of the left triangle. □

Taking adjoints we get:

Corollary 6.6.9. — *We have an isomorphism* $f^* \circ g^* \cong (g \circ f)^* \colon \mathrm{Sh}\mathbf{Z} \to \mathrm{Sh}\mathbf{X}$.

6.6.8. — We consider a topological stack X and the inclusion $j\colon \mathbf{X} \to \tilde{\mathbf{X}}$ which induces by Lemma 6.6.4 an equivalence of categories of sheaves

$$j^* \colon \mathrm{Sh}\tilde{\mathbf{X}} \leftrightarrows \mathrm{Sh}\mathbf{X} \colon j_* .$$

Note that the notion of flabbiness depends on the site.

Definition 6.6.10. — *We call a sheaf* $F \in \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}$ *strongly flabby if* $j_*(F)$ *is flabby.*

Since flabbiness is a condition to be checked for all covering families and since all covering families in $\mathbf{X}$ induce covering families in $\tilde{\mathbf{X}}$ it follows that a strongly flabby sheaf is flabby. Since injective sheaves are strongly flabby each sheaf admits a strongly flabby resolution.

6.6.9. — Let $f\colon X \to Y$ be a morphism of locally compact stacks.

Lemma 6.6.11. — *Strongly flabby sheaves are* f_**-acyclic.*

Proof. — In view of Lemma 6.6.6 it suffices to show that flabby sheaves in $\mathrm{Sh}_{\mathrm{Ab}}\tilde{\mathbf{X}}$ are $\tilde{f}_*$-acyclic. We now can write $\tilde{f}_* = \tilde{\imath}^\sharp \circ {}^p\tilde{f}_* \circ \tilde{\imath}$, where $\tilde{\imath}^\sharp$ and $\tilde{\imath}$ are the sheafification functor and the inclusion of sheaves into presheaves for the tilded sites, and ${}^p\tilde{f}_* = {}^p(\tilde{f}^\sharp)^* \colon \mathrm{Pr}\tilde{\mathbf{X}} \to \mathrm{Pr}\tilde{\mathbf{Y}}$. Since ${}^p\tilde{f}_*(F)(V \to Y) = F(V \times_Y X \to X)$ we see that ${}^p\tilde{f}_*$ is exact. Since strongly flabby sheaves are $\tilde{\imath}$-acyclic, and $\tilde{\imath}^\sharp$ is exact, it follows that strongly flabby sheaves are $\tilde{f}_*$-acyclic. □

Lemma 6.6.12. — *The functor*

$$f_*\colon \mathrm{Sh}_{\mathrm{Ab}}\mathbf{X} \to \mathrm{Sh}_{\mathrm{Ab}}\mathbf{Y}$$

preserves strongly flabby sheaves.

Proof. — We must show that $\tilde{f}_*$ preserves flabby sheaves. Let $F \in \mathrm{Sh}_{\mathrm{Ab}}\tilde{\mathbf{X}}$ and $\tau = (U_i \to U)$ be a covering family of $(U \to Y)$ in $\mathbf{Y}$. We must show that the Čech complex $C(\tau, \tilde{f}_*F)$ is acyclic. Note that $\tilde{f}_*F(V) = F(V \times_Y X)$. The family $f^\sharp(\tau) := (U_i \times_Y X \to U \times_Y X)$ is a covering family of $U \times_Y X$ in $\tilde{\mathbf{X}}$. We see that $C(\tau, \tilde{f}_*F) \cong C(f^\sharp(\tau), F)$. Since F is strongly flabby, the complex $C(f^\sharp\tau, F)$ is acyclic. □

6.6.10. — Consider again a sequence of composeable maps between locally compact stacks.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \;.$$

The following Lemma generalizes [**9**, Lemma 2.26], again by dropping the unnecessary assumptions that f has local sections or g is representable.

Lemma 6.6.13. — *We have an isomorphism of functors*

$$Rg_* \circ Rf_* \cong R(g \circ f)_*\colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{Z}).$$

Proof. — The isomorphism $(g \circ f)_* \to g_* \circ f_*$ induces a transformation $R(g \circ f)_* \to Rg_* \circ Rf_*$. Since injective sheaves are strongly flabby, f_* preserves strongly flabby sheaves, and strongly flabby sheaves are g_*-acyclic, this transformation is indeed an isomorphism. □

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